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Pseudo-differential operators and applications

Exercises 6

1. Let $p_j \in S^{m_j}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $j = 0, 1, 2, \ldots$, with $m_0 \geq \cdots \geq m_j \to -\infty$ as $j \to \infty$. Prove that there exists $p \in S^{m_0}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$p \sim \sum_{j=0}^{\infty} p_j, \text{ i.e. for all } N \in \mathbb{N},\ p - \sum_{j=0}^{N} p_j \in S^{m_{N+1}}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n).$$

2. Let $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. We define the principal symbol of $p(x, D_x)$ to be the equivalence class of p in $S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)/S^{m-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. One also calls any member of this equivalence class a principal symbol of $p(x, D_x)$.
Let A be a pseudodifferential operator on a manifold M with local symbol p.

   (a) Prove that the principal symbol of A transforms under diffeomorphisms like a function on the cotangent bundle T^*M.

   (b) Let p be a polyhomogeneous symbol, i.e. there exist $p_{m-j} \in S^{m-j}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $j = 0, 1, 2, \ldots$ with

$$p_{m-j}(x, t\xi) = t^{m-j} p_{m-j}(x, \xi), \text{ for all } t > 1,\ |\xi| \geq 1,$$

   such that $p \sim \sum_{j=0}^{\infty} p_{m-j}$. Show that the principal symbol of A is a well defined function

$$p_m : T^*M \setminus \{0\} \to \mathbb{C}.$$

3. Let $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be a smooth function with compact support. Prove that the operator

$$\begin{aligned} &\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \\ &\qquad u \mapsto \left(x \mapsto \int_{\mathbb{R}^n} K(x, y) u(y)\, dy \right) \end{aligned}$$

is in $\mathrm{OP}\, S^{-\infty}_{1,0}$.

4. Let M be a compact manifold. A parametrization $\phi : U \to V \subset M$ is a good parametrization if it can be extended to a parametrization $\widetilde{\phi} : \widetilde{U} \to \widetilde{V}$, such that U is a subset of $\widetilde{U}$ with compact closure (and thus the same holds for V and $\widetilde{V}$).

   (a) Show that a linear map $P : C^\infty(M) \to C^\infty(M)$ is a pseudo-differential operator if and only if for any good parametrization $\phi : U \to V$ the linear map P^ϕ given by the composition

   $$C_c^\infty(V) \overset{(\phi^{-1})^*}{\to} C_c^\infty(U) \hookrightarrow C^\infty(M) \overset{P}{\to} C^\infty(M) \overset{\phi^*}{\to} C^\infty(U)$$

   extends to a pseudo-differential operator on $\mathbb{R}^n$.

   (b) Assume that $\{\phi_1, \ldots, \phi_k\}$ is an atlas. Is it sufficient to check whether all P^{ϕ_i} extend to a pseudo-differential operator?