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Differential Geometry I
Exercise Sheet no. 5

Exercise 1

Let M be a smooth n-dimensional manifold and, for each point $p \in M$, $g_{|_p}$ be a Euclidean inner product on T_pM. Show that the following statements are equivalent:

1. For any smooth tangent vector fields X, Y on M, the map $M \to \mathbb{R}$, $p \mapsto g_{|_p}(X(p), Y(p))$, is smooth.

2. For any chart $\varphi : U_\varphi \to V_\varphi$ of M and all $1 \leq i, j \leq n$, the function $g_{ij}^\varphi : V_\varphi \to \mathbb{R}$ defined in the lecture is smooth.

Exercise 2

Let M^m, N^n be smooth manifolds and $\phi : M \to N$ be an immersion, that is, ϕ is a smooth map and $d\phi_{|_p} : T_pM \to T_{\phi(p)}N$ is injective for any $p \in M$. Show that, for any Riemannian metric h on N, the map $p \mapsto (d\phi_{|_p})^* h_{|_p}$ introduced in the lecture defines a Riemannian metric on M.

Exercise 3

Let M be a smooth n-dimensional manifold. Recall that a *derivation* on M is a linear map $\delta : C^\infty(M) \to C^\infty(M)$ which satisfies the product rule: for all $f_1, f_2 \in C^\infty(M)$,

$$\delta(f_1 f_2) = (\delta f_1) f_2 + f_1(\delta f_2).$$

Let X, Y are two smooth tangent vector fields on M.

1. Show that $[\partial_X, \partial_Y] := \partial_X \circ \partial_Y - \partial_Y \circ \partial_X$ defines a derivation on M. Here, ∂_X is the derivation associated to X as in the lecture.

2. Deduce that there exists a unique smooth tangent vector field on M, which we denote by $[X, Y]$, such that $\partial_{[X,Y]} = [\partial_X, \partial_Y]$.

3. Show that, for any $f \in C^\infty(M)$, one has $[X, fY] = \partial_X f \cdot Y + f[X, Y]$.

4. Show that, if $\varphi : U_\varphi \to V_\varphi$ is a chart of M, then $[\frac{\partial}{\partial \varphi^i}, \frac{\partial}{\partial \varphi^j}] = 0$ for all $1 \leq i, j \leq n$. Deduce that, if $X_{|_{U_\varphi}} = X^i \frac{\partial}{\partial \varphi^i}$ and $Y_{|_{U_\varphi}} = Y^i \frac{\partial}{\partial \varphi^i}$, then

$$[X, Y]_{|_{U_\varphi}} = \big(\partial_X(Y^i) - \partial_Y(X^i)\big) \frac{\partial}{\partial \varphi^i} = \Big(X^j \frac{\partial Y^i}{\partial \varphi^j} - Y^j \frac{\partial X^i}{\partial \varphi^j}\Big) \frac{\partial}{\partial \varphi^i}.$$

Exercise 4
Let $\langle\!\langle \cdot\,,\cdot \rangle\!\rangle$ denote the following bilinear form on $\mathbb{R}^{n+1}$:

$$\langle\!\langle x, y \rangle\!\rangle := -x_0 y_0 + \sum_{j=1}^{n} x_j y_j$$

for all $x = (x_0, x_1, \ldots, x_n)$ and $y = (y_0, y_1, \ldots, y_n)$ in $\mathbb{R}^{n+1}$.

1. Show that $\langle\!\langle \cdot\,,\cdot \rangle\!\rangle$ defines a non-degenerate symmetric bilinear form of index 1 on $\mathbb{R}^{n+1}$.

2. Let $\mathbb{H}^n := \{x \in \mathbb{R}^{n+1}\,,\, \langle\!\langle x, x \rangle\!\rangle = -1 \text{ and } x_0 > 0\} \subset \mathbb{R}^{n+1}$. Show that $\mathbb{H}^n$ is a smooth n-dimensional submanifold of $\mathbb{R}^{n+1}$.

3. Prove that, for any $p \in \mathbb{H}^n$, the tangent space of $\mathbb{H}^n$ at p can be canonically identified with $E_p := \{X \in \mathbb{R}^{n+1}\,,\, \langle\!\langle X, p \rangle\!\rangle = 0\}$.

4. Show that $\langle\!\langle \cdot\,,\cdot \rangle\!\rangle_{|E_p \times E_p}$ is positive-definite and deduce that $\langle\!\langle \cdot\,,\cdot \rangle\!\rangle$ induces a Riemannian metric on $\mathbb{H}^n$.

Abgabe der Lösungen: ***Montag, den 19.11.2012*** *vor der Vorlesung.*