Universität Regensburg, Mathematik
Prof. Dr. Bernd Ammann
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SoSe 2013
15.04.2013

Differential Geometry II
Exercise Sheet no. 1

Exercise 1
Assume (M, g) and (M', g') are surfaces with Riemannian metrics with negative Gauß curvature. Does the product metric on $M \times M'$ has everywhere negative sectional curvature?

Exercise 2
Let (M, g) be a Riemannian manifold, $p \in M$. For $r < \mathrm{injrad}(p)$, we define the chart $\varphi := (\exp_p|_{B_r(p)})^{-1}$, which defines the normal coordinates centered in p. As usual, we set

$$g_{ij}(x) := g_x(\frac{\partial}{\partial \varphi^i}|_x, \frac{\partial}{\partial \varphi^j}|_x), \qquad \text{for } x \in B_r(p).$$

i) Show that if $X = \sum_i X^i \frac{\partial}{\partial \varphi^i}$, then $\dot\gamma_X(t) = \sum_i X^i \frac{\partial}{\partial \varphi^i}|_{\gamma_X(t)}$.

ii) Show that the associated Christoffel symbols satisfy $\Gamma_{ij}^k(p) = 0$. (Hint: use the geodesic equation $\nabla_{\dot\gamma_X} \dot\gamma_X = 0$ to show that $\sum_{i,j} X^i X^j \Gamma_{ij}^k(p) = 0$, for any k and any $(X^1, \ldots, X^n) \in \mathbb{R}^n$).

iii) Deduce that there exists $c \in \mathbb{R}$ such that $|g_{ij}(x) - \delta_{ij}| \leq c \cdot (d(x,p))^2$, for all $x \in B_{\frac{r}{2}}(p)$. (Hint: use the Koszul formula for Γ_{ij}^k).

Exercise 3
Let (M, g) be a Riemannian manifold, $p, q \in M$. Assume that $\gamma_i : [0, L] \to M$, $i = 1, 2$, are two different shortest curves from p to q, parametrized by arc-length. Extend each geodesic γ_i to its maximal domain.

i) Show that $\dot\gamma_1(L) \neq \dot\gamma_2(L)$.

ii) Show that $\gamma_1|_{[0,L+\varepsilon]}$ is not a shortest curve for any $\varepsilon > 0$. (Hint: construct a shorter path from p to $\gamma_1(L+\varepsilon)$ by using a chart around q and the geodesic γ_2).

Exercise 4
Show that the following groups with the manifold structure induced from $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$ are Lie groups and determine their Lie algebras:

$$\mathrm{SO}(n), \mathrm{GL}(m, \mathbb{C}), \mathrm{U}(m), \mathrm{SU}(m), \text{ where } n = 2m.$$

Also determine the adjoint representations. Which of these Lie groups have a bi-invariant Riemannian metric?

Abgabe der Lösungen: ***Montag, den 22.04.2012*** *vor der Vorlesung.*

Universität Regensburg, Mathematik SoSe 2013
Prof. Dr. Bernd Ammann 22.04.2013
Dr. Mihaela Pilca

Differential Geometry II
Exercise Sheet no. 2

Exercise 1

Let Γ be a discrete group acting smoothly on a differentiable manifold M.

(a) Show that the action is proper if and only if both of the following conditions are satisfied:

(i) Each point $p \in M$ has a neighborhood U such that $(\gamma \cdot U) \cap U = \emptyset$, for all but finitely many $\gamma \in \Gamma$.

(ii) If $p, q \in M$ are not in the same Γ-orbit, there exist neighborhoods U of p and V of q such that $(\gamma \cdot U) \cap V = \emptyset$, for all $\gamma \in \Gamma$.

(b) If Γ acts moreover freely, then show that the action is proper if and only if for each $p, q \in M$ there exist neighborhoods U of p and V of q, such that for all $\gamma \in \Gamma$ with $q \neq \gamma \cdot p$ we have $(\gamma \cdot U) \cap V = \emptyset$.

Exercise 2

Let X be a left-invariant vector field on a Lie group G with unit element e .

i) Show that there exists a curve $\gamma : \mathbb{R} \to G$ satisfying $\gamma(0) = e$ and $\dot{\gamma}(t) = X_{\gamma(t)}$, for all $t \in \mathbb{R}$.

ii) Show that $\gamma(t + s) = \gamma(t) \cdot \gamma(s)$ and $\gamma(-t) = \gamma(t)^{-1}$, for all $s, t \in \mathbb{R}$.

