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Abstract

The Stefan problem with Gibbs-Thomson law describes solidification phenomena for pure substances. In applications the surface energy is anisotropic leading to an anisotropic Gibbs-Thomson law. We show the existence of weak solutions to the Stefan problem with anisotropic Gibbs-Thomson law using an implicit time discretization, and variational methods in an anisotropic BV setting. Our main result generalizes an existence result of Luckhaus to the anisotropic case.

Key words: Stefan problem, anisotropy, Gibbs-Thomson law, free boundary, implicit time discretization.
AMS-Classification: 35K55, 35R35, 49Q20, 73B40, 82B26, 58B20, 80A22.

1 Introduction

The Stefan problem describes solidification phenomena like the melting and solidification of a pure material. In the Stefan problem diffusion equations have to be solved in the liquid and solid and at the free boundary between solid and liquid the Stefan condition has to hold which guarantees energy conservation across the interface. In addition a thermodynamical equilibrium condition has to be prescribed at the interface and in the presence of surface tension this condition is given by the Gibbs-Thomson law. The Gibbs-Thomson law allows that the temperature at the free boundary differs from the melting temperature and hence allows for undercooling and superheating. In applications as e.g. the solidification of alloys or the growth of snowflakes the surface energy density usually depends on the local orientation of the interface, i.e. the surface energy is anisotropic. It turns out that in the Stefan problem with anisotropy the temperature at the free boundary is given as a

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multiple of an anisotropic curvature which reduces in the isotropic case to the mean curvature. For a derivation of the Stefan problem with anisotropic Gibbs-Thomson law in the context of rational thermodynamics we refer to Gurtin [13, 14].
Given a time interval $(0,T)$ and a bounded domain $\Omega \subset \mathbb{R}^n$ with C^1–boundary we define $\Omega_T := (0,T) \times \Omega$. We now seek for the temperature $u : \Omega_T \to \mathbb{R}$ and a phase function $\chi : \Omega_T \to \{0,1\}$ where the liquid phase is given as the set $\{(t,x) \in \Omega_T \,|\, \chi(t,x) = 1\}$ and the solid phase is given as $\{(t,x) \in \Omega_T \,|\, \chi(t,x) = 0\}$. Denoting by $f : \Omega_T \to \mathbb{R}$ given heat sources, the energy balance law is now given as

$$\partial_t(u + \chi) - \Delta u = f\,, \tag{1}$$

where this identity has to be understood in its distributional form. The strong formulation of (1) is given by

$$\partial_t u - \Delta u = f$$

in the solid and liquid phases together with the Stefan condition

$$V + [\nabla u]_s^l \cdot \nu = 0\,,$$

where ν is the unit normal to the interface Γ pointing into the liquid phase, V denotes the normal velocity of the interface and $[\nabla u]_s^l := \nabla u_{,s} - \nabla u_{,l}$ is the jump of ∇u across the interface, where $u_{,s}$ and $u_{,l}$ are respectively the temperature in the solid and liquid phase. At the interface between liquid and solid the *Gibbs-Thomson law* in its isotropic form is

$$u = H\,,$$

where H is the mean curvature of the interface which is defined to be the sum of the principal curvatures and we adopt the sign convention that H is negative for a convex solid phase. We refer to [21] for an introduction to the Stefan problem with Gibbs-Thomson law.
A fundamental global existence result for the Stefan problem with isotropic Gibbs-Thomson law is due to Luckhaus [15, 16], see also Röger [18]. Luckhaus formulates the Gibbs-Thomson condition $u = H$ in the following weak form

$$\int_0^T \int_\Omega \left(\operatorname{div} \xi - \frac{\nabla \chi}{|\nabla \chi|} \cdot D\xi \frac{\nabla \chi}{|\nabla \chi|} \right) d|\nabla \chi(t)| dt = \int_{\Omega_T} \operatorname{div}(u\xi)\, \chi\, d(t,x)\,, \tag{2}$$

which has to hold for all $\xi \in C^1(\overline{\Omega}_T, \mathbb{R}^n)$ with $\xi \cdot \nu_{\partial\Omega} = 0$ on $\partial\Omega$. Here $\nu_{\partial\Omega}$ is the outer unit normal to $\partial\Omega$, χ is assumed to be a function of bounded variation, see [3], and $\nabla \chi$ is the distributional derivative of χ which is assumed to be of bounded variation. In addition $\frac{\nabla \chi}{|\nabla \chi|}$ is the Radon-Nikodym derivative of $\nabla \chi$ with respect to the variation measure $|\nabla \chi|$.
If the interface is smooth and without boundary the equation (2) leads to

$$\int_0^T \int_{\Gamma(t)} \operatorname{div}_\Gamma \xi \, d\mathcal{H}^{n-1} dt = - \int_0^T \int_{\Gamma(t)} u\, \xi \cdot \nu \, d\mathcal{H}^{n-1} dt\,.$$

Here we denote by $\Gamma(t)$ the interface at time t, by $d\mathcal{H}^{n-1}$ integration with respect to the $(n-1)$–dimensional Hausdorff measure and by div_Γ the surface divergence on Γ. Using the Gauss theorem on manifolds, i.e.

