Chapter V

Lie groups and quotients

V.4 Quotient manifolds

Lemma V.4.4. *Let $f : X \to Y$ be a surjective submersion from the C^∞-manifold X to the C^∞-manifold Y, and let Z be a further C^∞-manifold. Let $h : Y \to Z$ be a map. Then h is smooth if and only if $h \circ f$ is smooth.*

Proof. It is obvious that $h \circ f$ is smooth if h is smooth, as every submersion is by definition a smooth map.

Now assume that $h \circ f$ is smooth. For a given $y \in Y$ we want to show that h is smooth on a neighborhood of y. As y may be arbitrarily chosen, this then implies that h is smooth.

Let $n := \dim X$ and $k := \dim Y$.

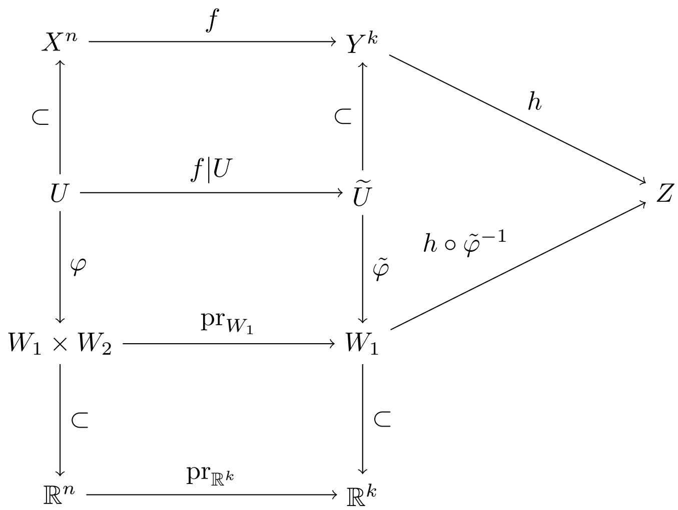
At first we choose a preimage $x \in X$ of y, i.e. $f(x) = y$. (Here we use the surjectivity of f.) We choose a chart $\tilde{\varphi}_0 : \widetilde{U}_0 \to \widetilde{V}_0$ of Y with $y \in \widetilde{U}_0$, then we choose a chart $\varphi_0 : U_0 \to V_0$ of X with $x \in U_0$

We obtain a smooth map $F : V_1 \to \widetilde{V}_0$, $F := \tilde{\varphi}_0 \circ f \circ \varphi_0^{-1}$, $V_1 := V_0 \cap \varphi_0(f^{-1}(\widetilde{U}_0))$. As $df|_x : T_xX \to T_yY$ is surjective, we see that $d(\tilde{\varphi}_0 \circ \varphi_o^{-1})|_{\varphi_0(x)}$ is surjective. The implicit function theorem thus says that there is a small neighborhood V_2 of $\varphi_0(x)$ in V_1, a diffeomorphism $\psi : V_2 \to W_1 \times W_2$, W_1 open in $\mathbb{R}^k$, W_2 open in $\mathbb{R}^{n-k}$, that there is an open neighborhood $\widetilde{V}_2$ of $\tilde{\varphi}_0(y)$ in $\widetilde{V}_0$ and a diffeomorphism $\tilde{\psi} : \widetilde{V}_2 \to W_1$, such that $\tilde{\psi} \circ F \circ \psi^{-1} : W_1 \times W_2 \to W_1$ is the projection to W_1, i.e. $\tilde{\psi} \circ F \circ \psi^{-1}(x_1, x_2) = x_1$ where $x_i \in W_i$.

In the following diagram all subset symbols denote open subsets.

$$\begin{array}{ccc}
X^n & \xrightarrow{\quad f \quad} & Y^k \\
\subset \big\uparrow & & \subset \big\uparrow \\
U_1 := U_0 \cap f^{-1}(\widetilde{U}_0) & \xrightarrow{\quad f|U_1 \quad} & \widetilde{U}_0 \\
\big\downarrow \varphi_0 & & \big\downarrow \tilde{\varphi}_0 \\
V_1 = V_0 \cap \varphi_0(f^{-1}(\widetilde{U}_0)) & \xrightarrow{\quad F \quad} & \widetilde{V}_0 \\
\subset \big\uparrow & & \subset \big\uparrow \\
V_2 & \xrightarrow{\quad F|_{V_2} \quad} & \widetilde{V}_2 \\
\big\downarrow \psi & & \big\downarrow \tilde{\psi} \\
W_1 \times W_2 & \xrightarrow{\quad \mathrm{pr}_{W_1} \quad} & W_1 \\
\big\downarrow \subset & & \big\downarrow \subset \\
\mathbb{R}^n & \xrightarrow{\quad \mathrm{pr}_{\mathbb{R}^k} \quad} & \mathbb{R}^k
\end{array}$$

We set $U := \varphi_0^{-1}(V_2)$, $\widetilde{U} := \tilde{\varphi}_0(\widetilde{V}_2)$, $\varphi := \psi \circ \varphi_0 : U \to W_1 \times W_2$, $\tilde{\varphi} := \tilde{\psi} \circ \tilde{\varphi}_0 : \widetilde{U} \to W_1$. Then $\varphi : U \to W_1 \times W_2$ and $\tilde{\varphi} : \widetilde{U} \to W_1$ are charts with $x \in U$ and $y \in \widetilde{U}$. Furthermore $\widetilde{\varphi} \circ f \circ \varphi^{-1} : W_1 \times W_2 \to W_1$ is the projection pr_{W_1} to W_1.



