EXAMPLES OF DIRAC-HARMONIC MAPS AFTER JOST-MO-ZHU

BERND AMMANN AND NICOLAS GINOUX

Abstract. We present the examples computed by J. Jost, X. Mo and M. Zhu in [2] and show that there are only few of them.

Contents

1. Introduction

Let (M^m, g) and (N^n, h) be an m-dimensional Riemannian (non-necessarily closed) spin manifold and an n-dimensional Riemannian manifold respectively. Denote by ΣM the corresponding spinor bundle of M. Given a smooth map $f : M \longrightarrow N$, one can define the twisted Dirac-operator $D^f := \sum_{j=1}^m e_j \cdot \nabla_{e_j}^{\Sigma M\otimes f^*TN}$ acting on $C^\infty(M, \Sigma M \otimes f^*TN)$, where $(e_j)_{1\leq j\leq m}$ is a local orthonormal frame on M and "$\cdot$" denotes Clifford multiplication $T^*M \otimes \Sigma M \otimes f^*TN \longrightarrow \Sigma M \otimes f^*TN$. Here $\Sigma M \otimes f^*TN$ is to be understood as the real tensor product of ΣM with f^*TN and is endowed with a natural Hermitian inner product $\langle\cdot\, ,\cdot\rangle$ making the Clifford action of each tangent vector skew-Hermitian. A pair $(f, \Phi) \in C^\infty(M, N)\times C^\infty(M, \Sigma M \otimes f^*TN)$ is called *Dirac-harmonic map* if and only if the identities

$$\left|\begin{array}{ll} D^f\Phi & = 0 \\ \mathrm{tr}_g(\nabla df) & = \frac{V_\Phi}{2} \end{array}\right. \tag{1}$$

hold on M, where $V_\Phi \in C^\infty(M, f^*TN)$ is the vector field defined by $h(V_\Phi, Y) := \sum_{j=1}^m \langle e_j \cdot R^N_{Y,f_*e_j}\Phi, \Phi\rangle$ for all $Y \in f^*TN$. Recall that, since each tangent vector to M and the curvature tensor R^N of (N, h) act in a skew-Hermitian (resp. skew-symmetric) way, the sum $\sum_{j=1}^m \langle e_j \cdot R^N_{Y,f_*e_j}\Phi, \Phi\rangle$ is real. Here and in the following the notation $e_j \cdot R^N_{Y,f_*e_j}\Phi$ stands for $(e_j \cdot\otimes R^N_{Y,f_*e_j})\Phi$. Our convention for curvature tensors is $R^N_{X,Y} = [\nabla^N_X, \nabla^N_Y] - \nabla^N_{[X,Y]}$.

As in [2], we look for solutions (f, Φ) to the Dirac-harmonic-map-equations (1) in the form

$$(f, \Phi := \sum_{j=1}^m e_j \cdot \psi \otimes f_*e_j + \varphi \otimes \nu), \tag{2}$$

where $\psi, \varphi \in C^\infty(M, \Sigma M)$ are (untwisted) spinor fields and $\nu \in C^\infty(M, f^*TN)$ is a unit vector field standing orthogonally onto TM at each point. Before stating the

Date: October 6, 2011.
Key words and phrases. Dirac harmonic maps, twistor-spinors.

main results, we write the Dirac-harmonic-map-equations for those (f,Φ) down explicitly. As usual, we denote by $D_M:\sum_{j=1}^m e_j\cdot\nabla^{\Sigma M}_{e_j}:C^\infty(M,\Sigma M)\to C^\infty(M,\Sigma M)$ the spin Dirac operator and by $P:C^\infty(M,\Sigma M)\to C^\infty(M,T^*M\otimes\Sigma M)$, $\psi\mapsto\nabla^{\Sigma M}\psi+\frac{1}{m}\cdot\sum_{j=1}^m e_j^\flat\otimes e_j\cdot D_M\psi$ the Penrose (or twistor) operator on M.

Lemma 1.1. *With the above assumptions and notations, one has*

$$\begin{aligned}D^f\Phi &= \sum_{j=1}^m\left(\frac{2-m}{m}e_j\cdot D_M\psi-2P_{e_j}\psi\right)\otimes f_*e_j-\psi\otimes\mathrm{tr}_g(\nabla df)\\ &\quad +(D_M\varphi)\otimes\nu+\sum_{j=1}^m e_j\cdot\varphi\otimes\nabla^N_{e_j}\nu\end{aligned}$$

and, for all $Y\in f^*TN$,

$$\begin{aligned}h(V_\Phi,Y) &= \sum_{j,k,l=1}^m h(R^N_{Y,f_*e_j}f_*e_k,f_*e_l)\Re e(\langle e_j\cdot e_k\cdot\psi,e_l\cdot\psi\rangle)\\ &\quad +2\sum_{j,k=1}^m h(R^N_{Y,f_*e_j}f_*e_k,\nu)\Re e(\langle e_j\cdot e_k\cdot\psi,\varphi\rangle).\end{aligned}$$

