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Differential Geometry II
Exercise Sheet no. 11

Exercise 1

Let M be a complete Riemannian manifold; let N be a submanifold and a closed subset of M. For any $p_0 \in M$ we define its distance to N as $d(p_0, N) := \inf_{q \in N} d(p_0, q)$. Show the following:

i) There exists a point $q_0 \in N$, such that $d(p_0, N) = d(p_0, q_0)$.

ii) If $p_0 \in M \setminus N$, then a minimizing geodesic joining p_0 and q_0 is orthogonal to N at q_0.
Hint: Use a variation of the geodesic with curves starting at p_0 and ending at points in N.

Exercise 2

Let N be a submanifold of a Riemannian manifold (M, g). The normal exponential map of N, $\exp^\perp : TN^\perp \to M$ is defined as the restriction of the exponential map $\exp : TM \to M$, $(p, v) \mapsto \exp_p v$ to points $q \in N$ and vectors $w \in (T_qN)^\perp$. Show that $p \in M$ is a focal point of $N \subset M$ if and only if p is a critical value of $\exp^\perp$.
*Hint: For "⇒" consider for a suitable variation $\gamma : (-\varepsilon, \varepsilon) \times [0, \ell] \to M$ with $\alpha(s) := \gamma(s, 0) \subset N$ and $V(s) := \frac{\nabla}{dt}\gamma|_{(s,0)}$ the curve $c(s) := (\alpha(s), \ell V(s))$.
For "⇐" consider for a suitable curve $c(s) = (\alpha(s), \ell V(s))$ in $TN^\perp$ the variation $\gamma(s, t) = \exp_{\alpha(s)}(tV(s))$.*

Exercise 3

Let N be a submanifold of a flat manifold (M, g) and γ be a geodesic in M with $\gamma(0) \in N$ and $\dot\gamma(0) \perp T_{\gamma(0)}N$. Show that $\gamma(\frac{1}{\lambda})$ is a focal point of N if and only if λ is a non-zero eigenvalue of $S_{\dot\gamma(0)}$.
Hint: For "⇒" consider $X(t) := (1 - \lambda t)E(t)$, where E is a parallel vector field along γ and $S_{\dot\gamma(0)}(E(0)) = \lambda E(0)$.

Hand in the solutions on ***Monday, July 1, 2013*** *before the lecture.*