Exercise 3

Let G and H be two Lie groups and e the unit element of G. If $f : G \to H$ is a smooth group homomorphism, then show that:

i) $d_e f : \mathfrak{g} \to \mathfrak{h}$ is surjective if and only if f is a submersion.

ii) $d_e f : \mathfrak{g} \to \mathfrak{h}$ is bijective if and only if f is locally diffeomorphic.

iii) If H is connected and $d_e f : \mathfrak{g} \to \mathfrak{h}$ is surjective, then f is surjective. (Hint: Show that $f(G)$ is open and closed. In order to prove that the image is closed one may cosider a sequence converging to any point in the closure of the imagine and translate it by left multiplication to the unit element of H.)

Exercise 4
For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, consider the following action of $\mathbb{R}$ on $M := S^1 \times S^1$:

$$\mathbb{R} \times M \to M, \quad (t,p) \mapsto f_t(p), \quad \text{where} \quad f_t(x,y) := (e^{it}x, e^{i\alpha t}y).$$

(a) Show that each orbit of this action is dense in M and is neither closed nor a submanifold.

(b) Is the map $\Theta : \mathbb{R} \times M \to M \times M$, $(t,p) \mapsto (f_t(p), p)$ closed? Is the action proper?

(c) Is $\mathbb{R}\backslash M$ (equipped with the quotient topology) a Hausdorff space?

Hand in the solutions on ***Monday, April 29, 2013*** *before the lecture.*

Universität Regensburg, Mathematik
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SoSe 2013
29.04.2013

Differential Geometry II
Exercise Sheet no. 3

Exercise 1

Let $\mathcal{H}_3 := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$ and $\Gamma := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{Z} \right\}$.

i) Show that $\mathcal{H}_3$ and Γ are Lie groups. Does $\mathcal{H}_3$ admit a bi-invariant Riemannian metric?

ii) Show that Γ acts on $\mathcal{H}_3$ by left multiplication and this action is free and proper.

iii) Consider the following action of $\mathbb{R}$ on $\mathcal{H}_3$:

$$\mathbb{R} \times \mathcal{H}_3 \to \mathcal{H}_3, \quad \left(\tilde{z}, \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right) \mapsto \begin{pmatrix} 1 & x & z + \tilde{z} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Show that this action descends to an action of $\mathbb{Z}\backslash\mathbb{R}$ on the quotient $\Gamma\backslash\mathcal{H}_3$ and the quotient manifold obtained by this action is the 2-dimensional torus.

Exercise 2

Let $S^{4n+3} \subset \mathbb{H}^{n+1}$ be the unit sphere in the $(n+1)$-dimensional quaternionic vector space.

i) Show that $S^3 \subset \mathbb{H}$ acts smoothly, freely and properly on S^{4n+3}.

ii) Give an atlas for the quotient manifold $\mathbb{H}\mathrm{P}^n := S^3\backslash S^{4n+3}$. The manifold $\mathbb{H}\mathrm{P}^n$ is called the n-dimensional quaternionic projective space.

Exercise 3

i) Determine the Lie bracket $[\cdot,\cdot]$ on $\mathfrak{gl}(n, \mathbb{R})$, the Lie algebra of the general linear group $GL(n, \mathbb{R})$.

ii) For any Lie group G with adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$, let $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ denote the differential of Ad at the unit element of G, $\mathrm{ad} := \mathrm{d}_{\mathbb{1}}\mathrm{Ad}$.
Show that for $GL(n, \mathbb{R})$, the map ad is given by $\mathrm{ad}(X)(Y) = [X, Y]$, for all $X, Y \in \mathfrak{gl}(n, \mathbb{R})$.

iii) Let $X \in \mathfrak{gl}(n, \mathbb{R})$, $\widetilde{X}$ the corresponding left-invariant vector field on $GL(n, \mathbb{R})$ and $\gamma : \mathbb{R} \to GL(n, \mathbb{R})$ be a curve with $\gamma(0) = \mathbb{1}_n$, $\dot{\gamma}(t) = \widetilde{X}_{\gamma(t)}$.
Show that $\gamma(t) = \sum_{n=0}^{\infty} \frac{1}{n!}(tX)^n$.

