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Differential Geometry II
Exercise Sheet no. 6

Exercise 1

The goal of this exercise is to show that there is no matrix $A \in \mathfrak{gl}(2, \mathbb{R}) \cong \mathbb{R}^{2\times 2}$ such that $\exp(A) = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$. Deduce a contradiction by considering the following cases:

i) A is diagonalizable.

ii) A is triagonalizable, but not diagonalizable.

iii) A has no real eigenvalues. Hint: Consider the eigenvalues of A and of $\exp(A)$.

Bonus points question: If G is a connected compact Lie group, is the Lie group exponential map surjective?

Exercise 2

We define

$$\mathcal{K} := \{J \in \mathrm{End}(\mathbb{R}^{2n}) \mid J^2 = -\mathrm{Id}\}.$$

The elements of $\mathcal{K}$ are called complex structures on $\mathbb{R}^{2n}$. The group $\mathrm{GL}(2n, \mathbb{R})$ acts by conjugation on $\mathrm{End}(\mathbb{R}^{2n})$. Show that $\mathcal{K}$ is an orbit of this action. Compute the isotropy group and write $\mathcal{K}$ as a homogeneous space.

Exercise 3

Let $(V, [\cdot,\cdot])$ be a Lie algebra over a field K. An ideal is a vector subspace W, such that $[x, y] \in W$, for all $x \in W$ and $y \in V$. Show the following:

i) The quotient space V/W carries a unique Lie bracket, such that the projection $V \to V/W$ is a Lie algebra homomorphism.

ii) The kernel of a Lie algebra homomorphism is an ideal and conversely, each ideal is the kernel of a Lie algebra homomorphism.

iii) (Bonus points) Let now $K = \mathbb{R}$. Let G be a Lie group and H a normal subgroup of G that is also a submanifold. Then the Lie algebra of H is an ideal of the Lie algebra of G.

*Hand in the solutions on **Monday, May 27, 2013** before the lecture.*