Proof. We set $\Psi:=\sum_{j=1}^m e_j\cdot\psi\otimes f_*e_j$ and compute

$$\begin{aligned}D^f\Phi &= \sum_{j=1}^m e_j\cdot\nabla^{\Sigma M\otimes f^*TN}_{e_j}(\sum_{k=1}^m e_k\cdot\psi\otimes f_*e_k)\\
&= \sum_{j,k=1}^m e_j\cdot\nabla^M_{e_j}e_k\cdot\psi\otimes f_*e_k+e_j\cdot e_k\cdot\nabla^{\Sigma M}_{e_j}\psi\otimes f_*e_k\\
&\qquad +e_j\cdot e_k\cdot\psi\otimes\nabla^{f^*TN}_{e_j}f_*e_k\\
&= -\sum_{j,k=1}^m e_k\cdot e_j\cdot\nabla^{\Sigma M}_{e_j}\psi\otimes f_*e_k-2\sum_{j,k=1}^m g(e_j,e_k)\nabla^{\Sigma M}_{e_j}\psi\otimes f_*e_k\\
&\quad +\underbrace{\sum_{j,k=1}^m e_j\cdot\nabla^M_{e_j}e_k\cdot\psi\otimes f_*e_k+e_j\cdot e_k\cdot\psi\otimes f_*(\nabla^M_{e_j}e_k)}_{0}\\
&\quad +\sum_{j,k=1}^m e_j\cdot e_k\cdot\psi\otimes\underbrace{(\nabla df)(e_j,e_k)}_{\text{symm. in }e_j,e_k}\\
&= -\sum_{k=1}^m e_k\cdot D_M\psi\otimes f_*e_k-2\sum_{j=1}^m\nabla^{\Sigma M}_{e_j}\psi\otimes f_*e_j\\
&\quad -\sum_{j=1}^m\psi\otimes(\nabla df)(e_j,e_j)\\
&= -\sum_{j=1}^m e_j\cdot D_M\psi\otimes f_*e_j-2\sum_{j=1}^m P_{e_j}\psi\otimes f_*e_j+\frac{2}{m}\sum_{j=1}^m e_j\cdot D_M\psi\otimes f_*e_j\\
&\quad -\psi\otimes\mathrm{tr}_g(\nabla df)\\
&= \frac{2-m}{m}\sum_{j=1}^m e_j\cdot D_M\psi\otimes f_*e_j-2\sum_{j=1}^m P_{e_j}\psi\otimes f_*e_j-\psi\otimes\mathrm{tr}_g(\nabla df).\end{aligned}$$

On the other hand,

$$\begin{aligned} D^f(\varphi\otimes\nu) &= \sum_{j=1}^m e_j\cdot\nabla^{\Sigma M\otimes f^*TN}_{e_j}(\varphi\otimes\nu) \\ &= \sum_{j=1}^m e_j\cdot\left(\nabla^{\Sigma M}_{e_j}\varphi\otimes\nu+\varphi\otimes\nabla^{f^*TN}_{e_j}\nu\right) \\ &= (D_M\varphi)\otimes\nu+\sum_{j=1}^m e_j\cdot\varphi\otimes\nabla^N_{e_j}\nu. \end{aligned}$$

This gives $D^f\Phi = D^f\Psi + D^f(\varphi\otimes\nu)$.
As for the vector field V_Φ, we recall that $\Phi\mapsto\sum_{j=1}^m e_j\cdot R^N_{Y,f_*e_j}\Phi$ is Hermitian, in particular

$$h(V_\Phi,Y)=h(V_\Psi,Y)+h(V_{\varphi\otimes\nu},Y)+2\Re e\left(\sum_{j=1}^m\langle e_j\cdot R^N_{Y,f_*e_j}\Psi,\varphi\otimes\nu\rangle\right)$$

for all $Y\in f^*TN$. We compute each term separately. First,

$$\begin{aligned} h(V_\Psi,Y) &= \sum_{j,k,l=1}^m \Re e\left(\langle e_j\cdot R^N_{Y,f_*e_j}(e_k\cdot\psi\otimes f_*e_k), e_l\cdot\psi\otimes f_*e_l\rangle\right) \\ &= \sum_{j,k,l=1}^m \Re e\left(\langle (e_j\cdot e_k\cdot\psi)\otimes R^N_{Y,f_*e_j}f_*e_k, e_l\cdot\psi\otimes f_*e_l\rangle\right) \\ &= \sum_{j,k,l=1}^m h(R^N_{Y,f_*e_j}f_*e_k, f_*e_l)\Re e\left(\langle e_j\cdot e_k\cdot\psi, e_l\cdot\psi\rangle\right). \end{aligned}$$

