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Differential Geometry I
Exercise Sheet no. 9

Exercise 1
Let $\gamma : [a,b] \longrightarrow M$ be a piecewise C^1 curve on a smooth Riemannian manifold (M,g).

(a) Prove that $L[\gamma]^2 \leq 2(b-a)\cdot E[\gamma]$, where $E[\gamma] := \frac{1}{2}\int_a^b g(\dot\gamma,\dot\gamma)dt$ is the *energy* of the curve γ.

(b) Show that $L[\gamma]^2 = 2(b-a)\cdot E[\gamma]$ holds iff γ is parametrized proportionally to arc-length.

Exercise 2
Let M be a smooth submanifold of $\mathbb{R}^k$.

(a) Show that, if M is closed, then M is complete.

(b) Show that the converse statement is wrong.

Exercise 3
Let (M,g) be a connected complete non-compact Riemannian manifold and $p \in M$ be a point.

(a) Show that there exists a sequence $(p_i)_{i\in\mathbb{N}}$ in M such that $d(p,p_i) \underset{i\to\infty}{\longrightarrow} \infty$.

(b) Show that, for each $i \in \mathbb{N}$, there exist $X_i \in T_pM$ and $r_i \in [0,\infty[$ with $g_p(X_i,X_i) = 1$ and $p_i = \exp_p(r_iX_i)$.

(c) Show that the sequence $\{X_i\}_{i\in\mathbb{N}}$ admits a converging subsequence and deduce that there exists a ray $\gamma : [0,\infty) \to M$ in (M,g) with $\gamma(0) = p$.

Exercise 4
Let M be a connected m-dimensional manifold, and assume that $N \subset M$ is an n-dimensional submanifold, i.e. for every $p \in N$ there is a chart $\phi : U \to V \subset \mathbb{R}^m$, $p \in U$ such that $\phi(U \cap N) = V \cap (\mathbb{R}^n \times \{0\})$. Let g be a Riemannian metric on M, such that (M,g) is complete, and assume that N is a closed (as a subset of M). Fix a point $q \in M$.

(a) Show the existence of a point $p \in N$ with $d(q,p) = d(q,N)$, where $d(q,N) := \inf_{x\in N}\{d(q,x)\}$. Is p unique? Justify your answer.

(b) Prove that there is a geodesic γ from q to p with length $L[\gamma] = d(q,p)$.

(c) Show that γ meets N orthogonally.

Abgabe der Lösungen: ***Montag, den 17.12.2012*** *vor der Vorlesung.*