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Differential Geometry I
Exercise Sheet no. 3

Exercise 1
Let M^m be an m-dimensional submanifold of $\mathbb{R}^k$ an $p \in M^m$ be a point. Prove that the tangent space of the manifold M^m at p as defined in the lecture can be identified with the tangent space of the submanifold M^m at p you already know from previous lectures.

Exercise 2
Show that any topological manifold carries an atlas with countably many charts.

Exercise 3
Let M be a set and $n \in \mathbb{N}$. Further, we assume that a family of bijective maps $(\phi_\alpha : U_\alpha \to V_\alpha)_{\alpha \in A}$ is given, where U_α is a subset of M and where V_α is an open subset of $\mathbb{R}^n$. This family is supposed to satisfy:

(i) $M = \bigcup_{\alpha \in A} U_\alpha$,

(ii) $\phi_\alpha(U_\alpha \cap U_\beta)$ is open in $\mathbb{R}^n$ for all $\alpha, \beta \in \mathbb{R}^n$

(iii) $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is continuous for all $\alpha, \beta \in A$.

Show that

(a) There is a unique topology on M such that all U_α are open and such that all ϕ_α are homeomorphisms.

(b) If $A_1 \subset A$ satisfies $M = \bigcup_{\alpha \in A_1} U_\alpha$, then $(\phi_\alpha : U_\alpha \to V_\alpha)_{\alpha \in A}$ and $(\phi_\alpha : U_\alpha \to V_\alpha)_{\alpha \in A_1}$ induce the same topology on M.

(c) The topology on M is second countable if A is countable.

(d) Suppose that for any $p, q \in M$ we have:

   (i) there is $\alpha \in A$ with $p, q \in U_\alpha$, **or**

   (ii) there are $\alpha, \beta \in A$ with $p \in U_\alpha$, $q \in U_\beta$, $U_\alpha \cap U_\beta = \emptyset$.

   Then the topology on M is Hausdorff.

Are the sufficient conditions in (c) resp. (d) for second countability resp. Hausdorff property also necessary?

Exercise 4
Let $\mathbb{C}\mathrm{P}^n := \mathbb{C}^{n+1} \setminus \{0\}/_{\sim}$ denote the complex projective space where, by definition, $x \sim y \iff x \in \mathbb{C} \cdot y$. We note by $[x^1, \ldots, x^{n+1}]$ the equivalence class of $(x^1, \ldots, x^{n+1}) \in \mathbb{C}^{n+1} \setminus \{0\}$. For $\alpha \in \{1, \ldots, n+1\}$ we let U_α be the subset of all $[x^1, \ldots, x^{n+1}] \in \mathbb{C}\mathrm{P}^n$ with $x^\alpha \neq 0$ and define the map

$$\begin{aligned} \phi_\alpha : U_\alpha &\longrightarrow \mathbb{C}^n \\ [x^1, \ldots, x^{n+1}] &\longmapsto \left(\frac{x_1}{x_\alpha}, \ldots, \widehat{x_\alpha}, \ldots, \frac{x_{n+1}}{x_\alpha} \right), \end{aligned}$$

where, as usual, $\widehat{x_\alpha}$ means that the α^{th} coordinate is omitted.

1. Show that U_α and ϕ_α are well-defined and that ϕ_α is bijective, for all $\alpha \in \{1, \ldots, n+1\}$.
2. Show with the help of Exercise 3 that $\mathcal{A} := \{(U_\alpha, \phi_\alpha), 1 \leq \alpha \leq n+1\}$ defines a structure of C^∞ manifold on $\mathbb{C}\mathrm{P}^n$.
3. Show that the underlying topology coincides with the quotient topology, where $\mathbb{C}^{n+1}$ carries the standard topology.

Abgabe der Lösungen: ***Montag, den 5.11.2012*** *vor der Vorlesung.*