Universität Regensburg, Mathematik WS 2012/13
Prof. Dr. Bernd Ammann 3.12.2012
Dr. Nicolas Ginoux

# Differential Geometry I
## Exercise Sheet no. 8

**Exercise 1**
Let $(M^n, g)$ be a smooth $n$-dimensional Riemannian manifold and $f : M \to \mathbb{R}$ be a smooth function on $M$.

(a) Show that there exists a unique smooth tangent vector field – which we denote by $\mathrm{grad} f$ and call the *gradient vector field* of $f$ – on $M$ such that

$$\partial_X f = g(\mathrm{grad} f, X)$$

holds for every $X \in \mathfrak{X}(M)$.

(b) Given any $p \in M$ and any orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_pM$ (i.e., $g_p(e_i, e_j) = \delta_i^j$), prove that $\mathrm{grad} f|_p = \sum_{j=1}^n \partial_{e_j} f \cdot e_j$.

(c) Given any chart $\varphi : U_\varphi \to V_\varphi$ of $M$, show that

$$\mathrm{grad} f_{|U_\varphi} = \sum_{i,j=1}^n g^{ij} \frac{\partial f}{\partial \varphi^i} \cdot \frac{\partial}{\partial \varphi^j},$$

where $(g^{ij})_{1 \le i,j \le n}$ is the inverse matrix of the matrix $(g_{kl})_{1 \le k,l \le n}$ of $g$ in the chart $\varphi$.

**Exercise 2** *(Poincaré half-space)*
Let $H := \{(x, y) \in \mathbb{R}^2\,,\, y > 0\}$ denote the upper half-space and define the Riemannian metric $g$ on $H$ by

$$g := \frac{dx^2 + dy^2}{y^2}.$$

(a) Compute the Christoffel symbols of the Levi-Civita connection in the canonical coordinates $x, y$ of $H$.

(b) Let $c : \mathbb{R} \to H$, $t \mapsto (t, 1)$ and $v_0 := (0, 1) \in \mathbb{R}^2 \cong T_{(0,1)}H$. Show that, for the parallel vector field $v$ along $c$ with $v(0) = v_0$, the vector $v(t)$ makes an angle equal to $t$ with the $y$-axis, for all $t$.

**Exercise 3** *(Surfaces of Revolution and Clairaut's theorem)*
For an interval $I$ and a positive smooth function $f : I \to \mathbb{R}^+$ we define

$$F(x, \phi) := \begin{pmatrix} x \\ f(x)\cos(\phi) \\ f(x)\sin(\phi) \end{pmatrix}.$$

The surface of revolution generated by $f$ is

$$M_f := \left\{ F(x, \phi) \,\middle|\, x \in I, \phi \in \mathbb{R} \right\}.$$

Then $\partial F/\partial x$ and $\partial F/\partial \phi$ define vector fields along $F : I \times \mathbb{R} \to M_f$.

(a) Construct vector fields $X_x$, $X_\phi \in \mathcal{X}(M_f)$ such that $X_x \circ F = \partial F/\partial x$ and $X_\phi \circ F = \partial F/\partial\phi$. Show that $X_x$ and $X_\phi$ are everywhere orthogonal.

(b) Show that $c_{x_0} : \mathbb{R} \to M_f$, $\phi \mapsto F(x_0, \phi)$ is a geodesic if and only if $f'(x_0) = 0$. *Hint: Exercise 2 from sheet no. 7 is helpful.*

(c) Let $\gamma(t) = F(x(t), \phi(t))$ be a curve in $M_f$. Verify

$$g\Big(\dot\gamma(t), \frac{\partial F}{\partial \phi}(x(t), \phi(t))\Big) = f(x(t))^2 \dot\phi(t) \tag{1}$$

(d) Verify the formula

$$\frac{d}{dt}\left(g\Big(\dot\gamma(t), \frac{\partial F}{\partial\phi}\Big)\right) = g\Big(\frac{\nabla}{dt}\dot\gamma(t), \frac{\partial F}{\partial\phi}\Big) + \dot\phi(t) g\Big(\dot\gamma(t), \frac{\partial^2 F}{\partial^2\phi}\Big) + \dot x(t) g\Big(\dot\gamma(t), \frac{\partial^2 F}{\partial\phi\,\partial x}\Big)$$

where we suppressed $(x(t), \phi(t))$ in the notation.

(e) Show that both $\|\dot\gamma(t)\|$ and $f(x(t))^2\dot\phi(t)$ are constant if $\gamma$ is a geodesic. *Hint: Derive both sides of* (1) *with respect to* $t$ *and use (d).*

(f) (Extra question, 2 bonus points.) Does "only if" hold in (e)?

**Exercise 4**

Let $\nabla$ be any connection on a smooth manifold $M$. Let $c : I \to M$ be any smooth curve and denote by $P_{s,t} : T_{c(s)}M \to T_{c(t)}M$ the parallel transport along $c$. Show that, for any smooth vector field $X$ along $c$ and any $t_0 \in I$, we have

$$\frac{\nabla X}{dt}(t_0) = \lim_{t\to t_0}\left(\frac{P_{t,t_0}(X(t)) - X(t_0)}{t - t_0}\right).$$

*Abgabe der Lösungen:* ***Montag, den 10.12.2012*** *vor der Vorlesung.*