$$\int_{\Gamma(t)} \mathrm{div}_\Gamma\, \xi \, d\mathcal{H}^{n-1} = -\int_{\Gamma(t)} H\xi \cdot \nu \, d\mathcal{H}^{n-1} ,$$

we obtain, using the fact that we can choose ξ arbitrary, that (2) is a weak formulation of $u = H$.
To formulate the Gibbs-Thomson law in its anisotropic form we need to introduce the anisotropic interfacial free energy

$$\mathcal{F}(\Gamma) := \int_\Gamma \gamma(\nu)\, d\mathcal{H}^{n-1}$$

for a hypersurface Γ. For the moment we require Γ to be smooth and define ν to be the unit normal to Γ pointing into the liquid phase. We assume that γ is a one-homogeneous, convex function. The free energy $\mathcal{F}(\Gamma)$ now depends on the local orientation of the interface since γ depends on the normal ν. The first variation of $\mathcal{F}$ for a hypersurface Γ in the direction of a vector field $\xi \in C_0^1(\overline{\Omega}, \mathbb{R}^n)$ with $\xi \cdot \nu_{\partial\Omega} = 0$ is given as, see [11],

$$\frac{\delta \mathcal{F}}{\delta \Gamma}(\Gamma)(\xi) = -\int_\Gamma H_\gamma (\xi \cdot \nu)\, d\mathcal{H}^{n-1}$$

with

$$H_\gamma := -\mathrm{div}_\Gamma \left(D\gamma(\nu) \right) ,$$

where $D\gamma$ is the gradient of γ.
The anisotropic Gibbs-Thomson law is now given as

$$u = H_\gamma \, .$$

In situations where Γ intersects the outer boundary $\partial\Omega$ one also has to require a boundary condition

$$D\gamma(\nu) \cdot \nu_{\partial\Omega} = 0 \quad \text{on} \quad \partial\Omega \cap \partial\Gamma(t),\ t \in [0,T] \, . \tag{3}$$

This condition generalizes the classical 90° angle condition which holds in the isotropic case, see e.g. [10],[21] .
If the interface is not smooth and the different phases are given by a function $\chi : \Omega \to \{0,1\}$ which is assumed to be of bounded variation we obtain the normal as the Radon-Nikodym derivative of $\nabla\chi$ with respect to the variation mesure $|\nabla\chi|$, i.e.

$$\overline{\nu} = \frac{\nabla\chi}{|\nabla\chi|} \, .$$

We will demonstrate later that a weak formulation of the anisotropic Gibbs-Thomson law $u = H_\gamma$ together with the boundary condition (3) is given as

$$\int_\Omega (\operatorname{div} \xi \, \gamma(\overline{\nu}) - \overline{\nu} \cdot D\xi D\gamma(\overline{\nu}))\, d|\nabla \chi| = \int_\Omega \operatorname{div}(u\xi)\chi \, dx \,,$$

which has to hold for all $\xi \in C^1(\overline{\Omega}, \mathbb{R}^n)$ with $\xi \cdot \nu_{\partial\Omega} = 0$.
Our approach will be based on a distributional definition of the anisotropic surface energy $\int_\Gamma \gamma(\nu) d\mathcal{H}^{n-1}$. Introducing a function $\gamma^0 : \mathbb{R}^n \to \mathbb{R}_0^+$ with the properties

$$\text{(G1)} \qquad \gamma^0 \in C^2(\mathbb{R}^n \setminus \{0\}),\ \gamma^0(p) > 0 \quad \text{for all} \quad p \in \mathbb{R}^n \setminus \{0\}\,, \tag{4}$$

$$\text{(G2)} \qquad \gamma^0 \quad \text{is positively homogeneous of degree 1, i.e.:}$$

$$\gamma^0(\lambda p) = \lambda \gamma^0(p) \quad \text{for all} \quad \lambda > 0 \text{ and } p \in \mathbb{R}^n \setminus \{0\}\,, \tag{5}$$

$$\text{(G3)} \qquad \text{there exists a } d_0 > 0 \text{ such that}$$

$$(D^2\gamma^0)(p)q \cdot q \geq d_0 |q|^2 \quad \text{for all} \quad p, q \in \mathbb{R}^n,\ |p| = 1,\ p \cdot q = 0\,, \tag{6}$$

we define for $f \in BV(\Omega)$

$$\int_\Omega |\nabla f|_\gamma := \sup \left\{ - \int_\Omega f \operatorname{div} \varphi \, dx \mid \varphi \in C_0^1(\Omega, \mathbb{R}^n),\ \gamma^0(\varphi(x)) \leq 1 \text{ a.e.} \right\} . \tag{7}$$

Assumption (G1) is a smoothness assumption on γ^0 where due to the homogeneity assumption we can expect γ^0 to be smooth only away from the point $p = 0$. The assumption (G3) is a strict convexity assumption for functions which are homogeneous of degree 1. We remark that due to the homogeneity we only require that the second derivative is positive in directions perpendicular to p, see also Giga [11]. In a direction p the function γ^0 has to be linear and hence strict convexity does not hold in this direction.
We now assume that γ is given as

$$\gamma(q) = \sup_{p \in \mathbb{R}^n \setminus \{0\}} \frac{p \cdot q}{\gamma^0(p)}\,, \tag{8}$$

and it turns out, see [1, 2] and Section 2, that for all $f \in BV(\Omega)$ we obtain

$$\int_\Omega |\nabla f|_\gamma = \int_\Omega \gamma(\nu_f)\, d|\nabla f|\,, \tag{9}$$

where $\nu_f = \frac{\nabla f}{|\nabla f|}$ for $|\nabla f|$ a.e. $x \in \Omega$.
The function γ is the dual function of γ^0 and under the assumptions made it will turn out that γ^0 is also the dual of γ. It is possible to visualize the anisotropy with the help of the Frank diagram $\mathcal{F}$ and the Wulff shape $\mathcal{W}$

$$\mathcal{F} = \{p \in \mathbb{R}^n \mid \gamma(p) \leq 1\}\,,\ \mathcal{W} = \{q \in \mathbb{R}^n \mid \gamma^0(q) \leq 1\}\,.$$