For $\varphi\otimes\nu$, using $h(R^N_{Y,f_*e_j}\nu,\nu)=0$, we obtain

$$\begin{aligned} h(V_{\varphi\otimes\nu},Y) &= \sum_{j=1}^m \Re e\left(\langle (e_j\cdot\varphi)\otimes R^N_{Y,f_*e_j}\nu, \varphi\otimes\nu\rangle\right) \\ &= \sum_{j=1}^m h(R^N_{Y,f_*e_j}\nu,\nu)\Re e\left(\langle e_j\cdot\varphi,\varphi\rangle\right) \\ &= 0. \end{aligned}$$

As for the cross term, we obtain

$$\begin{aligned} \Re e\left(\sum_{j=1}^m\langle e_j\cdot R^N_{Y,f_*e_j}\Psi,\varphi\otimes\nu\rangle\right) &= \sum_{j,k=1}^m \Re e\left(\langle (e_j\cdot e_k\cdot\psi)\otimes R^N_{Y,f_*e_j}f_*e_k,\varphi\otimes\nu\rangle\right) \\ &= \sum_{j,k=1}^m h(R^N_{Y,f_*e_j}f_*e_k,\nu)\Re e\left(\langle e_j\cdot e_k\cdot\psi,\varphi\rangle\right). \end{aligned}$$

The result follows. □

As a straightforward consequence of Lemma 1.1, Jost, Mo and Zhu [2, Thm. 2] obtain the

Corollary 1.2. *With the assumptions of* Lemma 1.1, *if furthermore $m = 2$, the spinor field ψ is a twistor-spinor, $\varphi = 0$ and the map f is harmonic, then (f,Φ) is a Dirac-harmonic map.*

Proof. The r.h.s. in the first identity of Lemma 1.1 vanishes and so does V_Φ since the Hermitian inner product $\langle e_j \cdot e_k \cdot \psi, e_l \cdot \psi\rangle$ is purely imaginary for all $j, k, l \in \{1, 2\}$. $\square$

2. Case of Riemannian hypersurfaces in spaceforms

We now specialize to the situation where f is an isometric immersion, $n = m + 1$, the manifold N is oriented and has constant sectional curvature $c \in \mathbb{R}$. Note that the orientations of M and N induce a global smooth unit normal vector field ν on M. Denote by $A := -\nabla^N \nu$ the corresponding Weingarten endomorphism-field of the hypersurface M and by $H := \frac{1}{m}\mathrm{tr}(A)$ its mean curvature. We reformulate Lemma 1.1:

Proposition 2.1. *With the assumptions above, one has*

$$\begin{aligned} D^f\Phi &= \sum_{j=1}^m \left(\frac{2-m}{m} e_j \cdot D_M\psi - 2P_{e_j}\psi - A(e_j)\cdot\varphi\right) \otimes f_* e_j \\ &\quad + (D_M\varphi - mH\psi)\otimes\nu \end{aligned}$$

and $V_\Phi = -2mc\Re e\left(\langle\psi,\varphi\rangle\right)\nu$.

Proof. Using $\nabla df = A \otimes \nu$, one has $\mathrm{tr}_g(\nabla df) = \mathrm{tr}(A)\nu = mH\nu$. Moreover, since $\nabla^N_X \nu = -A(X)$ and A is symmetric, Lemma 1.1 gives

$$\begin{aligned} D^f\Phi &= \sum_{j=1}^m \left(\frac{2-m}{m} e_j \cdot D_M\psi - 2P_{e_j}\psi\right) \otimes f_* e_j - mH\psi\otimes\nu \\ &\quad + (D_M\varphi)\otimes\nu - \sum_{j=1}^m e_j\cdot\varphi\otimes A(e_j) \\ &= \sum_{j=1}^m \left(\frac{2-m}{m} e_j \cdot D_M\psi - 2P_{e_j}\psi\right) \otimes f_* e_j + (D_M\varphi - mH\psi)\otimes\nu \\ &\quad - \sum_{j,k=1}^m g(A(e_j), e_k) e_j\cdot\varphi\otimes f_* e_k \\ &= \sum_{j=1}^m \left(\frac{2-m}{m} e_j \cdot D_M\psi - 2P_{e_j}\psi\right) \otimes f_* e_j + (D_M\varphi - mH\psi)\otimes\nu \\ &\quad - \sum_{k=1}^m A(e_k)\cdot\varphi\otimes f_* e_k \\ &= \sum_{j=1}^m \left(\frac{2-m}{m} e_j \cdot D_M\psi - 2P_{e_j}\psi - A(e_j)\cdot\varphi\right) \otimes f_* e_j \\ &\quad + (D_M\varphi - mH\psi)\otimes\nu, \end{aligned}$$

which proves the first identity. Since by assumption the curvature tensor of N is given by $h(R^N_{X,Y}Z, T) = c\cdot(h(X,T)h(Y,Z) - h(X,Z)h(Y,T))$ for all $X, Y, Z, T \in$

