Prof. Dr. H. Abels, Prof. Dr. B. Ammann
Dr. C. Neira Jimenez
Fakultät für Mathematik
Universität Regensburg

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Pseudo-differential operators and applications

Exercises 3

1. **(Elementary Composition)**
   Let $p_1(x,\xi) = \sum_{|\alpha|\le m_1} c_\alpha(x)\xi^\alpha$ be the symbol of a differential operator and let $p_2 \in S^{m_2}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. In this special case it is easy to prove that the composition of the assoziated operators is a pseudodifferential operator. Moreover, it is an elementary calculation to determine the symbol of the composition.
   More precisely, prove that

$$\begin{aligned} p_1(x,D_x)p_2(x,D_x) &= (p_1\#p_2)(x,D_x), \qquad \text{where} \\ p_1\#p_2(x,\xi) &= \sum_{|\beta|\le m_1} \frac{1}{\beta!}\partial_\xi^\beta p_1(x,\xi) D_x^\beta p_2(x,\xi). \end{aligned}$$

   In order to prove the statement you may use the identity

$$\binom{\alpha}{\beta}\xi^{\alpha-\beta} = \frac{1}{\beta!}\partial_\xi^\beta \xi^\alpha.$$

2. **(Properties of Amplitudes)**
   Let $a_j \in \mathcal{A}^{m_j}_{\tau_j}(\mathbb{R}^n \times \mathbb{R}^n)$, $j = 1, 2$, $\alpha, \beta \in \mathbb{N}_0^n$. Prove that:

   (a) $a_1 \cdot a_2 \in \mathcal{A}^{m_1+m_2}_{\tau_1+\tau_2}(\mathbb{R}^n \times \mathbb{R}^n)$ and for every $k \in \mathbb{N}$ there is a constant $C_k > 0$ independent of a_1, a_2 such that

$$|a_1 \cdot a_2|_{\mathcal{A}^{m_1+m_2}_{\tau_1+\tau_2},k} \le C_k |a_1|_{\mathcal{A}^{m_1}_{\tau_1},k} |a_2|_{\mathcal{A}^{m_2}_{\tau_2},k}.$$

   (b) $y^\alpha \cdot a_1 \in \mathcal{A}^{m}_{\tau_1+|\alpha|}(\mathbb{R}^n \times \mathbb{R}^n)$, $\eta^\alpha \cdot a_1 \in \mathcal{A}^{m+|\alpha|}_{\tau_1}(\mathbb{R}^n \times \mathbb{R}^n)$,

   (c) $\partial_y^\alpha \partial_\eta^\beta a_1 \in \mathcal{A}^{m_1}_{\tau_1}(\mathbb{R}^n \times \mathbb{R}^n)$.

3. **(Simple Properties of the Oscillatory Integrals)**

   (a) Let $A \in \mathbb{R}^{n\times n}$ such that $\det A \ne 0$ and let $a \in \mathcal{A}^m_\tau(\mathbb{R}^n \times \mathbb{R}^n)$, $m, \tau \in \mathbb{R}$. Prove that

$$\text{Os--}\iint e^{-iy\cdot A\eta} a(y, A\eta)|\det A|\, dx d\eta = \text{Os--}\iint e^{-iy\cdot\eta} a(y,\eta)\, dx d\eta.$$

(b) Let $a_1 \in \mathcal{A}^m_\tau(\mathbb{R}^n \times \mathbb{R}^n)$, $a_2 \in \mathcal{A}^m_\tau(\mathbb{R}^k \times \mathbb{R}^k)$, $m, \tau \in \mathbb{R}$, $n, k \in \mathbb{N}$ and let

$$a((y_1, y_2), (\eta_1, \eta_2)) := a_1(y_1, \eta_1) a_2(y_2, \eta_2) \quad \text{for all } (y_1, y_2), (\eta_1, \eta_2) \in \mathbb{R}^n \times \mathbb{R}^k.$$

Prove that

$$\begin{aligned} &\text{Os--} \iint_{(\mathbb{R}^n \times \mathbb{R}^k)^2} e^{-iy\cdot\eta} a(y, \eta)\, dy\, d\eta \\ &\quad = \text{Os--} \iint_{(\mathbb{R}^n)^2} e^{-iy_1\cdot\eta_1} a_1(y_1, \eta_1)\, dy_1\, d\eta_1\, \text{Os--} \iint_{(\mathbb{R}^k)^2} e^{-iy_2\cdot\eta_2} a_2(y_2, \eta_2)\, dy_2\, d\eta_2 \end{aligned}$$

where $y = (y_1, y_2), \eta = (\eta_1, \eta_2)$.