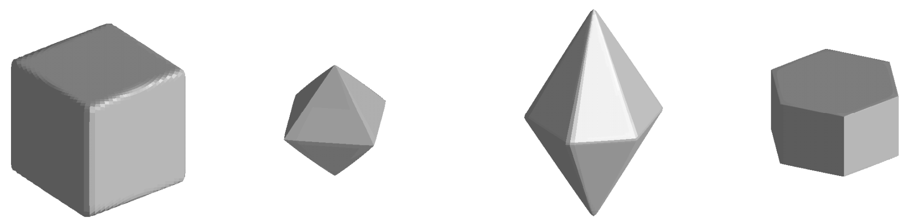

Figure 1: Frank diagrams and Wulff shapes for different surface energies. Cubic anisotropy (left) and hexagonal anisotropy (right).

Wulff's theorem, see Gurtin [14] and the references therein, states that $\mathcal{W}$ minimizes the surface energy $\mathcal{F}$ among all surfaces Γ enclosing the same volume as $\mathcal{W}$. Important surface energies have cubic or hexagonal symmetries which appear respectively in the solidification of metallic alloys and ice crystals, see Figure 1.
The main result of this paper is now given as follows.

Theorem 1.1 *Let the following assumptions hold:*

(A1) $\Omega \subset \mathbb{R}^n$ *is a bounded domain with* C^1*–boundary,* $T > 0$.

(A2) The initial data u_0, χ_0*, the boundary data* u^D *and the right hand side* f *fulfill*

$$\begin{aligned} u_0 &\in L^\infty(\Omega) \cap H^{1,2}(\Omega)\,, \\ \chi_0 &\in BV(\Omega; \{0,1\})\,, \\ u^D &\in H^{1,2}(\Omega), \\ f &\in L^\infty(\Omega_T)\,. \end{aligned}$$

(A3) The anisotropy γ *is given by (8), where* $\gamma^0 : \mathbb{R}^n \to \mathbb{R}$ *fulfills (4)-(6).*

Then there exist functions

$$\chi \in L^1(\Omega_T, \{0,1\})$$

such that $\operatorname{ess\,sup}_{t\in(0,T)} \int_\Omega |\nabla\chi|(t) < \infty$*, i.e. in particular* $\chi(t) \in BV(\Omega)$ *for almost all* t*, and*

$$u \in [u^D + L^2(0,T; H^{1,2}_0(\Omega))] \cap L^\infty(0,T; L^2(\Omega))$$

such that

$$(i) \quad \int_{\Omega_T} (u+\chi)\partial_t\varphi \, d(t,x) + \int_\Omega (u_0+\chi_0)\varphi(0)\, dx = \int_{\Omega_T} \nabla u \cdot \nabla\varphi \, d(t,x) - \int_{\Omega_T} f\varphi \, d(t,x)$$

for all $\varphi \in C^1_0([0,T) \times \Omega)$*, and*

$$(ii) \quad \int_0^T \int_\Omega (div\, \xi D\gamma(\overline{\nu}) \cdot \overline{\nu} - \overline{\nu} \cdot D\xi D\gamma(\overline{\nu}))\, d|\nabla\chi(t)| dt - \int_{\Omega_T} div(u\xi)\chi \, d(t,x) = 0$$

for all $\xi \in C^1(\overline{\Omega}_T, \mathbb{R}^n)$ *with* $\xi \cdot \nu_{\partial\Omega} = 0$ *on* $\partial\Omega$.

The outline of the paper is as follows.
In Section 2 we will discuss main facts about the anisotropic surface energy and in particular derive a weak formulation of the anisotropic Gibbs-Thomson law and its natural boundary condition (3). Then a time discrete approximation to the Stefan problem with anisotropic Gibbs Thomson law is introduced in Section 3. A variational structure of the time discrete problem is used to show existence of solutions as well as a priori estimates. Finally in Section 4 we will show that the time discrete solutions converge to a solution of the continuous problem. We will focus our presentation on the difficulties arising from the anisotropy. Arguments which are similar to the work of Luckhaus [15, 16] will only be sketched. It will turn out that the main difficulty will be to pass to the limit in the term

$$\int_\Omega (\operatorname{div}\xi\gamma(\overline{\nu}) - \overline{\nu}\cdot D\xi D\gamma(\overline{\nu}))\,d|\nabla\chi|\,.$$

In the isotropic case a lemma of Reshetnyak can be used to show that the approximate normals from the time discrete problems converge, see Luckhaus [15, 16]. In the anisotropic case such a reasoning is not possible and we will use the crucial fact that

$$\int_\Omega \gamma(\nu^h)\,d|\nabla\chi^h| \to \int_\Omega \gamma(\overline{\nu})\,d|\nabla\chi|\,,$$

ν^h being approximate normals, implies that

$$D\gamma(\nu^h) \to D\gamma(\overline{\nu})$$

in some appropriate sense. This fact will be important in order to pass to the limit in an approximate version of the weak form of the Gibbs-Thomson law. Finally we refer to related results for a static case, i.e. for a time independent situation, by Luckhaus, Modica [17], Garcke, Kraus [9] and Cialese, Nagase, Pisante [6]. We also refer to Barrett, Garcke, Nürnberg [4] for recent numerical simulations for the Stefan problem with Gibbs-Thomson law which demonstrate that the model can be used to describe realistic pattern formations in anisotropic solidification scenarios.

