Universität Regensburg, Mathematik — SoSe 2013
Prof. Dr. Bernd Ammann — 03.06.2013
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Differential Geometry II
Exercise Sheet no. 8

Exercise 1

Let G be a Lie group which acts isometrically, freely and properly on a Riemannian manifold (M, g). (An action is *isometric* if l_σ is an isometry for any $\sigma \in G$.) Show that there exists a metric on the quotient manifold $G\backslash M$ such that the projection $\pi : M \to G\backslash M$ is a Riemannian submersion. (A submersion $\pi : M \to N$ between Riemannian manifolds is called a *Riemannian submersion* if $d_x\pi$ is an isometry from the orthogonal complement of $\ker d_x\pi$ in T_xM to $T_{\pi(x)}N$ for any $x \in M$.)

Exercise 2

Let $\pi : (M, g) \to (N, h)$ be a Riemannian submersion.

i) Let γ be a geodesic in (N, h). Show that any horizontal lift of γ is a geodesic in (M, g).

ii) Let $\tau : [a, b] \to M$ be a geodesic in (M, g) such that $\dot\tau(a)$ is horizontal. Show that $\dot\tau(t)$ is horizontal for all $t \in [a, b]$. Conclude that if a horizontal lift $\widetilde\gamma$ of a curve γ is a geodesic in (M, g), then γ is a geodesic in (N, h).

iii) Let $\pi : S^{2n+1} \to \mathbb{C}P^n$ be the projection $z \mapsto [z]$, which defines the so-called *Hopf fibration*. Consider on $\mathbb{C}P^n$ the Riemannian metric that makes π a Riemannian submersion, where S^{2n+1} carries the standard metric. This means $\mathbb{C}P^n$ carries the metric defined vai Exercise 1. This metric on $\mathbb{C}P^n$ is called the *Fubini-Study* metric of $\mathbb{C}P^n$.
Show that the geodesics parametrized by arclength in $\mathbb{C}P^n$ are of the form $\gamma(t) = [\cos t\, v + \sin t\, w]$, where $v, w \in S^{2n+1} \subset \mathbb{C}^{n+1}$ with $\sum_{j=1}^{n+1} v_j \overline{w}_j = 0$. Show furthermore that in $\mathbb{C}P^1$ the points $[(1, 0)]$ and $[(0, 1)]$ are conjugated along a geodesic.

Exercise 3

i) Let V and W be two m-dimensional real vector spaces and A_t a smooth family of homomorphisms, where t is a real parameter. Let $A'_t = \frac{d}{dt}A_t$. Assume that

$$\mathrm{Im}(A_0) \oplus A'_0(\mathrm{Ker}(A_0)) = W.$$

Show that there exists an $\varepsilon > 0$, such that A_t has rank m for all $t \in (-\varepsilon, 0) \cup (0, \varepsilon)$.

ii) Let J_1 and J_2 be two Jacobi vector fields along a geodesic on a Riemannian manifold. Show that the function

$$t \mapsto \langle J_1(t), J'_2(t)\rangle - \langle J'_1(t), J_2(t)\rangle$$

is constant.

iii) Let $\gamma : [0, b) \to M$ be a geodesic on a Riemannian manifold. Show that the set

$$\{t \in [0, b) \,|\, t \text{ is conjugated to } 0\}$$

is closed and discrete in $[0, b)$. Hint: Use i) and ii).

Exercise 4

Let $\pi : (M, g) \to (N, h)$ be a Riemannian submersion. The vectors in the kernel of $d\pi$ are called vertical. For each $X \in \Gamma(TN)$, let $\overline{X}$ denote the horizontal lift of X, *i.e.* $\overline{X} \in \Gamma(TM)$ such that $d\pi \circ \overline{X} = X \circ \pi$ and $\overline{X}$ is orthogonal in each point to the kernel of $d\pi$.

i) Show that the vertical part of $[\overline{X}, \overline{Y}]$ in $p \in M$, denoted by $[\overline{X}, \overline{Y}]_p^v$, depends only on $\overline{X}(p)$ and $\overline{Y}(p)$.

ii) Let $X \in \Gamma(TN)$, $\eta \in \Gamma(TM)$ and η is vertical. Show that $[\eta, \overline{X}]$ is vertical.

iii) Compute $\overline{[X, Y]} - [\overline{X}, \overline{Y}]$ and $\nabla^M_{\overline{X}} \overline{Y} - \overline{\nabla^N_X Y}$, for $X, Y \in \Gamma(TN)$.

iv) Assume that $\overline{X}(p)$ and $\overline{Y}(p)$ are orthonormal. Let E be the plane spanned by $X(\pi(p))$ and $Y(\pi(p))$ and $\overline{E}$ be the plane spanned by $\overline{X}(p)$ and $\overline{Y}(p)$. Show the following formula for the sectional curvatures of (M, g) and (N, h):

$$K^{N,h}(E) = K^{M,g}(\overline{E}) + \frac{3}{4} \| [\overline{X}, \overline{Y}]_p^v \|^2.$$

Hand in the solutions on ***Monday, June 10, 2013*** *before the lecture.*