Hand in the solutions on ***Monday, May 6, 2013*** *before the lecture.*

Universität Regensburg, Mathematik
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06.05.2013

Differential Geometry II
Exercise Sheet no. 4

Exercise 1
Let $\pi : \overline{M} \to M$ be a covering of the manifold M, and let g be a Riemannian metric on M. We equip $\overline{M}$ with the metric π^*g defined as

$$\pi^* g_p(X,Y) := g_{\pi(p)}((\mathrm{d}_{\pi(p)}\pi)(X), (\mathrm{d}_{\pi(p)}\pi)(Y)), \quad \forall p \in \overline{M}, \forall X, Y \in T_p\overline{M}. \tag{1}$$

i) Show that if M is compact, then $(\overline{M}, \pi^*g)$ is complete.

ii) Is it still true that $(\overline{M}, \pi^*g)$ is complete when $\pi : \overline{M} \to M$ is only locally diffeomorphic and surjective?

Exercise 2
Let $\pi : \overline{M} \to M$ be a surjective map which is locally diffeomorphic and let g, resp. π^*g be Riemannian metrics on M, resp. $\overline{M}$, that are related by (1). We assume that $(\overline{M}, \pi^*g)$ is complete. Show that:

i) (M, g) is also complete.

ii) The map π is a covering. Hint: Use the Hopf-Rinow Theorem.

Exercise 3
Let G be a Lie group, let g a bi-invariant Riemannian metric on G, and let $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ be the map introduced in Exercise 3, ii) on Sheet no. 3.

i) Show that the map ad takes values into the skew-symmetric endomorphisms of $(\mathfrak{g} = T_{\mathbb{1}}G, g_{\mathbb{1}})$. Moreover, one can show that $\mathrm{ad}(X)(Y) = [X, Y]$, for all $X, Y \in \mathfrak{g}$ (we assume this result, it is not part of the exercise to prove it).

ii) Use i) and the Koszul formula to show that the Levi-Civita connection of g is given by $\nabla_X Y = \frac{1}{2}[X, Y]$, for all left-invariant vector fields X, Y.

iii) (Bonus points) Show that the sectional curvature of g is nonnegative. (Hint: First compute the Riemannian curvature tensor using ii): $R(X,Y)Z = -\frac{1}{4}[[X,Y],Z]$, for all left-invariant vector fields X, Y, Z. Use also the Jacobi identity).

Hand in the solutions on ***Monday, May 13, 2013*** *before the lecture.*

Universität Regensburg, Mathematik
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SoSe 2013
13.05.2013

Differential Geometry II
Exercise Sheet no. 5

Exercise 1

Let $S^3 \subset \mathbb{H}$ be the unit sphere in the quaternion algebra. Consider the following map:

$$\theta : S^3 \times S^3 \to \mathrm{Aut}(\mathbb{H})$$
$$(z, w) \mapsto (q \mapsto zq\overline{w}).$$

i) Show that θ defines a smooth action of $S^3 \times S^3$ on $\mathbb{H}$, which preserves the standard norm on $\mathbb{H} \cong \mathbb{R}^4$.

ii) Compute the kernel of θ.

iii) Show that the differential of θ at the identity element is bijective.

iv) Conclude that θ is the universal covering of $\mathrm{SO}(4)$.

Exercise 2

Let $\mathbb{Z}$ act on $\mathbb{R}^n$ by $k \cdot x := 2^k x$, for $k \in \mathbb{Z}$, $x \in \mathbb{R}^n$.

i) Is this action proper on $M_1 := \mathbb{R}^n$, on $M_2 := \mathbb{R}^n \setminus \{0\}$, on $M_3 := (0, \infty) \times (0, \infty) \times \mathbb{R}^{n-2}$?

ii) Are the quotients $\mathbb{Z} \backslash M_i$ Hausdorff? Are they compact?

Exercise 3

For $0 < m < n$, let $G(m, n)$ be the set of all m-dimensional subspaces in $\mathbb{R}^n$. Show that $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{O}(n, \mathbb{R})$ act transitively on $G(m, n)$. Determine the isotropy groups of $\mathbb{R}^m \times \{0\}$ for both actions, and write $G(m, n)$ as homogeneous space G/H where $G = \mathrm{GL}(n, \mathbb{R})$ or $G = \mathrm{O}(n, \mathbb{R})$.
What is the interpretation of

i) $\mathrm{O}(n, \mathbb{R})/(\mathrm{O}(m, \mathbb{R}) \times \mathrm{O}(n - m, \mathbb{R}))$,

ii) $\mathrm{SO}(n, \mathbb{R})/(\mathrm{SO}(m, \mathbb{R}) \times \mathrm{SO}(n - m, \mathbb{R}))$,

iii) $\mathrm{GL}_+(n, \mathbb{R})/(\mathrm{GL}_+(m, \mathbb{R}) \times \mathrm{GL}_+(n - m, \mathbb{R}))$,

iv) $\mathrm{GL}(n, \mathbb{R})/(\mathrm{GL}(m, \mathbb{R}) \times \mathrm{GL}(n - m, \mathbb{R}))$.

