Übungen zur Indextheorie

Universität Regensburg, Wintersemester 2016/17
Prof. Dr. Bernd Ammann/ PD Dr.habil. Olaf Müller

Exercise Sheet 12, due to 3.2.2017

Exercise 1: Generalized Mehler

Show the generalized Mehler's formula

$$k_t(x,y) = \sqrt{\frac{a}{2\pi \sinh(2at)}} \exp\left(\frac{1}{2} \cdot \left(-a(x^2+y^2)\coth(2at) + 2\frac{xy}{\sinh(2at)}\right)\right)$$

using the ansatz $k_t(x,y) = \alpha(t)\exp\Big(-\frac{1}{2}\beta(t)(x^2+y^2) - \gamma(t)xy\Big)$.

Exercise 2: Spinor bundle as a square root

Let M be a Riemannian spin manifold of even dimension. Show that there is an isomorphism of Clifford bundles $J : \Sigma M \otimes \Sigma M \to \Omega^*_{\mathbb{C}}(M) := \Omega^*(M) \otimes_{\mathbb{R}} \mathbb{C}$. Now on each factor ΣM we are free to choose a $\mathbb{Z}_2$-grading: either $G_{\omega_{\mathbb{C}}} := (\Sigma M^+, \Sigma M^-)$ given by the eigenvalue decomposition of $\omega_{\mathbb{C}}$ or the trivial grading $G_{\text{triv}} := (\Sigma M, \{0\})$. The four possible choices of $\mathbb{Z}_2$-gradings give four different $\mathbb{Z}_2$-gradings for $\Omega^*_{\mathbb{C}}(M)$. Describe these gradings for $\Omega^*_{\mathbb{C}}(M)$ and compare them to the grading into even and odd forms and to the grading given by the Hodge star operator.

Exercise 3: An index finally!

Show that the index of $(d+d^*)|_{\Omega^{\text{even}}(M)} \colon \Omega^{\text{even}}(M) \to \Omega^{\text{odd}}(M)$ is the Euler characteristic of M and that the index of $(d+d^*)|_{\Omega^+(M)} \colon \Omega^+(M) \to \Omega^-(M)$ is the signature of M. Here $(\Omega^+(M), \Omega^-(M))$ is the grading given by the Hodge star operator.

Remark to Exercise 2 and 3: depending on the convention that you use there might be some factors i or -1 to insert.