Universität Regensburg, Mathematik — WS 2012/13
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Differential Geometry I
Exercise Sheet no. 7

Exercise 1

Let (M, g) be a smooth compact Riemannian manifold. Show that every geodesic of (M, g) is defined in $\mathbb{R}$.

Exercise 2

1. Let M_1 and M_2 be two smooth surfaces in $\mathbb{R}^3$, and assume that we have a smooth curve $c : I \to \mathbb{R}^3$ with $c(I) \subset M_1 \cap M_2$. Further we assume that $T_{c(t)}M_1 = T_{c(t)}M_2$ for all $t \in I$. Show that the parallel transports along c in M_1 and in M_2 coincide.

2. Given $\theta \in]0, 2\pi[$ let $C := \{(r\cos\varphi, r\sin\varphi)\,,\ r \in]0,\infty[,\ \varphi \in]0,\theta[\} \subset \mathbb{R}^2$. Determine the parallel transport along the curve $c_r :]0,\theta[\to C$, $t \mapsto (r\cos t, r\sin t)$, where $r > 0$.

3. Deduce an explicit formula for the parallel transport along a circle of latitude $t \mapsto (\cos t\cos\varphi, \sin t\cos\varphi, \sin\varphi)$ in S^2, where $\varphi \in]-\frac{\pi}{2}, \frac{\pi}{2}[$.

Exercise 3 *(Geodesics in hyperbolic space)*

1. Let $\phi : M \to M$ be an isometry of (M, g). For any $X \in TM$ let $\gamma_X : I \to M$ be a geodesic with $\dot\gamma_X(0) = X$.
   Show: $\phi(\gamma_X(t)) = \gamma_X(t)$ for all $t \in I$ if and only if $d\phi(X) = X$. If $\gamma : I \to M$ is an arbitrary curve in M with $\phi(\gamma(t)) = \gamma(t)$ for all $t \in I$, then $d\phi(\frac{\nabla}{dt}\dot\gamma(t)) = \frac{\nabla}{dt}\dot\gamma(t)$ for all $t \in I$.

2. Let $\mathbb{H}^n := \{x \in \mathbb{R}^{n+1}\,,\ \langle\langle x, x\rangle\rangle = -1 \text{ and } x_0 > 0\}$ denote the n-dimensional hyperbolic space (see Exercise no. 4 in Sheet 5). We identify $T_x\mathbb{H}^n$ with $x^\perp := \{V \in \mathbb{R}^{n+1}\,,\ \langle\langle V, x\rangle\rangle = 0\}$. For V in $x^\perp$ we define $\|V\| := \sqrt{\langle\langle V, V\rangle\rangle}$. For $V \in x^\perp \setminus \{0\}$ show that
$$\gamma_{x,V}(t) := \cosh(\|V\|t)x + \sinh(\|V\|t)\frac{V}{\|V\|}$$
   is a curve in $\mathbb{H}^n$.

3. Determine a linear map $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with $\Phi^*\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle = \langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle$ such that its fixed point set is the plane spanned by $x \in \mathbb{H}^n$ and $V \in x^\perp \setminus \{0\}$. Show that its restriction to $\mathbb{H}^n$ defines an isometry $\phi : \mathbb{H}^n \to \mathbb{H}^n$. What is the fixed point set?

4. Conclude that $\frac{\nabla}{dt}\dot\gamma_{x,V}(t) = f(t)\dot\gamma_{x,V}(t)$ for all $t \in \mathbb{R}$ and for a suitable function f.

5. Show that $\gamma_{x,V}$ is a geodesic. *Hint: Calculate* $\|\dot{\gamma}_{x,V}(t)\|$. Are all non-constant geodesics in $\mathbb{H}^n$ of this form?

Exercise 4

Does there exist a Riemannian metric

1. on $\mathbb{R}^2$ such that all circles can be parametrized as geodesics?
2. on $\mathbb{R}^2 \setminus \{0\}$ such that all circles centered at 0 can be parametrized as geodesics?
3. on $\mathbb{R}^2 \setminus \{0\}$ such that all circles centered at 0 can be parametrized as geodesics but *no* ray through 0 is a geodesic?

Justify each of your answers.

Abgabe der Lösungen: ***Montag, den 3.12.2012*** *vor der Vorlesung.*