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Pseudo-differential operators and applications

Exercises 2

1. **(Product of Symbols)**
   Let $p_j \in S^{m_j}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $m_j \in \mathbb{R}$, $j = 1, 2$ and let $p(x, \xi) := p_1(x, \xi)p_2(x, \xi)$ for all $x, \xi \in \mathbb{R}^n$. Prove that $p \in S^{m_1+m_2}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ and that for every $k \in \mathbb{N}$ one has $|p|^{(m_1+m_2)}_k \leq C_k |p_1|^{(m_1)}_k |p_2|^{(m_2)}_k$, where C_k depends only on k and n.

2. **(Special Symbol)**
   Let $\langle\xi\rangle = \sqrt{1 + |\xi|^2}$ for $\xi \in \mathbb{R}^n$. Prove that for any $m \in \mathbb{R}$, $\alpha \in \mathbb{N}^n_0$ there is some $C_{m,\alpha} > 0$ such that

   $$|\partial^\alpha_\xi \langle\xi\rangle^m| \leq C_{m,\alpha}(1 + |\xi|)^{m-|\alpha|} \qquad \text{for all } \xi \in \mathbb{R}^n.$$

   Hint: Consider the function $f(a, x) := (a^2 + |x|^2)^{\frac{m}{2}}$, where $a \in \mathbb{R}$, $x \in \mathbb{R}^n$. Use that f is homogeneous of degree m, i.e., $f(ra, rx) = r^m f(a, x)$ for all $r > 0$, $a \in \mathbb{R} \setminus \{0\}$, $x \in \mathbb{R}^n \setminus \{0\}$.

3. **(A Simple Functional Calculus)**
   Let $\mathsf{L}^\infty(\mathbb{R}^n) := \{G : \mathbb{R}^n \to \mathbb{C} \,|\, G \text{ is bounded and measurable}\}$. For all functions $G \in \mathsf{L}^\infty(\mathbb{R}^n)$ set

   $$G(D_x)f := \mathcal{F}^{-1}[G(\xi)\hat{f}(\xi)] \qquad \text{for all } f \in L^2(\mathbb{R}^n). \tag{1}$$

   (a) Prove that for all $G \in \mathsf{L}^\infty(\mathbb{R}^n)$ we have $G(D_x) \in \mathcal{L}(L^2(\mathbb{R}^n))$ and the mapping

   $$\Phi\colon \mathsf{L}^\infty(\mathbb{R}^n) \ni G \mapsto G(D_x) \in \mathcal{L}(L^2(\mathbb{R}^n))$$

   is linear and bounded. Moreover, show that for every $G_j \in \mathsf{L}^\infty(\mathbb{R}^n)$, $j = 1, 2$, it holds that

   $$G_1(D_x) \circ G_2(D_x) = (G_1 \cdot G_2)(D_x). \tag{2}$$

   (b) Prove that if $G \in C^\infty_{\text{poly}}(\mathbb{R}^n)$, then $G(D_x)\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ (defined similarly as in (1)), is a bounded operator, and for $G_j \in C^\infty_{\text{poly}}(\mathbb{R}^n)$, $j = 1, 2$, Equation (2) holds as well.

(c) Let $p \in C^\infty_{\text{poly}}(\mathbb{R}^n)$. Prove that for all $\lambda \in \mathbb{C} \setminus \overline{p(\mathbb{R}^n)}$ we have $(\lambda - p(D_x))^{-1} \in \mathcal{L}(L^2(\mathbb{R}^n))$ and

$$(\lambda - p(D_x))(\lambda - p(D_x))^{-1} f = (\lambda - p(D_x))^{-1}(\lambda - p(D_x)) f = f$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$, where $p(D_x)f = \mathcal{F}^{-1}[p(\xi)\hat{f}(\xi)]$ for all $f \in \mathcal{S}(\mathbb{R}^n)$.

(d) For which $\lambda \in \mathbb{C}$ there exists $(\lambda - \Delta)^{-1}$ in the sense above?