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Differential Geometry I
Exercise Sheet no. 11

Exercise 1

(a) Let $V \to M$ be a complex vector bundle endowed with a Hermitian metric $\langle \cdot\,,\cdot\rangle$ over a smooth manifold. Show that the dual bundle $V^* \to M$ is isomorphic to the so-called *conjugate* vector bundle $\overline{V} \to M$, where $\overline{V}_x := V_x$ but where $\lambda \cdot v := \overline{\lambda} v$ for all $v \in V_x$, $x \in M$ and $\lambda \in \mathbb{C}$.

(b) Let $\tau \to \mathbb{C}\mathrm{P}^n$ be the tautological bundle as defined in Exercise 2 of Sheet no. 10. Using the canonical Hermitian inner product on $\mathbb{C}^{n+1}$, construct a Hermitian metric on τ.

Exercise 2

Let ∇ be any connection on the tangent bundle TM of a smooth manifold M and T be its torsion, that is, $T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$ for all $X, Y \in \mathfrak{X}(M)$. Show that T is a tensor on M, more precisely show that T defines a section of the vector bundle $T^*M \otimes T^*M \otimes TM \to M$.

Exercise 3

Let M be any smooth manifold.

(a) Given any vector bundles $E \to M$ and $F \to M$ with connections ∇^E and ∇^F respectively, prove that there exists a unique connection ∇ on the tensor product bundle $E \otimes F \to M$ such that $\nabla(s \otimes s') = (\nabla^E s) \otimes s' + s \otimes (\nabla^F s')$ for all sections s of E and s' of F.

(b) Let $E \to M$ be a vector bundle with connection ∇^E. In each fiber E_p the trace tr_p is an element of $\mathrm{Hom}_{\mathbb{K}}(E_p^* \otimes E_p, \mathbb{K}) \cong E_p \otimes E_p^*$. Show that $p \mapsto \mathrm{tr}_p$ is a smooth map from M to $E \otimes E^*$. Show that it is a parallel section of $E \otimes E^* \to M$.

(c) Given any real vector bundle $E \to M$ with Riemannian metric $\langle \cdot\,,\cdot\rangle$ and connection ∇^E, show that the Riemannian metric – as a section of the vector bundle $E^* \otimes E^* \to M$ – is parallel iff the connection ∇^E is metric.

Exercise 4

Let $V \to M$ be a real or complex line bundle over a smooth manifold. Show that the tensor vector bundle $V^* \otimes V \to M$ is trivial.

Abgabe der Lösungen: ***Montag, den 14.1.2013*** *vor der Vorlesung.*

Wir wünschen allen Teilnehmerinnen und Teilnehmern ein erfolgreiches neues Jahr!