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Differential Geometry I
Exercise Sheet no. 6

Exercise 1

Let $F : M \longrightarrow N$ be a smooth map between smooth manifolds M and N. Let X, Y (resp. $\tilde{X}, \tilde{Y}$) be (smooth) vector fields on M (resp. N). We say that X is *F-related* to $\tilde{X}$ iff $dF \circ X = \tilde{X} \circ F$ holds on M.
Show that, if X is F-related to $\tilde{X}$ and Y is F-related to $\tilde{Y}$, then $[X, Y]$ is F-related to $[\tilde{X}, \tilde{Y}]$.

Exercise 2

Let M^n be a smooth n-dimensional submanifold of $\mathbb{R}^k$. Given vector fields X, Y on M, we extend them to vector fields $\overline{X}, \overline{Y}$ in an open neighbourhood of M in $\mathbb{R}^k$ and set $(\nabla_X Y)_{|_p} := \mathrm{pr}_p\left((\partial_{\overline{X}}\overline{Y})_{|_p}\right)$ for every $p \in M$, where $\mathrm{pr}_p : \mathbb{R}^k \to T_pM$ denotes the orthogonal projection on T_pM (identified to a vector subspace of $\mathbb{R}^k$).
Show that ∇ defines a metric and torsion-free connection on M.

Exercise 3

Recall that an *isometry* between two smooth n-dimensional Riemannian manifolds (M, g) and (N, h) is a diffeomorphism $\varphi : M \to N$ which preserves the metric, that is, with $\varphi^* h = g$.

1. Show that, in the case $(M, g) = (N, h)$, the isometries of (M, g) form a group w.r.t. the composition of maps.

2. Let $\mathrm{Aff}(M, g)$ be the set of diffeomorphisms of M preserving the Levi-Civita connection ∇ of (M, g), that is,

$$\mathrm{Aff}(M, g) := \{\varphi : M \to M \text{ diffeo. with } \nabla_{\varphi_* X}\varphi_* Y = \varphi_*(\nabla_X Y)\ \forall X, Y \in \mathfrak{X}(M)\},$$

   where, for any $X \in \mathfrak{X}(M)$, the vector field $\varphi_* X \in \mathfrak{X}(M)$ is defined by $(\varphi_* X)(x) := d_{\varphi^{-1}(x)}\varphi(X(\varphi^{-1}(x)))$ for all $x \in M$. Show that $\mathrm{Aff}(M, g)$ is a group containing the group of isometries of (M, g).

3. Let $M = \mathbb{R}^n$ be endowed with the standard Riemannian metric g, i.e. the Euclidean metric. Determine all elements $\varphi \in \mathrm{Aff}(M, g)$ with $\varphi(0) = 0$.

Exercise 4
Let M be a smooth manifold. Given a 1-parameter group of diffeomorphisms $\varphi : M \times \mathbb{R} \to M$, $(x,t) \mapsto \varphi_t(x)$ on M, let X be associated tangent vector field on M as in Exercise no. 3 of Sheet 4. Show that, for any smooth tangent vector field Y on M,

$$\frac{d}{dt}_{|t=0} (\varphi_t)_* Y = -[X, Y],$$

where, for any $t \in \mathbb{R}$, $(\varphi_t)_* Y$ denotes the push-out tangent vector field of Y defined in the last exercise above.

Abgabe der Lösungen: ***Montag, den 26.11.2012*** *vor der Vorlesung.*