2 Anisotropic surface energy

In this section we derive results about the anisotropic surface energy (7) which will be needed later.
Suppose $\gamma^0 : \mathbb{R}^n \to \mathbb{R}$ fulfills (4)-(6). It is then possible to show that the statements (4)-(6) also hold for γ, see e.g. [11, 20]. Moreover, see [11, 20], the dual function of γ is γ^0, i.e.

$$\gamma^0(q) = \sup_{p\in\mathbb{R}^n\setminus\{0\}} \frac{p\cdot q}{\gamma(p)}\,.$$

Now we discuss some relevant properties for the anisotropy function γ.

Lemma 2.1 *Let γ fulfill (4)-(6). Then the identities*

(i) $D\gamma(p) \cdot p = \gamma(p)$,

(ii) $D\gamma(\lambda p) = D\gamma(p)$,

(iii) $D^2\gamma(p)p = 0$,

(iv) $D^2\gamma(\lambda p) = \frac{1}{\lambda} D^2\gamma(p)$

hold for all $p \in \mathbb{R}^n \setminus \{0\}$ and $\lambda > 0$.

For a proof we refer to Giga [11].
Using this lemma, we can prove the following statements.

Lemma 2.2 *Let γ fulfill (4)-(6). Then there exist constants C_1 and C_2 such that for all ν_1, $\nu_2 \in \mathbb{S}^{n-1}$ and $p \in \mathbb{R}^n \setminus \{0\}$ the following properties are satisfied:*

(i) $\gamma^0 \left(D\gamma(p) \right) = 1$,

(ii) $\gamma^0(p) D\gamma \left(D\gamma^0(p) \right) = p$,

(iii) $\gamma(\nu_1) - D\gamma(\nu_2) \cdot \nu_1 \geq C_1 \left| \nu_1 - \nu_2 \right|^2$,

(iv) $\left| D\gamma(\nu_1) - D\gamma(\nu_2) \right| \leq C_2 \left| \nu_1 - \nu_2 \right|$.

A proof of (i) and (ii) can be found in Bellettini and Paolini [5]. The properties (iii) and (iv) can be derived from Dziuk [7] and Giga [11]. The properties (i)-(iv) in Lemma 2.2 also hold if the roles of γ and γ^0 are interchanged.
The next lemma is necessary in Section 4 when we prove the convergence of the time discrete solutions to a solution of the continuous problem.

Lemma 2.3 *Let γ fulfill (4)-(6). Then there exists a constant $C > 0$ such that*

$$C \left| D\gamma(\nu) - p \right|^2 \leq \gamma(\nu) - p \cdot \nu$$

holds for all $\nu \in \mathbb{S}^{n-1}$ and $p \in \mathcal{W} = \{ q \in \mathbb{R}^n \mid \gamma^0(q) \leq 1 \}$.

Proof: Let $\nu \in \mathbb{S}^{n-1}$ and $p \in \mathbb{R}^n$ such that $\gamma^0(p) \leq 1$. Define

$$\begin{aligned} \tau(p, \nu) &:= \sup \left\{ t > 0 \,:\, \gamma^0(p + t\nu) \leq 1 \right\} , \\ C_\tau &:= \sup \left\{ \tau(p, \nu) \,:\, \nu \in \mathbb{S}^{n-1}, p \in \mathcal{W} \right\} \end{aligned}$$

and observe that $C_\tau < \infty$.
The Wulff shape $\mathcal{W} = \{ q \in \mathbb{R}^n \mid \gamma^0(q) \leq 1 \}$ is convex and it holds

$$\partial \mathcal{W} = \left\{ D\gamma(\tilde{\nu}) \mid \tilde{\nu} \in \mathbb{S}^{n-1} \right\} ,$$

cf. Giga [11]. We set

$$q = p + \tau(p,\nu)\nu.$$

Due to the continuity of γ^0, we obtain $\gamma^0(q) = 1$. Observe,

$$|q-p| = \tau(p,\nu) \quad \text{and} \quad q\cdot\nu = p\cdot\nu + \tau(p,\nu).$$

As $\gamma^0(q) = 1$, there exists some $\overline{\nu} \in \mathbb{S}^{n-1}$ such that

$$D\gamma(\overline{\nu}) = q.$$

Using Lemma 2.2 we obtain

$$\gamma(\nu) - D\gamma(\overline{\nu})\cdot\nu \geq C_3 \left|D\gamma(\nu) - D\gamma(\overline{\nu})\right|^2$$

for $C_3 \leq C_1/C_2^2$. This yields

$$\begin{aligned}\gamma(\nu) - p\cdot\nu &= \gamma(\nu) - q\cdot\nu + \tau(p,\nu)\\ &\geq C_3\left|D\gamma(\nu) - q\right|^2 + \tau(p,\nu).\end{aligned}$$

Furthermore, we have

$$\begin{aligned}\left|D\gamma(\nu) - p\right|^2 &= \left|D\gamma(\nu) - q + q - p\right|^2\\ &\leq 2\left(\left|D\gamma(\nu) - q\right|^2 + \left|q-p\right|^2\right)\\ &\leq 2\left(\left|D\gamma(\nu) - q\right|^2 + C_\tau\,\tau(p,\nu)\right)\\ &\leq C_4\left(C_3\left|D\gamma(\nu) - q\right|^2 + \tau(p,\nu)\right)\,.\end{aligned}$$

This shows that there exists a constant $C > 0$ (independent of $\nu \in \mathcal{S}^{n-1}$ and $p \in \mathcal{W}$) such that

$$\gamma(\nu) - p\cdot\nu \geq C\left|D\gamma(\nu) - p\right|^2.$$

□

The following lemma provides a weak formulation of the anisotropic Gibbs-Thomson law. We will denote by $\nabla_\Gamma f$ the surface gradient and $D_\Gamma\xi$ is the surface Jacobian of a vector valued function ξ, i.e. $D_\Gamma\xi$ is a matrix having the surface gradients of the components as rows, i.e. $(D_\Gamma\xi)_{ij} = (\nabla_\Gamma\xi_i)_j$ which is equivalent to $D_\Gamma\xi = D\xi - D\xi(\nu\otimes\nu)$ with $\nu\otimes\nu = \nu\nu^\top$. Here $^\top$ denotes the transpose.

