$T < 4E$ and the standard conjectures beyond positivity

Assume X is a simplicial complex on n vertices that allows for an embedding into R^2d? How many d-dimensional simplices can it have? This is a rather fundamental question. For $d = 1$, it goes back to Descartes and Euler, who established that planar simple graphs have at most three edges for every vertex. The case $d > 1$ remained elusive, and notoriously resisted modern topological and combinatorial techniques. I will discuss how this question is related to a deep problem in algebraic geometry, Grothendieck's hard Lefschetz conjecture, and indicate a new method to prove this conjecture for a case that was previously out of reach: beyond projectivity of the underlying variety. This has several interesting implications:

- We prove that for a simplicial complex that PL embeds into R^2d, the number of d-dimensional simplices exceeds the number of $(d-1)$-dimensional simplices by a factor of at most $d+2$. This generalizes a result going back to Descartes and Euler, and resolves the Gruenbaum-Kalai-Sarkaria conjecture.
- A consequence of this is a high-dimensional version of the celebrated crossing number inequality of Ajtai, Chvatal, Leighton, Newborn and Szemeredi: For a PL map of a simplicial complex X into R^2d, the number of pairwise intersections of d-simplices is at least

$$f_d^(d+2)(\Delta)/(d+3)^(d+2)f_{d-1}^{d+1}(\Delta)$$

$$\text{provided } f_d(\Delta) > (d+3)f_{d-1}(\Delta).$$

- We fully characterize the possible face numbers of simplicial rational homology spheres, resolving the g-conjecture of McMullen in full generality and generalizing Stanley's earlier proof for simplicial polytopes.
- We verify a conjecture of Kuehnel, proving tight lower bounds on the complexity of a triangulated manifold in terms of its Betti number. I intend to assume almost no background, and give a gentle introduction to the theory.