TN, one obtains for all $Y \in f^*TN$:

$$\begin{aligned} h(V_\Phi, Y) &= \sum_{j,k,l=1}^m h(R^N_{Y,f_*e_j} f_*e_k, f_*e_l)\Re e(\langle e_j \cdot e_k \cdot \psi, e_l \cdot \psi\rangle) \\ &\quad +2\sum_{j,k=1}^m h(R^N_{Y,f_*e_j} f_*e_k, \nu)\Re e(\langle e_j \cdot e_k \cdot \psi, \varphi\rangle) \\ &= c \cdot \sum_{j,k,l=1}^m \Bigg(h(Y, f_*e_l) \underbrace{h(f_*e_j, f_*e_k)}_{\delta_{jk}} - h(Y, f_*e_k) \underbrace{h(f_*e_j, f_*e_l)}_{\delta_{jl}} \Bigg) \Re e(\langle e_j \cdot e_k \cdot \psi, e_l \cdot \psi\rangle) \\ &\quad +2c \cdot \sum_{j,k=1}^m \Bigg(h(Y, \nu) \underbrace{h(f_*e_j, f_*e_k)}_{\delta_{jk}} - h(Y, f_*e_k) \underbrace{h(f_*e_j, \nu)}_{0} \Bigg) \Re e(\langle e_j \cdot e_k \cdot \psi, \varphi\rangle) \\ &= -mc \cdot \sum_{l=1}^m h(Y, f_*e_l) \underbrace{\Re e(\langle \psi, e_l \cdot \psi\rangle)}_{0} - mc \cdot \sum_{k=1}^m h(Y, f_*e_k) \underbrace{\Re e(\langle e_k \cdot \psi, \psi\rangle)}_{0} \\ &\quad -2mch(Y, \nu)\Re e(\langle \psi, \varphi\rangle), \end{aligned}$$

which concludes the proof. □

We can now characterize Dirac-harmonic maps of the form (2) in that setting (compare [2, Thm. 1]):

Theorem 2.2. *Let $f : M^m \longrightarrow N^{m+1}$ be an isometric immersion from a connected Riemannian spin manifold (M^m, g) into an oriented Riemannian manifold (N^{m+1}, h) with constant sectional curvature $c \in \mathbb{R}$. Let ν be a smooth unit normal vector field on M and $A := -\nabla^N \nu$ be the corresponding Weingarten-endomorphism-field with trace mH, where H is the mean curvature of f. For $\psi, \varphi \in C^\infty(M, \Sigma M)$ let $\Phi := \sum_{j=1}^m e_j \cdot \psi \otimes f_*e_j + \varphi \otimes \nu$, where $(e_j)_{1 \leq j \leq m}$ is any local orthonormal frame on M.*

- *i) If $m = 2$, then (f, Φ) is a Dirac-harmonic map with $\Phi \neq 0$ if and only if $H = 0$, $D_M \varphi = 0$, $c \cdot \Re e(\langle \psi, \varphi\rangle) = 0$ and $e_1 \cdot \nabla^{\Sigma M}_{e_1} \psi - e_2 \cdot \nabla^{\Sigma M}_{e_2} \psi = \kappa_1 \varphi$, where $A(e_1) = \kappa_1 e_1$ (the vector e_1 is pointwise an eigenvector for A associated to the principal curvature κ_1).*
- *ii) If $m \geq 3$* and f is totally umbilical, *then (f, Φ) is a Dirac-harmonic map if and only if $H = -c\Re e(\langle \psi, \varphi\rangle)$, $D_M \varphi = mH\psi$, $D_M \psi = -\frac{mH}{m-2}\varphi$ and $P\psi = 0$. If furthermore M is closed, then (f, Φ) is a Dirac-harmonic map with $\Phi \neq 0$ if and only if $A = 0$, $D_M \varphi = 0$, $\nabla^{\Sigma M} \psi = 0$ and $c \cdot \Re e(\langle \psi, \varphi\rangle) = 0$.*