Lemma 2.4 *For a smooth surface Γ and a smooth function u, the equation $(A3)(ii)$ in Theorem 1.1. is equivalent to*

$$u(t) = H_\gamma(t) \ \text{on}\ \Gamma(t) \quad \text{and} \quad D\gamma(\nu)\cdot\nu_{\partial\Omega} = 0 \ \text{on}\ \partial\Gamma\cap\partial\Omega\,. \tag{10}$$

Proof: Using Lemma 2.1 (i) and $D\xi = D_\Gamma \xi + D\xi(\nu \otimes \nu)$ we obtain

$$(\operatorname{div} \xi)\, D\gamma(\nu) \cdot \nu - \nu \cdot (D\xi\, D\gamma(\nu)) = (\operatorname{div}_\Gamma \xi)\gamma(\nu) - \nu \cdot (D_\Gamma \xi\, D\gamma(\nu)).$$

Also it is not difficult to show that

$$\nu \cdot (D_\Gamma \xi\, D\gamma(\nu)) = \operatorname{div}_\Gamma ((\xi \cdot \nu)\, D\gamma(\nu)) - \left((D_\Gamma \nu)^\top \xi \right) \cdot D\gamma(\nu) - (\xi \cdot \nu) \operatorname{div}_\Gamma D\gamma(\nu)\,.$$

We have by using the Gauss theorem on manifolds for vector fields $f : \Gamma \to \mathbb{R}^n$ which is given as $\int_\Gamma \operatorname{div}_\Gamma f \, d\mathcal{H}^{n-1} + \int_\Gamma f \cdot \nu \, H d\mathcal{H}^{n-1} = \int_{\partial\Gamma} f \cdot \nu_{con} \, d\mathcal{H}^{n-2}$, where ν_{con} is the outer conormal on $\partial\Gamma$:

$$\begin{aligned}
& \int_\Gamma ((\operatorname{div} \xi) D\gamma(\nu) \cdot \nu - \nu \cdot (D\xi\, D\gamma(\nu))) d\mathcal{H}^{n-1} \\
= & \int_\Gamma (\operatorname{div}_\Gamma \xi)\gamma(\nu) - \operatorname{div}_\Gamma ((\xi \cdot \nu)\, D\gamma(\nu))\, d\mathcal{H}^{n-1} \\
& + \int_\Gamma \left((D_\Gamma \nu)^\top \xi \right) \cdot D\gamma(\nu) + (\xi \cdot \nu) \operatorname{div}_\Gamma D\gamma(\nu)\, d\mathcal{H}^{n-1} \\
= & - \int_\Gamma \xi \cdot \nabla_\Gamma (\gamma(\nu)) + \gamma(\nu) H (\xi \cdot \nu) - H\, (\xi \cdot \nu)\, (D\gamma(\nu) \cdot \nu)\, d\mathcal{H}^{n-1} \\
& + \int_\Gamma \left((D_\Gamma \nu)^\top \xi \right) \cdot D\gamma(\nu) + (\xi \cdot \nu) \operatorname{div}_\Gamma D\gamma(\nu)\, d\mathcal{H}^{n-1} \\
& + \int_{\partial\Gamma} [(\xi \cdot \nu_{con})\, (D\gamma(\nu) \cdot \nu) - (\xi \cdot \nu)\, (D\gamma(\nu) \cdot \nu_{con})]\, d\mathcal{H}^{n-2} \\
= & - \int_\Gamma (\xi \cdot \nu)\, H_\gamma \, d\mathcal{H}^{n-1} + \int_{\partial\Gamma} [(\xi \cdot \nu_{con})\, (D\gamma(\nu) \cdot \nu) - (\xi \cdot \nu)\, (D\gamma(\nu) \cdot \nu_{con})]\, d\mathcal{H}^{n-2}\,.
\end{aligned}$$

For a smooth interface and the $\nabla\chi$-integrable function $\overline{\nu}(t) = \frac{\nabla\chi(t)}{|\nabla\chi(t)|}$, we get:

$$\begin{aligned}
& \int_0^T \int_\Omega (\operatorname{div} \xi\, D\gamma(\overline{\nu}) \cdot \overline{\nu} - \overline{\nu} \cdot D\xi\, D\gamma(\overline{\nu}))\, d|\nabla\chi(t)| dt \\
= & \int_0^T \int_{\Gamma(t)} (\operatorname{div} \xi\, D\gamma(\nu) \cdot \nu - \nu \cdot D\xi\, D\gamma(\nu))\, d\mathcal{H}^{n-1} dt \\
= & - \int_0^T \int_{\Gamma(t)} (\xi \cdot \nu)\, H_\gamma \, d\mathcal{H}^{n-1} dt \\
& + \int_0^T \int_{\partial\Gamma(t)} [(\xi \cdot \nu_{con})\, (D\gamma(\nu) \cdot \nu) - (\xi \cdot \nu)\, (D\gamma(\nu) \cdot \nu_{con})]\, d\mathcal{H}^{n-2} dt.
\end{aligned}$$

