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Algebraic Geometry I

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Exercise sheet 2

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Exercise 1. Let k be a field and $\bar{k}$ an algebraic closure of k. For each point $x \in \bar{k}^n$ consider the associated substitution homomorphism

$$\begin{array}{rcccc} \phi_x & : & k[X_1, \ldots, X_n] & \to & \bar{k}, \\ & & f & \mapsto & f(x). \end{array}$$

Show: The ideals of the form $\ker \phi_x$ are precisely the maximal ideals of $k[X_1, \ldots, X_n]$, i.e., the map

$$\begin{array}{ccc} \bar{k}^n & \to & \mathrm{Max}(k[X_1, \ldots, X_n]), \\ x & \mapsto & \ker \phi_x \end{array}$$

is well-defined and surjective.
Supplementary question: Under which conditions this map is injective?

(4 points)

Exercise 2. Let k be a field. Use Hilbert's Nullstellensatz to show: For every homomorphism of finitely generated k-algebras A, B

$$\psi : A \to B,$$

the preimages of maximal ideals $\mathfrak{m} \subseteq B$ are maximal ideals of A.

(4 points)

Exercise 3. a) Let k be a field and $\mathfrak{m} \subseteq k[X_1, \ldots, X_n]$ a maximal ideal. Show that there are n polynomials f_i $(1 \leq i \leq n)$ such that for each i, f_i is a monic polynomial in the indeterminate X_i with coefficients in $k[X_1, \ldots, X_{i-1}]$ and

$$\mathfrak{m} \cap k[X_1, \ldots, X_i] = < f_1, \ldots, f_i > .$$

In particular, $\mathfrak{m} = < f_1, \ldots, f_n >$ is generated by n elements.

b) Let R be a ring and let $\mathfrak{m} \subseteq R[X_1, \ldots, X_n]$ be a maximal ideal such that $\mathfrak{m} \cap R \subseteq R$ is a maximal ideal of R generated by m elements. Show that $\mathfrak{m}$ is generated by $m + n$ elements.

Hint: For the proof of part a) use induction as in part b) and make use of exercise 2.

(4 points)

Exercise 4. Let $V \subseteq \mathbb{R}^n$ be an algebraic subset. Show that there is a polynomial $f \in \mathbb{R}[X_1, \ldots, X_n]$ with $V = Z(f)$. Is this also true for $V \subseteq \mathbb{C}^n$ or $\mathbb{Q}^n$?

(4 points)