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Differential Geometry I
Exercise Sheet no. 15

Exercise 1
Right or wrong? Justify shortly each of your answers. In the whole exercise, V and W denote smooth $\mathbb{K}$-vector bundles over a smooth manifold M, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

(a) If V and W are trivial, then so are $V \oplus W$, $V \otimes W$, $\mathrm{Hom}(V, W)$ and $\Lambda^2 V$.

(b) For any $n \geq 3$, there exists an n-dimensional compact smooth manifold whose tangent bundle is trivial.

(c) Each complex vector bundle with connection over a 1-dimensional manifold is flat.

(d) Each real vector bundle over S^1 is trivial.

(e) If V and $V \oplus W$ are trivial, then so is W.

(f) Every complex line bundle with connection which admits a nowhere-vanishing parallel section is flat.

(g) Every real line bundle with connection is flat.

(h) There exists on every manifold a vector bundle with flat connection.

(i) The tangent bundle of every Riemannian manifold admits a unique metric connection.

(j) If a surface $M \subset \mathbb{R}^3$ contains a line (i.e., if there exists $p, v \in \mathbb{R}^3$, $v \neq 0$, with $p + \mathbb{R} \cdot v \subset M$), then $K \leq 0$ along that line.

Exercise 2
Let $M := S^2 \times \mathbb{R} \subset \mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$. Compute (after choosing a smooth unit normal field) the Weingarten map, the mean curvature, the Ricci-tensor and the scalar curvature of M.

Exercise 3
Let M and M' be compact surfaces.

(a) Assume M and M' to carry metrics with negative sectional (or Gauß) curvature. Does the connected sum $M \sharp M'$ carry such a metric? Justify your answer.

(b) Same question if M and M' carry a metric with positive sectional curvature.

Exercise 4
Let $V \to M$ be a real or complex vector bundle over an arbitrary smooth manifold. Show that there is a scalar product on V.
Hint: Construct at first a scalar product on trivial vector bundles. Then use local trivializations and a partition of unity (Lemma IV.6.2) to prove the general case.

Exercise 5
Given $n \in \mathbb{N}$, $n \geq 2$, let $\pi_N : S^n \setminus \{e_{n+1}\} \to \mathbb{R}^n$, $x \mapsto \frac{1}{1-x_{n+1}}(x_1, \ldots, x_n)$ and $\pi_S : S^n \setminus \{-e_{n+1}\} \to \mathbb{R}^n$, $x \mapsto \frac{1}{1+x_{n+1}}(x_1, \ldots, x_n)$, denote the stereographic projections from the North and South pole respectively. Recall the definition $f_*X = df \circ X \circ f^{-1}$ for a diffeomorphism $f : M \to N$ and a vector field X on M.

(a) Given $v \in \mathbb{R}^n \setminus \{0\}$, define the vector field X on $S^n \setminus \{e_{n+1}\}$ by $X := \pi_N^* v$, that is, $X_{|p} := (d_p\pi)^{-1}(v)$ for all $p \in S^n \setminus \{e_{n+1}\}$. Show that $((\pi_S)_*X)_{|x} = |x|^2 \cdot \left(v - 2\langle v, \frac{x}{|x|}\rangle \frac{x}{|x|}\right)$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

(b) Deduce that X can be extended uniquely as a smooth vector field $\widetilde{X}$ on S^n, and determine its zero points.

(c) From now on let $n = 2$. Apply the Poincaré-Hopf theorem to compute the index of $\widetilde{X}$ in e_{n+1}.

(d) In the case $n = 2$ check the result of (c) by a direct computation using part (a).

Abgabe der Lösungen: ***Montag, den 11.2.2013*** *bei Nicolas Ginoux (M122).*