Hint: Be cautious with the isotropy group of $\mathrm{GL}(n, \mathbb{R})$*, and its relation to iii) and iv).*

Hand in the solutions on ***Monday, May 20, 2013*** *before the lecture.*

Universität Regensburg, Mathematik SoSe 2013
Prof. Dr. Bernd Ammann 21.05.2013
Dr. Mihaela Pilca

Differential Geometry II
Exercise Sheet no. 6

Exercise 1
The goal of this exercise is to show that there is no matrix $A \in \mathfrak{gl}(2, \mathbb{R}) \cong \mathbb{R}^{2\times 2}$ such that $\exp(A) = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$. Deduce a contradiction by considering the following cases:

i) A is diagonalizable.

ii) A is triagonalizable, but not diagonalizable.

iii) A has no real eigenvalues. Hint: Consider the eigenvalues of A and of $\exp(A)$.

Bonus points question: If G is a connected compact Lie group, is the Lie group exponential map surjective?

Exercise 2
We define
$$\mathcal{K} := \{J \in \mathrm{End}(\mathbb{R}^{2n}) \mid J^2 = -\mathrm{Id}\}.$$
The elements of $\mathcal{K}$ are called complex structures on $\mathbb{R}^{2n}$. The group $\mathrm{GL}(2n, \mathbb{R})$ acts by conjugation on $\mathrm{End}(\mathbb{R}^{2n})$. Show that $\mathcal{K}$ is an orbit of this action. Compute the isotropy group and write $\mathcal{K}$ as a homogeneous space.

Exercise 3
Let $(V, [\cdot,\cdot])$ be a Lie algebra over a field K. An ideal is a vector subspace W, such that $[x, y] \in W$, for all $x \in W$ and $y \in V$. Show the following:

i) The quotient space V/W carries a unique Lie bracket, such that the projection $V \to V/W$ is a Lie algebra homomorphism.

ii) The kernel of a Lie algebra homomorphism is an ideal and conversely, each ideal is the kernel of a Lie algebra homomorphism.

iii) (Bonus points) Let now $K = \mathbb{R}$. Let G be a Lie group and H a normal subgroup of G that is also a submanifold. Then the Lie algebra of H is an ideal of the Lie algebra of G.

Hand in the solutions on ***Monday, May 27, 2013*** *before the lecture.*

Universität Regensburg, Mathematik
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SoSe 2013
27.05.2013

Differential Geometry II
Exercise Sheet no. 7

Exercise 1
The *Killing form* of a Lie algebra $\mathfrak{g}$ is the function defined by:

$$B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}, \quad B(X, Y) := \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)).$$

Show the following properties of the Killing form:

i) B is a symmetric bilinear form on $\mathfrak{g}$.

ii) If $\mathfrak{g}$ is the Lie algebra of the Lie group G, then B is Ad-invariant:

$$B(\mathrm{Ad}(\sigma)X, \mathrm{Ad}(\sigma)Y) = B(X, Y), \quad \forall \sigma \in G, \forall X, Y \in \mathfrak{g}.$$

Hint: Show first that if α is an automorphism of $\mathfrak{g}$, i.e. a linear isomorphism α satisfying $\alpha([X, Y]) = [\alpha(X), \alpha(Y)]$ for all $X, Y \in \mathfrak{g}$, then $\mathrm{ad}(\alpha(X)) = \alpha \circ \mathrm{ad}(X) \circ \alpha^{-1}$, for any $X \in \mathfrak{g}$.

iii) For each $Z \in \mathfrak{g}$, $\mathrm{ad}(Z)$ is skew-symmetric with respect to B:

$$B(\mathrm{ad}(Z)X, Y) = -B(X, \mathrm{ad}(Z)X), \forall X, Y \in \mathfrak{g}.$$

Exercise 2
Let (M, g) be a Riemannian manifold of constant sectional curvature κ and let $\gamma : [0, \ell] \to M$ be a geodesic parametrized by arc-length. Let J be a vector field along γ, normal to γ'.

i) Show that the Jacobi equation can be written as $J'' + \kappa J = 0$.

ii) Let V be a parallel unit vector field along γ normal to γ'. Determine the Jacobi vector field J satisfying the initial conditions $J(0) = 0$ and $J'(0) = V(0)$.

