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Differential Geometry I
Exercise Sheet no. 13

Exercise 1
Let (M^n, g) be a connected n-dimensional Riemannian manifold. Assume $n \geq 3$ and that, for each $p \in M$ and any two planes $E, E' \subset T_pM$, we have $K(E) = K(E')$, where $K(E)$ denotes the sectional curvature of the plane E.

(a) Show the existence of a smooth function $\kappa : M \to \mathbb{R}$ such that, for all $X, Y, Z, T \in \mathfrak{X}(M)$,

$$\langle R(X,Y)Z, T\rangle = \kappa \cdot \Big(g(Y,Z)(gX,T) - g(X,Z)g(Y,T)\Big)$$

holds on M.

(b) Deduce that $\mathrm{ric} = (n-1)\kappa g$ and that (M^n, g) has constant sectional curvature, i.e. that κ is constant.

Exercise 2 *(Möbius strip)*
Let $F : \mathbb{R} \times]-1,1[\to \mathbb{R}^3$ be the map defined by

$$F(x,y) := \begin{pmatrix} (1 + \frac{y}{2}\cos(\frac{x}{2}))\cos(x) \\ (1 + \frac{y}{2}\cos(\frac{x}{2}))\sin(x) \\ \frac{y}{2}\sin(\frac{x}{2}) \end{pmatrix}$$

and let $M := F(\mathbb{R} \times]-1,1[) \subset \mathbb{R}^3$.

(a) Show that M is a smooth 2-dimensional submanifold of $\mathbb{R}^3$.

(b) Show that, for every $(x,y) \in \mathbb{R} \times]-1,1[$, the vector $\frac{\frac{\partial F}{\partial x}(x,y) \times \frac{\partial F}{\partial y}(x,y)}{\|\frac{\partial F}{\partial x}(x,y) \times \frac{\partial F}{\partial y}(x,y)\|} \in \mathbb{R}^3$ has unit norm and is orthogonal to $T_{F(x,y)}M$. Here "$\times$" denotes the cross product for vectors in $\mathbb{R}^3$.

(c) Show that no continuous unit normal field exists on M and deduce that M is not orientable.

Exercise 3
Let $C := \{x = (x_0, x_1, x_2, x_3, x_4) \in \mathbb{R}^5 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = \cosh(x_0)^2\}$. Compute the second fundamental form (in $\mathbb{R}^5$), the Ricci-tensor and the scalar curvature of (M, g), where g is the Riemannian metric induced by the standard Euclidean inner product.
Hint: Show that the second fundamental form has pointwise two eigenvalues, κ with multiplicity 1 and $-\kappa$ with multiplicity 3.

Exercise 4
Let $M \subset \widehat{M}$ be a submanifold of the Riemannian manifold $\widehat{M}$. It is called totally geodesic iff $\mathbb{I} \equiv 0$.

(a) Show that M is a totally geodesic iff every geodesic of M is also a geodesic of $\hat{M}$.

(b) Assume additionally that M is complete. Show that M is totally geodesic iff every geodesic $\gamma : I \to \hat{M}$, $0 \in I$, of $\hat{M}$ with $\dot{\gamma}(0) \in TM$ is contained in M.

Abgabe der Lösungen: ***Montag, den 28.1.2013*** *vor der Vorlesung.*