Proof. Proposition 2.1 implies that (f, Φ) is a Dirac-harmonic map if and only if $D_M \varphi - mH\psi = 0$, $\frac{2-m}{m} e_j \cdot D_M \psi - 2P_{e_j} \psi - A(e_j) \cdot \varphi = 0$ for all $1 \leq j \leq m$ and $mH \cdot \nu = \frac{V_\Phi}{2} = -mc\Re e\left(\langle \psi, \varphi\rangle\right) \nu$. In other words, (f, Φ) is a Dirac-harmonic map if and only if $D_M \varphi = mH\psi$, $H = -c\Re e\left(\langle \psi, \varphi\rangle\right)$ and

$$\frac{2-m}{m} X \cdot D_M \psi - 2P_X \psi - A(X) \cdot \varphi = 0 \tag{3}$$

for all $X \in TM$. Note that, plugging $X = e_j$, taking the Clifford product of (3) with e_j and summing over j gives, using the symmetry of A,

$$\begin{aligned} 0 &= \frac{2-m}{m} \sum_{j=1}^m e_j \cdot e_j \cdot D_M \psi - 2 \underbrace{\sum_{j=1}^m e_j \cdot P_{e_j} \psi}_{0} - \sum_{j=1}^m e_j \cdot A(e_j) \cdot \varphi \\ &= (m-2) D_M \psi + mH\varphi. \end{aligned} \tag{4}$$

Case $m = 2$: Then it follows from (4) that $H\varphi = 0$. Since on the open set $\Omega := \{x \in M \,|\, H(x) \neq 0\}$ the spinor φ has to vanish, so does ψ on Ω because of $D_M\varphi = mH\psi$, so that $\Phi = 0$ on Ω and therefore on M by the unique continuation property for elliptic self-adjoint differential operators. Since we look for a pair (f,Φ) with $\Phi \neq 0$, we necessarily have $\Omega = \varnothing$, that is, $H = 0$ on M. The identities $D_M\varphi = mH\psi$, $H = -c\Re e\left(\langle\psi,\varphi\rangle\right)$ become $D_M\varphi = 0$ and $c\Re e\left(\langle\psi,\varphi\rangle\right) = 0$ respectively. As for (3), putting $X = e_j$, taking its Clifford product with X and remembering the definition of P, one obtains

$$\begin{aligned} e_j \cdot A(e_j) \cdot \varphi &= -2e_j \cdot P_{e_j}\psi \\ &= -2e_j \cdot \nabla^{\Sigma M}_{e_j}\psi + D_M\psi \end{aligned}$$

for both $j \in \{1,2\}$. The difference of both identities for $j = 1$ and 2 yields $e_2 \cdot A(e_2) \cdot \varphi - e_1 \cdot A(e_1) \cdot \varphi = 2(e_1 \cdot \nabla^{\Sigma M}_{e_1}\psi - e_2 \cdot \nabla^{\Sigma M}_{e_2}\psi)$. Take now $(e_j)_{1\leq j\leq 2}$ to be a pointwise orthonormal basis of TM made of eigenvectors for A. With the condition $H = 0$ one can write $A(e_1) = \kappa_1 e_1$ and $A(e_2) = -\kappa_1 e_2$, therefore one obtains

$$2(e_1 \cdot \nabla^{\Sigma M}_{e_1}\psi - e_2 \cdot \nabla^{\Sigma M}_{e_2}\psi) = 2\kappa_1\varphi,$$

this identity implying trivially (3). This shows *i)*.
Case $m \geq 3$: It follows from (4) that $D_M\psi = -\frac{mH}{m-2}\varphi$. As a consequence, the assumption $A = H \cdot \mathrm{Id}$ (total umbilicity of f) makes (3) equivalent to $P\psi = 0$. This proves the general case. If moreover M is closed, then $D_M^2\psi = -\frac{mH}{m-2}D_M\varphi = -\frac{m^2H^2}{m-2}\psi$. Here we use the fact any $m(\geq 2)$-dimensional totally umbilical hypersurface in an Einstein manifold has constant mean curvature: it is an elementary consequence of $\delta A = -mdH + \mathrm{Ric}^N(\nu)^T$, which itself follows from the Codazzi-Mainardi-identity (the 1-form $\mathrm{Ric}^N(\nu)^T \in T^*M$ is defined by $\mathrm{Ric}^N(\nu)^T(X) = h(\mathrm{Ric}^N(\nu), X)$ for all $X \in TM$). Analogously $D_M^2\varphi = -\frac{m^2H^2}{m-2}\varphi$. Since D_M^2 is a non-negative operator, it does not have any negative eigenvalue on a closed manifold, therefore $\psi = \varphi = 0$ unless $H = 0$, which is the only possibility because of $\Phi \neq 0$. Therefore H - hence A - has to vanish on M. Since both $D_M\psi = 0$ and $P\psi = 0$, one obtains $\nabla^{\Sigma M}\psi = 0$ (hence ψ is actually parallel). This shows *ii)* and concludes the proof. □

At this point we notice a mistake in [2, Thm. 1]: even in the case $m \geq 3$ the authors assume the spinor field φ to be harmonic (i.e., $D_M\varphi = 0$), which with Theorem 2.2 yields $H = 0$, $c\Re e(\langle\psi,\varphi\rangle) = 0$ and $\nabla^{\Sigma M}\psi = 0$. In particular no example with non-harmonic map f can be produced with their result.