Exercise 3

i) Let (M, g) be a Riemannian manifold and $\gamma : I \to M$ a geodesic. Show that if M is 2-dimensional, then the relation for points of γ to be conjugated to each other along γ is transitive. More precisely, for any $t_i \in I$, $i = 1, 2, 3$, such that $\gamma(t_1)$ is conjugated to $\gamma(t_2)$ and $\gamma(t_2)$ is conjugated to $\gamma(t_3)$, it follows that $\gamma(t_1)$ is conjugated to $\gamma(t_3)$.

ii) Show that the statement in i) is not true for higher dimensions, by considering for instance the Riemannian manifold $(S^2 \times S^2, g_{std} \oplus g_{std})$, that is the Riemannian product of two spheres with the standard metric and the following geodesic $\gamma(t) = (\cos(t), 0, \sin(t), \cos(\pi t), 0, \sin(\pi t)) \in S^2 \times S^2 \subset \mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6$.

Hand in the solutions on ***Monday, June 3, 2013*** *before the lecture.*

Universität Regensburg, Mathematik
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SoSe 2013
03.06.2013

Differential Geometry II
Exercise Sheet no. 8

Exercise 1

Let G be a Lie group which acts isometrically, freely and properly on a Riemannian manifold (M, g). (An action is *isometric* if l_σ is an isometry for any $\sigma \in G$.) Show that there exists a metric on the quotient manifold $G\backslash M$ such that the projection $\pi : M \to G\backslash M$ is a Riemannian submersion. (A submersion $\pi : M \to N$ between Riemannian manifolds is called a *Riemannian submersion* if $d_x\pi$ is an isometry from the orthogonal complement of $\ker d_x\pi$ in T_xM to $T_{\pi(x)}N$ for any $x \in M$.)

Exercise 2

Let $\pi : (M, g) \to (N, h)$ be a Riemannian submersion.

i) Let γ be a geodesic in (N, h). Show that any horizontal lift of γ is a geodesic in (M, g).

ii) Let $\tau : [a, b] \to M$ be a geodesic in (M, g) such that $\dot{\tau}(a)$ is horizontal. Show that $\dot{\tau}(t)$ is horizontal for all $t \in [a, b]$. Conclude that if a horizontal lift $\widetilde{\gamma}$ of a curve γ is a geodesic in (M, g), then γ is a geodesic in (N, h).

iii) Let $\pi : S^{2n+1} \to \mathbb{C}P^n$ be the projection $z \mapsto [z]$, which defines the so-called *Hopf fibration*. Consider on $\mathbb{C}P^n$ the Riemannian metric that makes π a Riemannian submersion, where S^{2n+1} carries the standard metric. This means $\mathbb{C}P^n$ carries the metric defined vai Exercise 1. This metric on $\mathbb{C}P^n$ is called the *Fubini-Study* metric of $\mathbb{C}P^n$.
Show that the geodesics parametrized by arclength in $\mathbb{C}P^n$ are of the form $\gamma(t) = [\cos t\, v + \sin t\, w]$, where $v, w \in S^{2n+1} \subset \mathbb{C}^{n+1}$ with $\sum_{j=1}^{n+1} v_j \overline{w}_j = 0$. Show furthermore that in $\mathbb{C}P^1$ the points $[(1, 0)]$ and $[(0, 1)]$ are conjugated along a geodesic.

Exercise 3

i) Let V and W be two m-dimensional real vector spaces and A_t a smooth family of homomorphisms, where t is a real parameter. Let $A'_t = \frac{d}{dt} A_t$. Assume that
$$\mathrm{Im}(A_0) \oplus A'_0(\mathrm{Ker}(A_0)) = W.$$
Show that there exists an $\varepsilon > 0$, such that A_t has rank m for all $t \in (-\varepsilon, 0) \cup (0, \varepsilon)$.

ii) Let J_1 and J_2 be two Jacobi vector fields along a geodesic on a Riemannian manifold. Show that the function
$$t \mapsto \langle J_1(t), J'_2(t)\rangle - \langle J'_1(t), J_2(t)\rangle$$

is constant.

iii) Let $\gamma : [0, b) \to M$ be a geodesic on a Riemannian manifold. Show that the set

$$\{t \in [0, b) \,|\, t \text{ is conjugated to } 0\}$$

is closed and discrete in $[0, b)$. Hint: Use i) and ii).

