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Pseudo-differential operators and applications

Exercises 1

1. **(Concrete Fourier Transformations)**

   (a) Compute the Fourier transformation of the following functions:

   $$\begin{aligned} f_1(x) &= \begin{cases} e^{-ax} & \text{for } x \geq 0, \\ 0 & \text{else,} \end{cases} \\ f_2(x) &= e^{-a|x|}, \\ f_3(x) &= \chi_{[-a,a]}(x), \end{aligned}$$

   where $a > 0$.

   (b) Compare the properties of the functions f_j (continuity, differentiability, analyticity, and the decay for $|x| \to \infty$) with the corresponding properties of $\widehat{f_j}$.

2. In the following, for $f \in \mathcal{S}$ and $m \in N$ let

   $$|f|_{m,\mathcal{S}} := \sup_{|\alpha|+|\beta|\leq m} \sup_{x\in\mathbb{R}^n} \left|x^\alpha \partial_x^\beta f(x)\right|.$$

   Prove that for every $\alpha \in \mathbb{N}_0^n$ and $m \in \mathbb{N}$ there are constants $C_{m,\alpha}, C'_{m,\alpha} > 0$ such that

   $$|x^\alpha f|_{m,\mathcal{S}} \leq C_{m,\alpha} |f|_{m+|\alpha|,\mathcal{S}} \qquad \text{and} \qquad |\partial_x^\alpha f|_{m,\mathcal{S}} \leq C'_{m,\alpha} |f|_{m+|\alpha|,\mathcal{S}}$$

   uniformly in $f \in \mathcal{S}(\mathbb{R}^n)$.

3. Let $C^\infty_{\text{poly}}(\mathbb{R}^n)$ be the set of all smooth functions $m\colon \mathbb{R}^n \to \mathbb{C}$ of *polynomial growth*, i.e., for every $\alpha \in \mathbb{N}_0^n$ there exist a $k(\alpha) \in \mathbb{N}$ and $C_\alpha > 0$ with

   $$|\partial_x^\alpha m(x)| \leq C_\alpha (1 + |x|)^{k(\alpha)}, \qquad \text{for all } x \in \mathbb{R}^n.$$

   Moreover, let $m \in C^\infty_{\text{poly}}(\mathbb{R}^n)$ and let $(Mf)(x) := m(x)f(x)$ for all $f \in \mathcal{S}(\mathbb{R}^n)$. Prove that $M\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a bounded operator, i.e., for all $k \in \mathbb{N}$ there exist $n(k) \in \mathbb{N}$ and $C > 0$ such that

   $$|Mf|_{k,\mathcal{S}} \leq C |f|_{n(k),\mathcal{S}}.$$

As a corollary prove the following:
For any pair of functions $f, g \in \mathcal{S}(\mathbb{R}^n)$ the product fg lies in $\mathcal{S}(\mathbb{R}^n)$.

4. **(Supplementary exercise)**
Let $(Mf)(x) := m(x)f(x)$ for $f \in \mathcal{S}(\mathbb{R}^n)$, where $m\colon \mathbb{R}^n \to \mathbb{C}$ is a smooth function. Prove that $m \in C^\infty_{\text{poly}}(\mathbb{R}^n)$ if $M\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a bounded operator.
Hint: First of all

$$\sup_{x\in\mathbb{R}^n} |m(x)f(x)| \leq C|f|_{k,\mathcal{S}}, \qquad f \in \mathcal{S}(\mathbb{R}^n)$$

for some $k \in \mathbb{N}$. Then consider $f(x) = (1+|x|^2)^{-k/2}e^{-\varepsilon|x|^2/2}$, $\varepsilon > 0$.