We now describe explicit examples fitting into Theorem 2.2. From now on we denote by $N^{m+1}(c)$ any Riemannian spaceform of constant sectional curvature c and by $\widetilde{N}^{m+1}(c)$ *the* simply-connected complete Riemannian spaceform of constant sectional curvature c. Without loss of generality (up to rescaling the metric h, which does not affect any existence result), we can and shall assume that the sectional curvature c of N lies in $\{-1,0,1\}$, i.e., $\widetilde{N}^{m+1}(c) = \mathbb{H}^{m+1}(-1)$, $\mathbb{R}^{m+1}$ and $\mathbb{S}^{m+1}(1)$ for $c = -1$, 0 and 1 respectively.

Proposition 2.3. *With the above notations, one has the following:*

i) Case $m = 2$ and M is closed:
  1) *For any conformally minimal immersion f from $M := \mathbb{S}^2$ into $N := N^3(1)$, there exists a non-zero $\Phi \in C^\infty(M, \Sigma M \otimes f^*TN)$ such that (f,Φ) is a Dirac-harmonic map.*
  2) *For any conformally minimal immersion f from $M := \mathbb{T}^2$ endowed with the trivial spin structure into $N := N^3(1)$ or compact $N^3(0)$,*

there exists a non-zero $\Phi \in C^\infty(M, \Sigma M \otimes f^*TN)$ *such that* (f, Φ) *is a Dirac-harmonic map.*

ii) Case $m = 2$ *and* M *is non-compact: for any conformally minimal immersion* f *from any non-empty open subset* $M \subset \mathbb{R}^2$ *into* $N := \widetilde{N}^3(c)$*, there exists a non-zero* $\Phi \in C^\infty(M, \Sigma M \otimes f^*TN)$ *such that* (f, Φ) *is a Dirac-harmonic map.*

iii) Case $m \geq 3$ *and* $A = 0$*: for the inclusion map* f *of any non-empty open subset* M *of any hyperplane in* $N := \widetilde{N}^{m+1}(0) = \mathbb{R}^{m+1}$*, there exists a non-zero* $\Phi \in C^\infty(M, \Sigma M \otimes f^*TN)$ *such that* (f, Φ) *is a Dirac-harmonic map. The same holds when* $M := \mathbb{T}^m$ *with flat metric and spin structure is embedded totally geodesically into* $N := \mathbb{T}^{m+1}$.

iv) Case $m \geq 3$ *and* $A = H \cdot \mathrm{Id}$ *with* $H \neq 0$*: for the inclusion map* f *of any non-empty open subset* M *of any hyperplane* $\mathbb{H}^m(-\frac{4}{m+2})$ *in* $N := \widetilde{N}^{m+1}(-1) = \mathbb{H}^{m+1}$*, there exists a non-zero* $\Phi \in C^\infty(M, \Sigma M \otimes f^*TN)$ *such that* (f, Φ) *is a Dirac-harmonic map.*

Proof. Note first that, if we let $\varphi = 0$ in case $m = 2$, then Theorem 2.2 states that the pair (f, Φ) is a non-trivial Dirac-harmonic map if and only if $H = 0$ (that is, f is harmonic) and ψ is a non-zero twistor-spinor (for $e_1 \cdot \nabla^{\Sigma M}_{e_1}\psi - e_2 \cdot \nabla^{\Sigma M}_{e_2}\psi = e_1 \cdot P_{e_1}\psi - e_2 \cdot P_{e_2}\psi$, as we have seen above). Therefore the first result by Jost, Mo and Zhu [2, Thm. 2] is recovered in that particular setting.

For $m = 2$ we remind the reader of the conformal invariance of the Dirac-harmonic-map equation: a pair (f, Φ) is Dirac-harmonic on (M^2, g) if and only if $(f, e^{-\frac{u}{2}}\Phi)$ is Dirac-harmonic on $(M^2, e^{2u}g)$, whatever $u \in C^\infty(M, \mathbb{R})$ is. In case the surface M^2 is closed, the only possibility for it to carry non-trivial twistor-spinors is to be conformally spin diffeomorphic to $\mathbb{S}^2$ or to $\mathbb{T}^2$ with trivial spin structure. Combining that fact with the preceding remarks, we deduce that, given any immersion f from such a surface into a spaceform, if there is a conformal metric on N such that the immersion is minimal, then for that metric the pair (f, Φ) made out of a non-zero twistor spinor ψ and with $\varphi = 0$ is Dirac-harmonic; by conformal invariance there is a non-trivial Dirac-harmonic map for the original metric. Note that the Gauß equation for scalar curvature implies $c > 0$ (where $N = N^3(c)$) in case $M = \mathbb{S}^2$ and $c \geq 0$ in case $M = \mathbb{T}^2$. There is anyway no closed example in $\mathbb{R}^3$ or $\mathbb{H}^3$ since there is no closed minimal hypersurface in those spaceforms. This shows *i*). The proof of *ii*) follows the same lines since any (non-empty) open subset of $\mathbb{R}^2$ has an infinite-dimensional space of twistor-spinors, whatever the metric it carries.