Exercise 4

Let $\pi : (M, g) \to (N, h)$ be a Riemannian submersion. The vectors in the kernel of $d\pi$ are called vertical. For each $X \in \Gamma(TN)$, let $\overline{X}$ denote the horizontal lift of X, *i.e.* $\overline{X} \in \Gamma(TM)$ such that $d\pi \circ \overline{X} = X \circ \pi$ and $\overline{X}$ is orthogonal in each point to the kernel of $d\pi$.

i) Show that the vertical part of $[\overline{X}, \overline{Y}]$ in $p \in M$, denoted by $[\overline{X}, \overline{Y}]_p^v$, depends only on $\overline{X}(p)$ and $\overline{Y}(p)$.

ii) Let $X \in \Gamma(TN)$, $\eta \in \Gamma(TM)$ and η is vertical. Show that $[\eta, \overline{X}]$ is vertical.

iii) Compute $\overline{[X,Y]} - [\overline{X}, \overline{Y}]$ and $\nabla^M_{\overline{X}} \overline{Y} - \overline{\nabla^N_X Y}$, for $X, Y \in \Gamma(TN)$.

iv) Assume that $\overline{X}(p)$ and $\overline{Y}(p)$ are orthonormal. Let E be the plane spanned by $X(\pi(p))$ and $Y(\pi(p))$ and $\overline{E}$ be the plane spanned by $\overline{X}(p)$ and $\overline{Y}(p)$. Show the following formula for the sectional curvatures of (M, g) and (N, h):

$$K^{N,h}(E) = K^{M,g}(\overline{E}) + \frac{3}{4} \| [\overline{X}, \overline{Y}]_p^v \|^2.$$

Hand in the solutions on ***Monday, June 10, 2013*** *before the lecture.*

Universität Regensburg, Mathematik SoSe 2013
Prof. Dr. Bernd Ammann 10.06.2013
Dr. Mihaela Pilca

Differential Geometry II
Exercise Sheet no. 9

Exercise 1

Let (M, g) be a connected, complete and simply-connected Riemannian manifold with sectional curvature $K \leq 0$. Show that there is a unique geodesic between any two points on M. Hint: use Cartan-Hadamard Theorem.

Exercise 2

Let M be a connected manifold and $p \in M$. We consider the map defined in the lecture between the fundamental group of M and the set of free homotopy classes of loops:

$$F : \pi_1(M, p) \to \pi_o \mathcal{L}(M),$$
$$[\gamma] \mapsto [\gamma]_{\text{free}}.$$

Show the following:

i) F is surjective.

ii) F induces a well-defined map on the set of conjugacy classes in $\pi_1(M, p)$, that is $[\gamma\tau\gamma^{-1}]_{\text{free}} = [\tau]_{\text{free}}$, for any $\gamma, \tau \in \pi_1(M, p)$.

iii) The map induced by F on the set of conjugacy classes in $\pi_1(M, p)$ is injective.

Exercise 3

We consider the Hopf fibration and the Fubini-Study metric on $\mathbb{C}P^n$ introduced in Exercise 2, (iii) on Sheet no. 8. We use the same notation as in this exercise, and again X^v is the vertical part of X. The vertical vectors of the Hopf fibration in the point $z \in S^{2n+1}$ are of the form $\lambda i z$, $\lambda \in \mathbb{R}$.
For $X, Y \in \mathbb{C}^{n+1}$, we define $\langle X, Y \rangle_{\mathbb{C}} := \sum_{j=1}^{n+1} X_j \overline{Y}_j$ and $\langle X, Y \rangle_{\mathbb{R}} := \mathrm{Re}(\sum_{j=1}^{n+1} X_j \overline{Y}_j)$.
Then it holds $\langle X, Y \rangle_{\mathbb{C}} = \langle X, Y \rangle_{\mathbb{R}} + i \langle X, iY \rangle_{\mathbb{R}}$. Show the following:

i) For any $\widetilde{X}_0 \in \mathbb{C}^{n+1}$, the map $w \mapsto \widetilde{X}_w := \widetilde{X}_0 - \langle \widetilde{X}_0, w \rangle_{\mathbb{C}} w$ is a well-defined vector field on S^{2n+1}.

ii) $\widetilde{X}$ is horizontal everywhere.

iii) Each point $p \in \mathbb{C}P^n$ admits an open neighborhood U and a smooth map $f : \pi^{-1}(U) \to S^1$, such that $f(\lambda z) = \lambda f(z)$, for all $z \in \pi^{-1}(U)$ and $\lambda \in S^1$.

iv) $f\widetilde{X}$ is a horizontal lift of a vector field $X \in \Gamma(TU)$.

