Universität Regensburg, Mathematik — WS 2012/13
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**Differential Geometry I**
**Exercise Sheet no. 12**

**Exercise 1** Let $(M^n, g)$ be a Riemannian manifold and $(U_\varphi, \varphi)$ be a chart on $M$.

(a) Show that the components $R^l_{ijk} := d\varphi^l\left(R(\frac{\partial}{\partial\varphi^i}, \frac{\partial}{\partial\varphi^j})\frac{\partial}{\partial\varphi^k}\right)$ of the curvature tensor $R$ of $g$ on $U_\varphi$ are given in terms of the Christoffel symbols associated to $(U_\varphi, \varphi)$ by

$$R^l_{ijk} = \frac{\partial\Gamma^l_{kj}}{\partial\varphi^i} - \frac{\partial\Gamma^l_{ki}}{\partial\varphi^j} + \sum_{m=1}^n \left(\Gamma^l_{mi}\Gamma^m_{kj} - \Gamma^l_{mj}\Gamma^m_{ki}\right).$$

(b) Deduce that, if $\mathrm{ric} = \sum_{i,j=1}^n \mathrm{ric}_{ij} d\varphi^i \otimes d\varphi^j$ denotes the decomposition of the Ricci-tensor of $g$, then

$$\mathrm{ric}_{ij} = \sum_{k=1}^n R^k_{kij} = \sum_{k=1}^n \left(\frac{\partial\Gamma^k_{ji}}{\partial\varphi^k} - \frac{\partial\Gamma^k_{jk}}{\partial\varphi^i} + \sum_{m=1}^n \left(\Gamma^k_{mk}\Gamma^m_{ji} - \Gamma^k_{mi}\Gamma^m_{jk}\right)\right).$$

**Exercise 2**
Let $(M := M_1 \times M_2, g := g_1 \oplus g_2)$ be the product of two Riemannian manifolds as in Exercise 3.(b) of Sheet no. 10.

(a) Show that the Levi-Civita connection $\nabla$ of $(M, g)$ is given by

$$\nabla_{(X_1,X_2)}(Y_1, Y_2) = \nabla^{M_1}_{X_1} Y_1 + \nabla^{M_2}_{X_2} Y_2 + \partial_{X_1} Y_2 + \partial_{X_2} Y_1,$$

for all $X_1, Y_1 \in \Gamma(\pi_1^* TM_1)$, $X_2, Y_2 \in \Gamma(\pi_2^* TM_2)$ and where $\partial_{X_1} Y_2$ (resp. $\partial_{X_2} Y_1$) denotes the usual derivative (make sense of this).

(b) Deduce that the curvature tensor $R$ of $(M, g)$ is given by

$$R^M_{X,Y} Z = R^{M_1}_{X_1,Y_1} Z_1 + R^{M_2}_{X_2,Y_2} Z_2,$$

for all $X_i, Y_i, Z_i \in T_{x_i} M_i$, $i = 1, 2$, where $X := X_1 + X_2$, $Y := Y_1 + Y_2$ and $Z := Z_1 + Z_2$. (Here we write $R_{X,Y}$ instead of $R(X, Y)$ as this is better for typesetting in this context.)

(c) Calculate the Ricci tensor and the scalar curvature of $M$ in terms of the Ricci tensor and the scalar curvature of $M_1$ and $M_2$.

**Exercise 3**
Let $E, F \to M$ be (real or complex) vector bundles with connections $\nabla^E, \nabla^F$ over a given manifold $M$ and $x \in M$ be a point. Prove the following identities:

(a) The curvature tensor of the connection $\nabla^E \oplus \nabla^F$ on $E \oplus F \to M$ is given by

$$R^{E\oplus F}_{X,Y} = R^E_{X,Y} \oplus R^F_{X,Y},$$

for all $X, Y \in T_xM$.

(b) The curvature tensor of the tensor connection on $E \otimes F \to M$ as defined in Exercise 3 of Sheet no. 11 is given by

$$R^{E\otimes F}_{X,Y} = R^E_{X,Y} \otimes \mathrm{Id}_F + \mathrm{Id}_E \otimes R^F_{X,Y},$$

for all $X, Y \in T_xM$.

(c) The curvature tensor of the dual bundle $E^* \to M$ endowed with the induced connection is given by

$$(R^{E^*}_{X,Y}\alpha)(V) = -\alpha(R^E_{X,Y}V),$$

for all $X, Y \in T_xM$, for all $V \in E_x$ and $\alpha \in E^*_x$.

**Exercise 4**

Let $(M^n, g)$ be a Riemannian manifold and denote by $\nabla$ resp. $R$ the Levi-Civita connection resp. the Riemannian curvature tensor of $(M^n, g)$. Let the $(0,4)$-tensor $\tilde{R}$ be defined by $\tilde{R}(X,Y,Z,W) := g(R(X,Y)Z,W)$, for all $X, Y, Z, W \in T_xM$ and $x \in M$.

(a) Let $x \in M$ be a point. Prove that the following identities are satisfied: for all $X, Y, Z, T, U \in T_pM$,

$$(\nabla_X\tilde{R})(Y,Z,T,U) = -(\nabla_X\tilde{R})(Z,Y,T,U) = -(\nabla_X\tilde{R})(Y,Z,U,T) = (\nabla_X\tilde{R})(T,U,Y,Z).$$

(b) For a given tensor field $A \in \Gamma(T^*M \otimes T^*M)$ let the *divergence* of $A$ be defined by

$$\mathrm{div}(A)(X) := \sum_{j=1}^{n} (\nabla_{E_j}A)(E_j, X) \qquad \forall X \in TM,$$

where $\{E_j\}_{1\leq j\leq n}$ is a local orthonormal basis of $TM$, that is, $g(E_i, E_j) = \delta_i^j$. Prove using the second Bianchi identity:

$$\mathrm{div}(\mathrm{ric}) = \frac{1}{2} d\,\mathrm{scal}.$$

*(Hint: for a given point $x \in M$, the basis $\{E_j\}_{1\leq j\leq n}$ can be chosen such that $(\nabla E_i)_{|x} = 0$ holds; how can this be done?)*

(c) *Application*: Assuming $n \geq 3$, the manifold $M$ connected and the existence of a smooth function $f : M \to \mathbb{R}$ with $\mathrm{ric} = f \cdot g$ on $M$, prove that $f$ is constant on $M$. Such a Riemannian manifold is then called *Einstein*.

*Abgabe der Lösungen:* ***Montag, den 21.1.2013*** *vor der Vorlesung.*