In case $m \geq 3$ and $A = 0$, any (non-empty) open subset of $\mathbb{R}^m$ with flat metric carrying non-zero parallel (hence harmonic) spinors, one can choose ψ and φ to be parallel spinors and obtains a non-zero Φ such that (f, Φ) is a Dirac-harmonic map. Note that those examples with $\psi \neq 0$ have to be Ricci-flat hence flat hypersurfaces M in $N^{m+1}(0)$. Closed examples for M can be similarly obtained by choosing a flat $M := \mathbb{T}^m$ totally geodesically sitting in $N = \mathbb{T}^{m+1}$, provided $\mathbb{T}^m$ carries the trivial spin structure (otherwise no non-zero parallel spinor is available). This proves *iii*).

As for the remaining case where $m \geq 3$ and $A = H \cdot \mathrm{Id} \neq 0$, recall that M has to be non-compact (Theorem 2.2). Since $P\psi = 0$, we know that $D_M^2\psi = \frac{mS_g}{4(m-1)}\psi$, where S_g is the scalar curvature of (M^m, g). Comparing with $D_M^2\psi = -\frac{m^2H^2}{m-2}\psi$ and assuming $\psi \neq 0$ (otherwise $\varphi = 0$ hence $\Phi = 0$, as we have seen above), we obtain $\frac{mS_g}{4(m-1)} = -\frac{m^2H^2}{m-2}$ and Gauß equation $S_g = m(m-1)c + m^2H^2 - |A|^2 = m(m-1)(H^2 + c)$ implies $H^2 = -\frac{m-2}{m+2}c$, in particular c has to be negative, w.l.o.g. $c = -1$. We consider the case where $N = \widetilde{N}^{m+1}(-1) = \mathbb{H}^{m+1}(-1)$. Then M has to be a totally umbilical (but non-totally geodesic) hyperbolic hyperplane of constant sectional curvature $H^2+c=\frac{4}{m+2}c=-\frac{4}{m+2}$. Up to changing ν into $-\nu$, one can assume H to be positive, so that $H=\sqrt{\frac{m-2}{m+2}}$. Now the space of twistor-spinors on any hyperbolic space is explicitly known: it is the direct sum of the space of Killing spinors for the opposite (imaginary) Killing constants. More precisely $\ker(P)=\mathcal{K}_p\oplus\mathcal{K}_m$ on M, where $\mathcal{K}_p:=\{\psi\in C^\infty(M,\Sigma M)\,|\,\nabla^{\Sigma M}_X\psi=\frac{i}{\sqrt{m+2}}X\cdot\psi\;\forall X\in TM\}$ and $\mathcal{K}_m:=\{\psi\in C^\infty(M,\Sigma M)\,|\,\nabla^{\Sigma M}_X\psi=-\frac{i}{\sqrt{m+2}}X\cdot\psi\;\forall X\in TM\}$. Looking for ψ in the form $\psi=\psi_p+\psi_m$ with *a priori* arbitrary $(\psi_p,\psi_m)\in\mathcal{K}_p\oplus\mathcal{K}_m$, we write the equations of Theorem 2.2 down: one has $D_M\psi=-\frac{im}{\sqrt{m+2}}(\psi_p-\psi_m)$, in particular one has to choose $\varphi:=-\frac{m-2}{mH}D_M\psi=i\sqrt{m-2}(\psi_p-\psi_m)$. Then $D_M\varphi=i\sqrt{m-2}\cdot(-\frac{im}{\sqrt{m+2}}(\psi_p+\psi_m))=m\sqrt{\frac{m-2}{m+2}}\psi=mH\psi$. The only remaining condition having to be satisfied is $H=-c\cdot\Re e(\langle\psi,\varphi\rangle)$, that is,

$$\begin{aligned}\sqrt{\frac{m-2}{m+2}} &= \sqrt{m-2}\cdot\Re e(-i\langle\psi_p+\psi_m,\psi_p-\psi_m\rangle)\\ &= \sqrt{m-2}\cdot\Im m(|\psi_p|^2-|\psi_m|^2+\langle\psi_m,\psi_p\rangle-\langle\psi_p,\psi_m\rangle)\\ &= -2\sqrt{m-2}\cdot\Im m(\langle\psi_p,\psi_m\rangle),\end{aligned}$$