Futhermore, we have the equation

$$\int_0^T \int_\Omega \operatorname{div}(u\, \xi)\, \chi dx dt = - \int_0^T \int_{\Gamma(t)} u\, (\xi \cdot \nu)\, d\mathcal{H}^{n-1} dt.$$

Altogether we get from (A3)(ii)

$$
\begin{aligned}
0 \;=\; & -\int_0^T \int_{\Gamma(t)} [(\xi \cdot \nu)\, H_\gamma - u\,(\xi \cdot \nu)]\, d\mathcal{H}^{n-1} dt \\
& + \int_0^T \int_{\partial\Gamma(t)} [(\xi \cdot \nu_{con})\,(D\gamma(\nu)\cdot \nu) - (\xi \cdot \nu)\,(D\gamma(\nu)\cdot \nu_{con})]\, d\mathcal{H}^{n-2} dt.
\end{aligned}
$$

Suppose $\xi(t) = \nu(t)\,\varphi$ on $\Gamma(t)$, where $\varphi \in C_0^1(\Omega_T)$ is arbitrary. Since φ can be chosen arbitrarily we obtain

$$
u(t) = H_\gamma(t) \text{ on } \Gamma(t).
$$

Hence,

$$
0 = \int_0^T \int_{\partial\Gamma(t)} [(\xi \cdot \nu_{con})\,(D\gamma(\nu)\cdot \nu) - (\xi \cdot \nu)\,(D\gamma(\nu)\cdot \nu_{con})]\, d\mathcal{H}^{n-2} dt. \tag{11}
$$

Now our aim is to show the force balance condition $D\gamma(\nu)\cdot\nu_{\partial\Omega} = 0$ on $\partial\Omega \cap \partial\Gamma(t)$. Let $\{\nu, \nu_{con}, \tau_1, \ldots, \tau_{n-2}\}$ be an orthonormal basis of $\mathbb{R}^n$. Introducing the rotation Q define via

$$
Q(\tau_i) = \tau_i\,,\; Q(\nu_{con}) = -\nu\,,\; Q(\nu) = \nu_{con}
$$

and the orthogonal projection P onto span $\{\nu, \nu_{con}\}$, we obtain from (11)

$$
0 = \int_0^T \int_{\partial\Gamma(t)} \xi \cdot (P\,Q\,D\gamma(\nu))\, d\mathcal{H}^{n-2} dt = \int_0^T \int_{\partial\Gamma(t)} (Q^\top P\,\xi)\cdot D\gamma(\nu)\, d\mathcal{H}^{n-2} dt\,. \tag{12}
$$

Since $\tau_1, \ldots, \tau_{n-2}$ are tangent to $\partial\Omega$ and to $\partial\Gamma$ we obtain that $\nu_{\partial\Omega}$ lies in span $\{\nu, \nu_{con}\}$. Hence (12), the definition of Q and the fact that ξ with $\xi\cdot\nu_{\partial\Omega} = 0$ can be chosen arbitrarily, imply $D\gamma(\nu)\cdot\nu_{\partial\Omega} = 0$.
Showing that (10) implies (A3)(ii) is now straightforward by using the above calculations. □

3 Time discretization and a priori estimate

In order to prove Theorem 1.1, we approximate the Stefan problem by time discrete problems. Choosing a time step $h = \frac{T}{N}$, $N \in \mathbb{N}$, we use an inductive procedure. Moreover, we define $f^h(t) := \int_{t-h}^t f(s)\,ds$ for $t = h, \ldots, Nh$ and we set $\chi^h(t) = \chi_0$ and $u^h(t) = u_0$ for all $t \le 0$.
Now we construct functions χ^h and u^h for the times $t \in (0, T]$. Suppose that we already know the functions $\chi^h(t-h)$ and $u^h(t-h)$. For what follows it will be useful

to define the two elliptic solution operators $u_{h,t}$ and L^0_h.

$$\begin{aligned} u_{h,t} : L^2(\Omega) \longrightarrow H^{1,2}(\Omega)\,, &\qquad g \mapsto v \text{ by} \\ (v+g) - \big(u^h(t-h) + \chi^h(t-h)\big) - h\Delta v &= h\, f^h(t) \text{ in } \Omega, \\ v &= u^D \text{ on } \partial\Omega, \end{aligned} \tag{13}$$

and

$$\begin{aligned} L^0_h : L^2(\Omega) \longrightarrow H^{1,2}_0(\Omega)\,, &\qquad g \mapsto v^0 \text{ by} \\ v^0 - h\Delta v^0 &= -g \text{ in } \Omega, \\ v^0 &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{14}$$

Standard arguments show that for $g_1, g_2 \in L^2(\Omega)$ we have

$$\|u_{h,t}(g_1) - u_{h,t}(g_2)\|_{L^2(\Omega)} \leq \|g_1 - g_2\|_{L^2(\Omega)}\,. \tag{15}$$

Similar as in Luckhaus [15, 16] we first construct $\chi^h(t)$ as a minimum of a suitable functional $F_{h,t}$ and then determine $u^h(t)$ as a solution of the following time discrete variant of (1), namely of

$$\partial_t^{-h}\left(u^h + \chi^h\right)(t) - \Delta u^h(t) = f^h(t), \tag{16}$$

i.e. $u^h(t) = u_{h,t}(\chi^h(t))$. Here and it what follows we define $\partial_t^{-h} w := (w(t) - w(t-h))/h$. We introduce the functional

$$F_{h,t}(\chi) := \int_\Omega |\nabla\chi|_\gamma + \int_\Omega u_{h,t}(\chi)\left[-\tfrac{1}{2}u_{h,t}(\chi) - \chi + u^h(t-h) + \chi^h(t-h)\right] dx$$

for all $\chi \in BV(\Omega; \{0,1\})$. We remark that the above functional differs from the one in [16] and our choice will simplify the a priori estimates. In order to show existence of a minimizer we have to show that $F_{h,t}$ is lower semicontinuous. Therefore we need the following lemma.

