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Differential Geometry II
Exercise Sheet no. 1

Exercise 1
Assume (M, g) and (M', g') are surfaces with Riemannian metrics with negative Gauß curvature. Does the product metric on $M \times M'$ has everywhere negative sectional curvature?

Exercise 2
Let (M, g) be a Riemannian manifold, $p \in M$. For $r < \mathrm{injrad}(p)$, we define the chart $\varphi := (\exp_p|_{B_r(p)})^{-1}$, which defines the normal coordinates centered in p. As usual, we set

$$g_{ij}(x) := g_x(\frac{\partial}{\partial \varphi^i}|_x, \frac{\partial}{\partial \varphi^j}|_x), \qquad \text{for } x \in B_r(p).$$

i) Show that if $X = \sum_i X^i \frac{\partial}{\partial \varphi^i}$, then $\dot{\gamma}_X(t) = \sum_i X^i \frac{\partial}{\partial \varphi^i}|_{\gamma_X(t)}$.

ii) Show that the associated Christoffel symbols satisfy $\Gamma_{ij}^k(p) = 0$. (Hint: use the geodesic equation $\nabla_{\dot{\gamma}_X} \dot{\gamma}_X = 0$ to show that $\sum_{i,j} X^i X^j \Gamma_{ij}^k(p) = 0$, for any k and any $(X^1, \ldots, X^n) \in \mathbb{R}^n$).

iii) Deduce that there exists $c \in \mathbb{R}$ such that $|g_{ij}(x) - \delta_{ij}| \leq c \cdot (d(x,p))^2$, for all $x \in B_{\frac{r}{2}}(p)$. (Hint: use the Koszul formula for Γ_{ij}^k).

Exercise 3
Let (M, g) be a Riemannian manifold, $p, q \in M$. Assume that $\gamma_i : [0, L] \to M$, $i = 1, 2$, are two different shortest curves from p to q, parametrized by arclength. Extend each geodesic γ_i to its maximal domain.

i) Show that $\dot{\gamma}_1(L) \neq \dot{\gamma}_2(L)$.

ii) Show that $\gamma_1|_{[0,L+\varepsilon]}$ is not a shortest curve for any $\varepsilon > 0$. (Hint: construct a shorter path from p to $\gamma_1(L+\varepsilon)$ by using a chart around q and the geodesic γ_2).

Exercise 4
Show that the following groups with the manifold structure induced from $\mathbb{R}^{n \times n} \cong \mathbb{R}^{n^2}$ are Lie groups and determine their Lie algebras:

$$\mathrm{SO}(n), \mathrm{GL}(m, \mathbb{C}), \mathrm{U}(m), \mathrm{SU}(m), \text{ where } n = 2m.$$

Also determine the adjoint representations. Which of these Lie groups have a bi-invariant Riemannian metric?

Abgabe der Lösungen: ***Montag, den 22.04.2012*** *vor der Vorlesung.*