v) For a fixed $z \in S^{2n+1}$ assume that $\langle \widetilde{X}_0, z\rangle_{\mathbb{C}} = \langle \widetilde{Y}_0, z\rangle_{\mathbb{C}} = 0$. For the Levi-Civita connection ∇ of S^{2n+1} it holds:

$$\nabla_{\widetilde{Y}_w} \widetilde{X}_w|_{w=z} = -(\mathrm{Im}(\langle \widetilde{X}_0, \widetilde{Y}_0\rangle_{\mathbb{C}}))iz$$

vi) Choose f such that $f(z_0) = 1$ for a $z_0 \in \pi^{-1}(p)$. Conclude that $[f\widetilde{Y}, f\widetilde{X}]^v|_{z_0} = -2(\mathrm{Im}\langle \widetilde{X}_0, \widetilde{Y}_0\rangle_{\mathbb{C}})iz_0$.

vii) The sectional curvature K of $\mathbb{C}P^n$ satisfies: $1 \leq K \leq 4$. For which planes is $K = 4$ and for which planes is $K = 1$?

Hand in the solutions on ***Monday, June 17, 2013*** *before the lecture.*

Universität Regensburg, Mathematik — SoSe 2013
Prof. Dr. Bernd Ammann — 17.06.2013
Dr. Mihaela Pilca

Differential Geometry II
Exercise Sheet no. 10

Exercise 1
Determine $\mathcal{C}_p^{\tan} M$, and $\mathcal{C}_p M$ for

(a) $M = \mathbb{R}^2/\Gamma$, where Γ is the subgroup of $\mathbb{R}^2$ generated by $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\2\end{pmatrix}$, and $p := [0]$.

(b) $M = \mathbb{R}P^m = S^m/\{\pm 1\}$ with the quotient metric, and $p := [e_1]$.

Exercise 2
Let $M = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = e^{-z^2}\}$. Show that M is a smooth surface, and that M is complete, $\mathrm{vol}(M) < \infty$, $\mathrm{injrad}(M) = 0$, $\mathrm{diam}(M) = \infty$.

Exercise 3
Let M be a complete connected Riemannian manifold, $p \in M$ fixed. We define $\mathrm{diam}\, M := \sup\{d(x,y) \,|\, x, y \in M\}$. Show

(a) $\mathrm{diam}\, M = \sup_{X \in SM} s(X)$

(b) $\mathrm{injrad}(p) = \min_{X \in S_pM} s(X)$

(c) $\mathrm{injrad}(M) = \inf_{X \in SM} s(X)$

(d) $\sup_{X \in SM} s(X) = \infty$ if and only if there is for all $p \in M$ an $X \in S_pM$ with $s(X) = \infty$.
Hint: Use Exercise no. 3 on Sheet no. 9 of Differential Geometry I

(e) Give an example of a complete Riemannian manifold such that $\sup_{X \in S_pM} s(X)$ depends on p.

Exercise 4
We consider $S^3 \subset \mathbb{C}^2$ endowed with the standard metric, and $\Gamma := \{1, i, -1, -i\}$ which acts freely und isometrically on S^3. Let $M := S^3/\Gamma$, $\pi : S^3 \to M$ the corresponding projection and $p := \pi(e_1) = e_1 \mod \Gamma \in M$. Show that for the cut locus $\mathcal{C}_p$ the following holds:

$$\mathcal{C}_p = \{\pi(x) \,|\, x \in S^3 \text{ with } d(x, e_1) = d(x, ie_1)\}$$
$$= \left\{ \pi\left(\frac{(1+i)r}{\sqrt{2}} e_1 + v e_2 \right) \,\middle|\, r \in [0,1], \quad v \in \mathbb{C} \text{ with } r^2 + |v|^2 = 1 \right\}.$$

Answer without justification: Where are the minima and maxima of the function $s : S_pM \to (0, \infty)$?
Bonus question: Where is $\mathcal{C}_p$ a smooth hypersurface and where not?

Hand in the solutions on ***Monday, June 24, 2013*** *before the lecture.*

Universität Regensburg, Mathematik
Prof. Dr. Bernd Ammann
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SoSe 2013
24.06.2013

Differential Geometry II
Exercise Sheet no. 11

Exercise 1
Let M be a complete Riemannian manifold; let N be a submanifold and a closed subset of M. For any $p_0 \in M$ we define its distance to N as $d(p_0, N) := \inf_{q \in N} d(p_0, q)$. Show the following:

i) There exists a point $q_0 \in N$, such that $d(p_0, N) = d(p_0, q_0)$.

ii) If $p_0 \in M \setminus N$, then a minimizing geodesic joining p_0 and q_0 is orthogonal to N at q_0.
Hint: Use a variation of the geodesic with curves starting at p_0 and ending at points in N.

