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WS 2012/13
17.12.2012

Differential Geometry I
Exercise Sheet no. 10

Exercise 1
Show that the map $p : S^n \to \mathbb{R}\mathrm{P}^n$, $x \mapsto \mathbb{R}x$ is a local diffeomorphism, i.e. every $x \in S^n$ is in an open set U such that $p(U)$ is open in $\mathbb{R}\mathrm{P}^n$ and such that $p|_U$ is a diffeomorphism from U to $p(U)$. Show that $\mathbb{R}\mathrm{P}^n$ carries a metric g_0 such that p^*g_0 is the standard metric on S^n. This metric g_0 is called the *standard metric of* $\mathbb{R}\mathrm{P}^n$. Determine the injectivity radius of $(\mathbb{R}\mathrm{P}^n, g_0)$.

Exercise 2
The *tautological bundle* on the n-dimensional real projective space $\mathbb{R}\mathrm{P}^n$ is given by $L := \{(\ell, y) \in \mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1}, y \in \ell\}$ together with the projection map $\pi : L \to \mathbb{R}\mathrm{P}^n$, $(\ell, y) \mapsto \ell$. Prove that there does not exist any continuous and nowhere vanishing section s of $\pi : L \to \mathbb{R}\mathrm{P}^n$. *(Hint: Interprete such a section as a map* $\mathbb{R}\mathrm{P}^n \to \mathbb{R}^{n+1} \setminus \{0\}$*; considering the composition with the map* $S^n \to \mathbb{R}\mathrm{P}^n$*, get a map* $S^n \to S^n$ *which has to be* $\pm\mathrm{Id}$*; conclude.)*

Exercise 3 *(Geodesics and distance function on products)*

(a) Let $\gamma : [a, b] \to M$ be a piecewise C^1 curve on a smooth Riemannian manifold (M, g). Prove that γ minimizes the energy functional $E : c \mapsto \frac{1}{2}\int_a^b g(\dot{c}, \dot{c})\, dt$ among all piecewise C^1 curves $c : [a, b] \to M$ with $c(a) = p$ and $c(b) = q$ iff γ is a minimal geodesic.

(b) From now on let $(M, g) := (M_1 \times M_2, g_1 \oplus g_2)$, where (M_i, g_i) is a smooth Riemannian manifold and the product manifold $M_1 \times M_2$ (see Exercise no. 1 in Sheet 2) is equipped with the *product metric* $g_1 \oplus g_2$, which is defined at $p = (p_1, p_2) \in M_1 \times M_2$ by:

$$(g_1 \oplus g_2)|_{(p_1,p_2)}\left((X_1, X_2), (Y_1, Y_2)\right) := g_1|_{p_1}(X_1, Y_1) + g_2|_{p_2}(X_2, Y_2)$$

for all $X_i, Y_i \in T_{p_i}M$, $i = 1, 2$. Show that, if $\gamma_i : [a, b] \to M_i$ is a piecewise C^1 curve, $i = 1, 2$, then $\gamma := (\gamma_1, \gamma_2) : [a, b] \to M_1 \times M_2$ is a piecewise C^1 curve with $E(\gamma) = E(\gamma_1) + E(\gamma_2)$.

(c) Show that γ is a minimal geodesic iff γ_1 and γ_2 are minimal geodesics.

(d) Deduce that the distance function d associated to $g = g_1 \oplus g_2$ is given by

$$d\left((x_1, x_2), (y_1, y_2)\right) = \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2},$$

for all $(x_1, x_2), (y_1, y_2) \in M_1 \times M_2$, where d_i is the distance function associated to the metric g_i on M_i.

Exercise 4 *(Sufficient criterion for the existence of a line)*
Let (M,g) be a complete smooth Riemannian manifold.

(a) Let $(X_k)_{k\in\mathbb{N}}$ be a sequence in TM converging to some X and $a,b\in\mathbb{R}$ with $a<b$. Show that, if $\gamma_{X_k}|_{[a,b]}:[a,b]\to M$ is a shortest curve, then so is $\gamma_X|_{[a,b]}:[a,b]\to M$. Here and as usual, for any $Y\in TM$, we denote by $\gamma_Y:\mathbb{R}\to M$ the unique geodesic with $\gamma_Y(0)=\pi(Y)\in M$ and $\dot\gamma_Y(0)=Y$.

(b) Assume the existence of two sequences $(x_k)_{k\in\mathbb{N}}$, $(y_k)_{k\in\mathbb{N}}$ in M, of a point $p\in M$ and of an $R\in\,]0,\infty[$ with $d(x_k,p)\underset{k\to\infty}{\longrightarrow}\infty$, $d(y_k,p)\underset{k\to\infty}{\longrightarrow}\infty$ and such that every shortest curve from x_k to y_k meets the ball $B_R(p)$. Show that there exists a line in (M,g).
(Hint: construct a limit of a sequence of shortest curves. Exercise no. 3 of Sheet 9 may be helpful.)

Abgabe der Lösungen: ***Montag, den 7.1.2013*** *vor der Vorlesung.*

Wir wünschen allen Teilnehmerinnen und Teilnehmern frohe Weihnachten und einen guten Rutsch!