Lemma 3.1 *Let $f,\ f_k \in BV(\Omega)$ for all $k \in \mathbb{N}$ and $f_k \to f$ in $L^1_{loc}(\Omega)$, then it holds:*

$$\int_\Omega |\nabla f|_\gamma = \int_\Omega \gamma(\nu_f)\; d|\nabla f| \leq \liminf_{k\to\infty} \int_\Omega \gamma(\nu_{f_k})\; d|\nabla f_k| = \liminf_{k\to\infty} \int_\Omega |\nabla f_k|_\gamma.$$

Proof: We will use the identity in (7). Let $\varphi \in C^1_0(\Omega, \mathbb{R}^n)$ with $\gamma^0(\varphi) \leq 1$, then we have:

$$-\int_\Omega f \operatorname{div}\varphi = \lim_{k\to\infty} \int_\Omega (-f_k \operatorname{div}\varphi) \leq \liminf_{k\to\infty} \int_\Omega |\nabla f_k|_\gamma.$$

Taking the supremum over all φ now gives the claim. □

The following lemma gives the existence of the time discrete solutions.

Lemma 3.2 *The minimum problem $F_{h,t}(\chi) \to min!$ in the class $BV(\Omega;\{0,1\})$ has at least one solution.*

Proof: We now write $u_i = u(ih)$, $\chi_i = \chi(ih)$ for all $i = 1,\dots,N$. Using (15) we obtain for $\chi \in BV(\Omega;\{0,1\})$:

$$\begin{aligned}
F_{h,t}(\chi) &= \int_\Omega |\nabla\chi|_\gamma + \int_\Omega [-\frac{1}{2}(u_{h,t}(\chi))^2 - u_{h,t}(\chi)\,\chi + u_{h,t}(\chi)\,(u_{i-1}+\chi_{i-1})]dx \\
&= \int_\Omega |\nabla\chi|_\gamma + \int_\Omega [-\frac{1}{2}(u_{h,t}(\chi) - u_{h,t}(0))^2 + \frac{1}{2}(u_{h,t}(0))^2]dx \\
&\quad - \int_\Omega (u_{h,t}(\chi) - u_{h,t}(0))\,(u_{h,t}(0) + \chi - u_{i-1} - \chi_{i-1})\,dx \\
&\quad - \int_\Omega u_{h,t}(0)\,(u_{h,t}(0) + \chi - u_{i-1} - \chi_{i-1})\,dx \\
&\geq \int_\Omega |\nabla\chi|_\gamma - \frac{1}{2}\|\chi\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega (u_{h,t}(0))^2\,dx \\
&\quad - \Big(\|\chi\|_{L^2(\Omega)} + \|u_{h,t}(0)\|_{L^2(\Omega)}\Big)\,\|u_{h,t}(0) + \chi - u_{i-1} - \chi_{i-1}\|_{L^2(\Omega)}\,.
\end{aligned}$$

Since $\|\chi\|_{L^2(\Omega)} \leq C$, $F_0 := \inf F_{h,t}(\chi)$ exists. Let $(\chi_k)_{k\in\mathbb{N}} \subset BV(\Omega;\{0,1\})$ be a minimizing sequence. The strict positivity of γ on $\mathbb{S}^{n-1}$ and (9) imply that there is a constant $C > 0$ such that

$$\|\chi_k\|_{BV} \leq C \quad \text{for all } k \in \mathbb{N}.$$

Hence, there exists a function $\chi \in BV(\Omega,\{0,1\})$ such that $\chi_k \to \chi$ in $L^1(\Omega)$ and almost everywhere, cf [8]. Lemma 3.1 now gives

$$\int_\Omega |\nabla\chi|_\gamma \leq \liminf_{k\to\infty} \int_\Omega |\nabla\chi_k|_\gamma.$$

Since all other terms in $F_{h,t}$ are continuous with respect to L^2 convergence, we obtain the existence of a minimizer. □

We hence constructed time discrete solutions for all $t > 0$. We always choose the time discrete solution to be constant in time on time intervals $((t-1)h, th]$. In order to obtain a solution of the continuous problem, we need a priori estimates of the functions u^h and χ^h.

Theorem 3.1 (energy estimate) *For the time discrete solutions the following a priori estimates are satisfied:*

$$(i) \quad \operatorname*{ess\,sup}_{0\leq t\leq T} \left(\int_\Omega u^h(t)^2 + \int_\Omega \left|\nabla\chi^h(t)\right|_\gamma \right) + \int_0^T \int_\Omega \left|\nabla u^h\right|^2 \leq C\,,$$

$$(ii) \quad \int_0^T \left\|\partial_t^{-h}(u^h + \chi^h)(t)\right\|^2_{H^{-1,2}(\Omega)} \leq C\,.$$

Here $\|\,.\,\|_{H^{-1,2}}$ *is the norm in the dual space* $H^{-1,2}(\Omega) := (H_0^{1,2}(\Omega))^*$ *and* C *does not depend on* h.