that is, $\Im m(\langle\psi_p,\psi_m\rangle)=-\frac{1}{2\sqrt{m+2}}$. Note that the inner product $\langle\psi_p,\psi_m\rangle$ is anyway constant on M (its first derivative vanishes). Since $(\psi_p,\psi_m)\mapsto\langle\psi_p,\psi_m\rangle$ is sesquilinear, it suffices to find a pair (ψ_p,ψ_m) with $\langle\psi_p,\psi_m\rangle\neq 0$ (then multiply ψ_m by a suitable complex constant to obtain the desired imaginary part). This can be achieved in an elementary way, taking ψ_p to be arbitrary (non-vanishing) and setting $\psi_m(x):=x\cdot e_{m+1}\cdot\psi_p(x)$ for all $x\in\mathbb{H}^m(-1)$ (if the result is true for $\mathbb{H}^m(-1)$, then it is obviously true for $\mathbb{H}^m(-\frac{4}{m+2})$), where e_{m+1} is the last canonical basis vector in $\mathbb{R}^{m+1}\supset\mathbb{H}^m(-1)$ and here "$\cdot$" denotes the Lorentzian Clifford multiplication in $\mathbb{R}^{m+1}$ with Minkowski-metric. It is namely a straightforward computation to show that $\langle\psi_p,\psi_m\rangle=\|\psi_p\|^2$, where $\|\cdot\|$ denotes the *positive-definite* Hermitian inner product on the space of spinors of $\mathbb{R}^{m+1}$. In particular, $\langle\psi_p,\psi_m\rangle\neq 0$, which is what we wanted.[1] This shows *iv*) and concludes the proof. □

It may be interesting to know whether 2-dimensional examples with $\varphi\neq 0$ can be obtained. Namely if one considers the Clifford torus $M^2:=\mathbb{S}^1(\frac{1}{\sqrt{2}})\times\mathbb{S}^1(\frac{1}{\sqrt{2}})$ sitting canonically in $N:=\mathbb{S}^3$, then the inclusion map is minimal (with principal curvatures 1 and -1) but a short computation shows that the only Dirac-harmonic maps (f,Φ) in the form (2) have vanishing φ-component.

Note that, in case $N=\mathbb{H}^{m+1}(-1)$, we have actually shown in the proof of Proposition 2.3 that the example described is the *only* one with (f,Φ) in the particular form (2). Even in the case where $m=2$ no non-trivial example of Dirac-harmonic maps from a closed hyperbolic surface can be obtained with that approach, since those do not carry non-zero twistor-spinors. In that setting, examples can be produced with the help of index-theoretical methods, see e.g. [1]. Curvature conditions implying the vanishing of the Φ defined in (2) have been investigated by X. Mo [3] and confirm that only few examples of that special form can be expected.

For higher codimensions the same approach can probably be carried out, the existence of a global unit normal ν already restricting the generality. On the other hand, there are in that case obvious examples of Dirac-harmonic maps which are *not* in the

[1] Probably it is cleverer to show this by *trivializing* the spinor bundle of $\mathbb{H}^m$ by $\frac{i}{2}$- as well as by $-\frac{i}{2}$-Killing spinors. Then it is no problem, one just have to choose ψ_p and ψ_m so that they *coincide* at one point. (N.)

form (2): take e.g. $M := \mathbb{S}^2 = \mathbb{C}\mathrm{P}^1$ embedded totally geodesically into $N = \mathbb{C}\mathrm{P}^2$, then we know by the index-theorem (see e.g. [1]) that $\dim_{\mathbb{C}}(\ker(D^f)) \equiv 2\ (4)$ and is at least 4-dimensional by [2] (the space of twistor-spinors on $\mathbb{S}^2$ injects into $\ker(D^f)$), so that it is at least - actually exactly - 6-dimensional. Now if $\Phi \in \ker(D^f)$, then it is an easy remark that w.r.t. the canonical splitting $\Phi = \Phi_+ + \Phi_-$ one has $D^f\Phi_\pm = 0$ and $V_{\Phi_\pm} = 0$, in particular (f, Φ_+) and (f, Φ_-) are Dirac-harmonic maps; since $\dim_{\mathbb{C}}(\ker(D^f_\pm)) \geq 3$ and the space of pure twistor-spinors is complex 2-dimensional, there are at least one non-trivial $\Phi_+ \in \ker(D^f_+)$ and one non-trivial $\Phi_- \in \ker(D^f_-)$ such that $(f, \Phi_\pm)$ are Dirac-harmonic but do not come from any twistor-spinor on $\mathbb{S}^2$.

References

[1] B. Ammann, N. Ginoux, *Dirac-harmonic maps from index theory*, in preparation.
[2] J. Jost, X. Mo, M. Zhu, *Some explicit constructions of Dirac-harmonic maps*, J. Geom. Phys. **59** (2009), no. 11, 1512–1527.
[3] X. Mo, *Some rigidity results for Dirac-harmonic maps*, Publ. Math. Debrecen **77** (2010), no. 3-4, 427–442.

Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany
E-mail address: bernd.ammann@mathematik.uni-regensburg.de

Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany
E-mail address: nicolas.ginoux@mathematik.uni-regensburg.de