Universität Regensburg, Mathematik
Prof. Dr. Bernd Ammann
Dr. Nicolas Ginoux

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Differential Geometry I
Exercise Sheet no. 14

Exercise 1
Let (M^n, g) be a connected Riemannian manifold and $f : M \to M$ be an isometry. Show that, if at a point $p \in M$, one has $f(p) = p$ and $d_p f = \mathrm{Id}_{T_pM}$, then $f = \mathrm{Id}_M$.

Exercise 2
We denote as usual by $\mathbb{RP}^n$ the n-dimensional real projective space.

(a) Given any $p \in S^n$, we define $\omega \in \Gamma(\Lambda^n T_p^* S^n)$ as $\omega_p(V_1, V_2, \ldots, V_n) = \det(p, V_1, \ldots, V_n)$. Show that ω defines a trivialization of $\Lambda^n T_p^* S^n$ and an orientation of S^n.

(b) Show $(-\mathrm{Id}_{S^n})^*\omega = (-1)^{n+1}\omega$. In other words, show that

$$\omega_{-p}(d(-\mathrm{Id}_{S^n})(V_1), d(-\mathrm{Id}_{S^n})(V_2), \ldots, d(-\mathrm{Id}_{S^n})(V_n)) = (-1)^{n+1}\omega_p(V_1, V_2, \ldots, V_n)$$

for all $p \in S^n$ and all $V_1, \ldots, V_n \in T_pS^n$.

(c) Deduce that $\mathbb{RP}^n$ is orientable iff n is odd.
(Hint: The proof that it is non-orientable for n even can be carried out similarly to Exercise 2 of Sheet no. 10.)

Exercise 3
Let M be a compact (orientable) surface in $\mathbb{R}^3$. Let $\overline{B}_r(0)$ be the closed ball of radius r around 0 in $\mathbb{R}^3$, and let $S_r(0) = \partial\overline{B}_r(0)$ be its boundary.

(a) Show that the infimum $R := \inf\{r > 0 \,|\, M \subset \overline{B}_r(0)\} > 0$ is attained, and conclude that $M \cap S_R(0)$ is not empty.

(b) Show that T_xM is the orthogonal complement of x for any $x \in M \cap S_R(0)$. Show for any such x, that the two principal curvatures in x have the same sign (for a given choice of a unit normal field).

(c) Deduce that the Gauss curvature of M is positive in any $x \in M \cap S_R(0)$.

(d) Are there compact minimal surfaces M in $\mathbb{R}^3$? Justify your answer. *(We want to recall the fact that a surface is called minimal if the mean curvature H vanishes on all of M.)*

Exercise 4
Let $M \subset \mathbb{R}^3$ be a compact and connected surface. We accept without proof that $\mathbb{R}^3 \setminus M$ has two connected components, one bounded and one unbounded.

(a) Show the existence of a smooth map $N : M \to S^2$ with $N(p) \perp T_pM$ for all $p \in M$.

(b) Deduce that M is orientable.

(c) Show that N is surjective.
(Hint: show that any 2-dimensional vector subspace of $\mathbb{R}^3$ is the tangent space of M at two or more points of M.)
Comment: Such a map N is called "the" Gauß map of the surface. It is unique up to a sign.

Abgabe der Lösungen: ***Montag, den 4.2.2013*** *vor der Vorlesung.*