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Pseudo-differential operators and applications

Exercises 7

1. Let M be compact manifold. A parametrization $\phi : U \to V \subset M$ is a good parametrization if it can be extended to a parametrization $\widetilde{\phi} : \widetilde{U} \to \widetilde{V}$, such that U is a subset of $\widetilde{U}$ with compact closure (and thus the same holds for V and $\widetilde{V}$).

   (a) Show that a linear map $P : C^\infty(M) \to C^\infty(M)$ is a pseudo-differential operator if and only if for any good parametrization $\phi : U \to V$ the linear map P^ϕ given by composition

   $$C_c^\infty(V) \overset{(\phi^{-1})^*}{\to} C_c^\infty(U) \hookrightarrow C^\infty(M) \overset{P}{\to} C^\infty(M) \overset{\phi^*}{\to} C^\infty(U)$$

   extends to a pseudo-differential operator on $\mathbb{R}^n$.

   (b) Assume that $\{\phi_1, \ldots, \phi_k\}$ is an atlas. Is it sufficient to check whether all P^{ϕ_i} extend to a pseudo-differential operator?

2. Let M be a compact manifold with a fixed volume form, and E and F two vector bundls over M. Let $P : \Gamma(E) \to \Gamma(F)$ be a pseudo-differential operator of symbol class $S^m_{1,0}$. Then there exists an pseudo-differential operator $P^* : \Gamma(F^*) \to \Gamma(E^*)$ which is (formally) adjoint to P, i.e.

   $$(\psi, P\phi)_{L^2} = (P^*\psi, \phi)_{L^2}.$$

3. Let u be a distribution on M, which is a compact manifold or Euclidean space. Show that u is smooth on an open subset U of M in the sense of the lecture if and only if for all $x \in U$ there is an open neighborhood V_x of x in U such that u is smooth on V_x.

4. Let P be an elliptic pseudodifferential operator on Euclidean space or on a compact manifold. Show for all $u \in \mathcal{D}'$

   $$\text{sing supp } (Pu) = \text{sing supp } u.$$

   Note: This property is called *hypo-ellipticity*.