Universität Regensburg, Mathematik
Prof. Dr. Bernd Ammann
Dr. Mihaela Pilca

SoSe 2013
22.04.2013

Differential Geometry II
Exercise Sheet no. 2

Exercise 1

Let Γ be a discrete group acting smoothly on a differentiable manifold M.

(a) Show that the action is proper if and only if both of the following conditions are satisfied:

(i) Each point $p \in M$ has a neighborhood U such that $(\gamma \cdot U) \cap U = \emptyset$, for all but finitely many $\gamma \in \Gamma$.

(ii) If $p, q \in M$ are not in the same Γ-orbit, there exist neighborhoods U of p and V of q such that $(\gamma \cdot U) \cap V = \emptyset$, for all $\gamma \in \Gamma$.

(b) If Γ acts moreover freely, then show that the action is proper if and only if for each $p, q \in M$ there exist neighborhoods U of p and V of q, such that for all $\gamma \in \Gamma$ with $q \neq \gamma \cdot p$ we have $(\gamma \cdot U) \cap V = \emptyset$.

Exercise 2

Let X be a left-invariant vector field on a Lie group G with unit element e .

i) Show that there exists a curve $\gamma : \mathbb{R} \to G$ satisfying $\gamma(0) = e$ and $\dot{\gamma}(t) = X_{\gamma(t)}$, for all $t \in \mathbb{R}$.

ii) Show that $\gamma(t+s) = \gamma(t) \cdot \gamma(s)$ and $\gamma(-t) = \gamma(t)^{-1}$, for all $s, t \in \mathbb{R}$.

Exercise 3

Let G and H be two Lie groups and e the unit element of G. If $f : G \to H$ is a smooth group homomorphism, then show that:

i) $d_e f : \mathfrak{g} \to \mathfrak{h}$ is surjective if and only if f is a submersion.

ii) $d_e f : \mathfrak{g} \to \mathfrak{h}$ is bijective if and only if f is locally diffeomorphic.

iii) If H is connected and $d_e f : \mathfrak{g} \to \mathfrak{h}$ is surjective, then f is surjective. (Hint: Show that $f(G)$ is open and closed. In order to prove that the image is closed one may cosider a sequence converging to any point in the closure of the imagine and translate it by left multiplication to the unit element of H.)

Exercise 4

For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, consider the following action of $\mathbb{R}$ on $M := S^1 \times S^1$:

$$\mathbb{R} \times M \to M, \quad (t,p) \mapsto f_t(p), \quad \text{where} \quad f_t(x,y) := (e^{it}x, e^{i\alpha t}y).$$

(a) Show that each orbit of this action is dense in M and is neither closed nor a submanifold.

(b) Is the map $\Theta : \mathbb{R} \times M \to M \times M$, $(t,p) \mapsto (f_t(p), p)$ closed? Is the action proper?

(c) Is $\mathbb{R}\backslash M$ (equipped with the quotient topology) a Hausdorff space?

Hand in the solutions on ***Monday, April 29, 2013*** *before the lecture.*