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SoSe 2012
05.06.2012

Pseudo-differential operators and applications

Exercises 5

1. (a) Let $p \in S^m_{1,0}(\mathbb{R}^{2n} \times \mathbb{R}^n)$ be a symbol of an operator in (x,y)-form such that $p(x,x,\xi) = 0$ for all $x, \xi \in \mathbb{R}^n$. Prove that $p(x, D_x, x) = p_L(x, D_x)$, where $p_L \in S^{m-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$.

   (b) Prove that the adjoint of an operator $a(x, D_x, x)$ is $a^*(x, D_x, x)$ with $a^*(x,y,\xi) = \overline{a(y,x,\xi)}$ for all x, y, ξ.

2. Let $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Prove that there exists $q \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$p(x, D_x) = q(D_x, x).$$

3. Let X^s, $s \in \mathbb{R}$, be Banach spaces such that

   (i) $\mathcal{S}(\mathbb{R}^n) \subseteq X^s \subseteq \mathcal{S}'(\mathbb{R}^n)$ with continuous embeddings, where functions and regular distributions are identified in the standard way.

   (ii) $\langle D_x \rangle^m : X^{s+m} \to X^s$ is a bounded linear operator for all $s, m \in \mathbb{R}$.

   (iii) $p(x, D_x) : X^0 \to X^0$ for all $p \in S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$.

   Prove that

   (a) $X^s = \{u \in \mathcal{S}'(\mathbb{R}^n) : \langle D_x \rangle^s u \in X^0\}$ and $X^s \subseteq X^t$ if $t \leq s$ with continuous embedding.

   (b) $p(x, D_x) : X^{s+m} \to X^s$ for all $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $s, m \in \mathbb{R}$.

   (c) If $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ is an elliptic symbol and

$$p(x, D_x)u = f$$

   with $f \in X^s$ and $u \in X^{-\infty} := \bigcup_{s \in \mathbb{R}} X^s$, then $u \in X^{s+m}$.