Exercise 2
Let N be a submanifold of a Riemannian manifold (M, g). The normal exponential map of N, $\exp^{\perp} : TN^{\perp} \to M$ is defined as the restriction of the exponential map $\exp : TM \to M$, $(p, v) \mapsto \exp_p v$ to points $q \in N$ and vectors $w \in (T_q N)^{\perp}$. Show that $p \in M$ is a focal point of $N \subset M$ if and only if p is a critical value of $\exp^{\perp}$.
Hint: For "⇒" consider for a suitable variation $\gamma : (-\varepsilon, \varepsilon) \times [0, \ell] \to M$ with $\alpha(s) := \gamma(s, 0) \subset N$ and $V(s) := \frac{\nabla}{dt}\gamma|_{(s,0)}$ the curve $c(s) := (\alpha(s), \ell V(s))$.
For "⇐" consider for a suitable curve $c(s) = (\alpha(s), \ell V(s))$ in $TN^{\perp}$ the variation $\gamma(s, t) = \exp_{\alpha(s)}(tV(s))$.

Exercise 3
Let N be a submanifold of a flat manifold (M, g) and γ be a geodesic in M with $\gamma(0) \in N$ and $\dot{\gamma}(0) \perp T_{\gamma(0)}N$. Show that $\gamma(\frac{1}{\lambda})$ is a focal point of N if and only if λ is a non-zero eigenvalue of $S_{\dot{\gamma}(0)}$.
Hint: For "⇒" consider $X(t) := (1 - \lambda t)E(t)$, where E is a parallel vector field along γ and $S_{\dot{\gamma}(0)}(E(0)) = \lambda E(0)$.

Hand in the solutions on ***Monday, July 1, 2013*** *before the lecture.*

Universität Regensburg, Mathematik SoSe 2013
Prof. Dr. Bernd Ammann 01.07.2013
Dr. Mihaela Pilca

Differential Geometry II
Exercise Sheet no. 12

Exercise 1
Let (M, g) be a Riemannian manifold, whose sectional curvature K satisfies the inequalities:

$$0 < L \leq K \leq H,$$

for some positive constants L and H. For a geodesic $\gamma : [0, \ell] \to M$, parametrized by arclength, we define

$$d := \min\{t > 0 \,|\, \gamma(t) \text{ is conjugated to } \gamma(0) \text{ along } \gamma|_{[0,t]}\}.$$

Show

$$\frac{\pi}{\sqrt{H}} \leq d \leq \frac{\pi}{\sqrt{L}}.$$

Hint: Use the First Rauch Comparison Theorem.

Exercise 2
Let (M, g) be a complete Riemannian manifold with sectional curvature $K \geq 0$. Let Γ be a discrete group without 2-torsion (*i.e.* $\gamma^2 \neq e$, for any $\gamma \in \Gamma \setminus \{e\}$, where e is the identity element of Γ), acting isometrically, freely and properly on M. For a point $p \in M$, let $\gamma_0 \in \Gamma$ be an element with $d(p, \gamma_0 p) = \min\limits_{\gamma \in \Gamma \setminus \{e\}} d(p, \gamma p)$.
We choose a minimal geodesics c_1 which connects p to $\gamma_0 p$, and a geodesic c_2 which connects p to $\gamma_0^{-1} p$. Show that c_1 and c_2 form at p an angle $\alpha \geq \frac{\pi}{3}$.

Exercise 3
Let (M, g) be a complete Riemannian manifold with sectional curvature $K \geq 0$ and let $\gamma, \sigma : [0, \infty) \to M$ be two geodesics, parametrized by arclength, with $\gamma(0) = \sigma(0)$. We assume that γ is a ray and that $\alpha := \sphericalangle(\dot\gamma(0), \dot\sigma(0)) < \frac{\pi}{2}$.
Show that $\lim\limits_{t \to \infty} d(\sigma(0), \sigma(t)) = \infty$.
Hint: Using the triangle inequality, show first that it is enough to prove: $\lim\limits_{s \to \infty} \left(d(\gamma(s), \sigma(t)) - d(\gamma(s), \gamma(0))\right) \geq t \cos \alpha$, *for any fixed* $t \geq 0$. *Then apply Toponogov's Theorem (A).*

Hand in the solutions on ***Monday, July 15, 2013*** *before the lecture.*