Now as $h \circ f$ is smooth, $h \circ f \circ \varphi^{-1} : W^1 \times W_2 \to Z$ is smooth as well. As the map

$$h \circ f \circ \varphi^{-1} = (h \circ \tilde{\varphi}^{-1}) \circ \mathrm{pr}_{W_1} : W^1 \times W_2 \to Z$$

is smooth, it is in particular smooth in the W_1 direction (for fixed element in W_2), but this is just the map $h \circ \tilde{\varphi}^{-1}) \to W_1$, which is thus smooth. This implies that $h|_{\widetilde{U}}$ is smooth. $\square$

Chapter VI

Interpretation of Curvature

VI.11 Toponogov's comparison theorem

Convention $\pi/\sqrt{\kappa} = \infty$ if $\kappa \leq 0$. In other words, for $\ell \geq 0$, the inequality $\ell \leq \pi/\sqrt{\kappa}$ is read as (i.e. defined as) the inequality $\ell^2\kappa \leq \pi^2$.

Theorem VI.11.4 (Toponogov)**.** *Let M be a complete Riemannian manifold, $K \geq \kappa$.*

(A) Let $(\gamma_1, \gamma_2, \alpha)$ be a hinge in M with γ_1 minimal and $\ell_2 := L(\gamma_2) \leq \pi/\sqrt{\kappa}$. Then any comparison hinge $(\overline{\gamma}_1, \overline{\gamma}_2, \alpha)$ in $\mathbb{M}_\kappa$ satisfies

$$d(\gamma_1(\ell_1), \gamma_2(\ell_2)) \leq d(\overline{\gamma}_1(\ell_1), \overline{\gamma}_2(\ell_2)).$$

(B) Let $(\gamma_1, \gamma_2, \gamma_3)$ be a geodesic triangle in M, let γ_1 and γ_2 be minimal and assume $L(\gamma_3) \leq \pi/\sqrt{\kappa}$.

- *(i) Then a comparison triangle $(\overline{\gamma}_1, \overline{\gamma}_2, \overline{\gamma}_3)$ in $\mathbb{M}_\kappa$ exists.*
- *(ii) The comparison triangle in (i) can be chosen such that $\overline{\alpha}_1 \leq \alpha_1$ and $\overline{\alpha}_2 \leq \alpha_2$.*
- *(iii) The comparison triangle in (i) is unique up to isometries of $\mathbb{M}_\kappa$ iff $L(\gamma_i) < \pi/\sqrt{\kappa}$ for $i = 1, 2, 3$.*

Lemma VI.11.7. *Fix ℓ_1, ℓ_2 with $0 < \ell_1, \ell_2 < \pi/\sqrt{\kappa}$. For $\alpha \in [0, \pi]$ choose a hinge $(\overline{\gamma}_1, \overline{\gamma}_2, \alpha)$ in $\mathbb{M}_\kappa$, $L(\overline{\gamma}_j) = \ell_j$. Define $f(\alpha) := d(\overline{\gamma}_1(\ell_1), \overline{\gamma}_2(\ell_2))$. Then $f : [0, \pi] \to \mathbb{R}$ increases strictly monotonically from $|\ell_2 - \ell_1|$ to D, where we set*

$$D := \min\left\{\frac{2\pi}{\sqrt{\kappa}} - \ell_1 - \ell_2, \ell_1 + \ell_2\right\}$$

for $\kappa > 0$ and $D := \ell_1 + \ell_2$ for $\kappa \leq 0$.

Lemma VI.11.8. *In $\mathbb{M}_\kappa$ a triangle $(\overline{\gamma}_1, \overline{\gamma}_2, \overline{\gamma}_3)$ with side lengths $\ell_i < \pi/\sqrt{\kappa}$ is determined by the ℓ_i up to isometries of $\mathbb{M}_\kappa$.*

Corollary VI.11.10 (Corollary of Rauch II. Stronger version of Cor. 9.8.)**.** *Let* M, $\overline{M}$ *be complete Riemannian manifolds,* $K^M \geq K^{\overline{M}}$, $\dim M \geq \dim \overline{M}$. *Let* $\tau, \overline{\tau} : [a,b] \to M, \overline{M}$ *be geodesics, parametrized by arclength. Let* $E \in \mathfrak{X}(\tau)$ *and* $\overline{E} \in \mathfrak{X}(\overline{\tau})$ *be parallel with* $\|E(t)\| = \|\overline{E}(t)\| = cst$ *and* $\langle E(t), \dot{\tau}(t) \rangle = \langle \overline{E}(t), \dot{\overline{\tau}}(t) \rangle = cst$. *Let* $f : [a,b] \to \mathbb{R}$ *be a smooth function.*

Assume that there are no focal points along

$$\eta_t : [0, f(t)] \to M$$
$$\eta_t(s) := \exp_{\tau(t)} (sE(t)) .$$

We define

$$c(t) := \exp_{\tau(t)} (f(t)E(t)) = \eta_t(f(t))$$
$$\overline{c}(t) := \exp_{\overline{\tau}(t)} \left(f(t)\overline{E}(t)\right) .$$

Then

$$L(\overline{c}) \geq L(c)$$

Proof. Almost identical to the proof of Cor. 9.8. □

Remark: In contrast to Cor 9.8 we did no require $E \perp \dot{\tau}$, $\overline{E} \perp \dot{\overline{\tau}}$ and $\|E\| = \|\overline{E}\| = 1$.