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Pseudo-differential operators and applications

Exercises 4

1. **(Ellipticity for homogeneous symbols)**
   Let $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}$, be a symbol that is *homogeneous of degree m for* $|\xi| \geq 1$, i.e.,
   $$p(x, r\xi) = r^m p(x,\xi) \qquad \text{for all } r, |\xi| \geq 1.$$
   Show that p is elliptic if and only if $p(x,\xi) \neq 0$ for all $|\xi| = 1$, $x \in \mathbb{R}^n$ and
   $$\inf_{x\in\mathbb{R}^n, |\xi|=1} |p(x,\xi)| > 0.$$

2. **(Commutator of Pseudodifferential Operators)**
   Let $p_j \in S^{m_j}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $j = 1, 2$. Show that there is some $r \in S^{m_1+m_2-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ such that
   $$p_1(x, D_x)p_2(x, D_x) - p_2(x, D_x)p_1(x, D_x) = r(x, D_x).$$

3. **(Ellipticity stable under lower order symbols)**
   Let $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}$, be an elliptic symbol, and let $q \in S^{m-\varepsilon}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ for some $\varepsilon > 0$. Prove that $p(x,\xi) + q(x,\xi)$ is an elliptic symbol of order m.

4. **(Products of polyhomogeneous symbols)**
A symbol $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ is called *polyhomogeneous* if p admits a symbolic expansion

$$p(x,\xi) \sim \sum_{k=0}^{\infty} p_k(x,\xi) \ :\Leftrightarrow\ p(x,\xi) - \sum_{k=0}^{N} p_k(x,\xi) \in S^{m-N-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n), \forall N \in \mathbb{N},$$

where $p_k \in S^{m-k}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ are homogeneous of degree $m-k$ for $|\xi| \geq 1$. Moreover, $p_0(x,\xi)$ is called *principal symbol* of $p(x,\xi)$.

(a) Show that $p(x,\xi)q(x,\xi)$ is a polyhomogeneous symbol of order m_1+m_2 if p and q are polyhomogeneous symbols of order m_1, m_2, respectively and that

$$p(x,\xi)q(x,\xi) \sim \sum_{k=0}^{\infty} \sum_{j+l=k} p_j(x,\xi)q_l(x,\xi),$$

where $p(x,\xi) \sim \sum_{j=0}^{\infty} p_j(x,\xi)$ and $q(x,\xi) \sim \sum_{l=0}^{\infty} q_l(x,\xi)$ and p_j and q_l are homogeneous of degree $m_1 - j$, $m_2 - l$, respectively, in $|\xi| \geq 1$.

(b) Show that $p\#q(x,\xi)$ is a polyhomogeneous symbol of order m_1+m_2 if p and q are polyhomogeneous symbols of order m_1, m_2, respectively, and that

$$(p\#q)(x,\xi) \sim \sum_{k=0}^{\infty} \sum_{|\alpha|+j+l=k} \frac{1}{\alpha!}\partial_\xi^\alpha p_j(x,\xi) D_x^\alpha q_l(x,\xi).$$