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WS 2012/13
5.11.2012

Differential Geometry I
Exercise Sheet no. 4

Exercise 1
Let X be a tangent vector field on a smooth manifold M, that is, X is a map $M \to TM$ with $\pi \circ X = \mathrm{Id}_M$, where $\pi : TM \to M$ is the projection map. Recall that, given a chart $\varphi : U \to V$ of M, the associated coordinate vector fields $\{\frac{\partial}{\partial \varphi^1}, \ldots, \frac{\partial}{\partial \varphi^n}\}$ form a basis of TM in each point of U, in particular any tangent vector field X on M can be written in the form $X = X^i \frac{\partial}{\partial \varphi^i}$ on U, where $X^1, \ldots, X^n : U \to \mathbb{R}$ are functions.
Show that X is smooth as a map between manifolds if and only if the functions $X^1, \ldots, X^n : U \to \mathbb{R}$ are smooth for any chart.

Exercise 2
Let $f : M \to \mathbb{R}$ be a C^1 map on a compact smooth n-dimensional manifold, where $n \geq 1$.

1. Let $p \in M$ be a point. Show that, if f reaches a maximum or a minimum at p, then $d_p f = 0$.

2. Show that the differential map of f vanishes in at least two points in M.

3. Show that f has exactly one critical value if and only if f is constant.

Exercise 3
Let M be a compact smooth n-dimensional manifold. By definition, a *one-parameter group of diffeomorphisms* on M is a smooth map $\varphi : M \times \mathbb{R} \to M$, $(x, t) \mapsto \varphi_t(x)$, with $\varphi_0 = \mathrm{Id}_M$ and $\varphi_t \circ \varphi_s = \varphi_{t+s}$ for all $s, t \in \mathbb{R}$.

1. Show that, given any one-parameter group of diffeomorphisms $(\varphi_t)_t$ on M, the map $X(x) := \frac{d}{dt}|_{t=0}(\varphi_t(x))$ defines a smooth tangent vector field on M.

2. Conversely, show that, given any smooth vector field X on M, there exists a unique one-parameter group of diffeomorphisms $(\varphi_t)_t$ on M such that $\frac{d}{dt}|_{t=0}(\varphi_t(x)) = X(x)$ for all $x \in M$.
   Hint: First construct $\varphi_t(x)$ for fixed x and t close to 0 using the theorem of Picard-Lindelöff; then show that $t \mapsto \varphi_t(x)$ can be extended on $\mathbb{R}$.

Abgabe der Lösungen: ***Montag, den 12.11.2012*** *vor der Vorlesung.*