Proof: (i): Testing the weak formulation of

$$(u_i + \chi_i) - (u_{i-1} + \chi_{i-1}) - h\Delta u_i = h\, f_i$$

with $\left(u_i - u^D\right) \in H_0^{1,2}(\Omega)$ gives

$$\begin{aligned} &\int_\Omega \left[\left(u_i^2 + u_i\,\chi_i\right) - \left(\frac{1}{2}\,u_i^2 + u_i\,\chi_i\right) + \left(\frac{1}{2}\,u_i^2 + u_i\,\chi_i\right) - u_i\,(u_{i-1} + \chi_{i-1}) \right] \\ &+ \int_\Omega \left[-u^D\,(u_i + \chi_i) + u^D\,(u_{i-1} + \chi_{i-1}) + h\nabla u_i \cdot \nabla\left(u_i - u^D\right)\right] \\ = &\int_\Omega h\, f_i\left(u_i - u^D\right). \end{aligned} \tag{17}$$

Since χ_i is a minimizer of $F_{h,t}$ by definition, we have:

$$F_{h,t}(\chi_i) \leq F_{h,t}(\chi_{i-1}).$$

Using the definition of $F_{h,t}$, we derive:

$$\begin{aligned} &\int_\Omega |\nabla\chi_i|_\gamma + \int_\Omega \left[-\frac{1}{2}\,u_i^2 - u_i\,\chi_i + u_i\,(u_{i-1} + \chi_{i-1})\right] \\ \leq\; &\int_\Omega |\nabla\chi_{i-1}|_\gamma + \int_\Omega \left[-\frac{1}{2}\,(u_{h,t}(\chi_{i-1}))^2 - u_{h,t}(\chi_{i-1})\,\chi_{i-1} + u_{h,t}(\chi_{i-1})\,(u_{i-1} + \chi_{i-1})\right] \\ \leq\; &\int_\Omega |\nabla\chi_{i-1}|_\gamma + \int_\Omega \frac{1}{2}u_{i-1}^2. \end{aligned} \tag{18}$$

Taking (17) into account, we get

$$\begin{aligned} &\int_\Omega |\nabla\chi_i|_\gamma - \int_\Omega |\nabla\chi_{i-1}|_\gamma + \int_\Omega \left[\frac{1}{2}\,u_i^2 - \frac{1}{2}\,u_{i-1}^2 - u^D\,(u_i + \chi_i) + u^D\,(u_{i-1} + \chi_{i-1})\right] \\ &\quad + \int_\Omega h\nabla u_i \cdot \nabla\left(u_i - u^D\right) \leq \int_\Omega h\, f_i\left(u_i - u^D\right). \end{aligned}$$

Hölder's inequality now gives

$$\begin{aligned} &\int_\Omega |\nabla\chi_i|_\gamma - \int_\Omega |\nabla\chi_{i-1}|_\gamma + \int_\Omega \left[\frac{u_i^2}{2} - \frac{u_{i-1}^2}{2} - u^D\,(u_i + \chi_i) + u^D\,(u_{i-1} + \chi_{i-1}) + h\,|\nabla u_i|^2\right] \\ &\quad \leq\; C\,h\left(\|u_i\|_{L^2(\Omega)} + \|\nabla u_i\|_{L^2(\Omega)} + 1\right). \end{aligned}$$

For $t_0 = k_0 h$ we sum up from $i = 1$ to $i = k_0$ and get (using that the solutions are piecewise constant)

$$\begin{aligned} &\int_\Omega |\nabla\chi_{k_0}|_\gamma - \int_\Omega |\nabla\chi_0|_\gamma + \int_\Omega \left[\frac{u_{k_0}^2}{2} - \frac{u_0^2}{2} - u^D\,(u_{k_0} + \chi_{k_0}) + u^D\,(u_0 + \chi_0)\right] \\ &+ \int_0^{t_0}\int_\Omega |\nabla u(t)|^2 \leq C\int_0^{t_0}\left[\|u(t)\|_{L^2(\Omega)} + \|\nabla u(t)\|_{L^2(\Omega)} + 1\right]. \end{aligned}$$

Using $u^D \in H^{1,2}(\Omega)$ we now obtain the existence of a constant $C > 0$ such that

$$\begin{aligned} &\operatorname*{ess\,sup}_{t\in[0,T]} \left(\int_\Omega u(t)^2 + \int_\Omega |\nabla\chi(t)|_\gamma \right) + \int_0^T \int_\Omega |\nabla u(t)|^2 \\ &\leq \; C \int_0^T \Big[\|u(t)\|_{L^2(\Omega)} + \|\nabla u(t)\|_{L^2(\Omega)} + 1 \Big]. \end{aligned}$$

Now statement (i) easily follows.
(ii): According to equation (16) $\partial_t^{-h} (u+\chi)(t) \in H^{-1,2}(\Omega)$ is given by

$$\left\langle \partial_t^{-h} (u+\chi)(t), \zeta \right\rangle = -\int_\Omega \nabla u(t) \cdot \nabla \zeta + \int_\Omega f(t)\, \zeta$$

for all $\zeta \in H_0^{1,2}(\Omega)$. Consequently,

$$\left| \left\langle \partial_t^{-h} (u+\chi)(t), \zeta \right\rangle \right| \leq \left(\|\nabla u(t)\|_{L^2} + \|f(t)\|_{L^2} \right) \|\zeta\|_{H_0^{1,2}(\Omega)}$$

and hence

$$\|\partial_t^{-h} (u+\chi)\|^2_{H^{-1,2}(\Omega} \leq C \left(\|\nabla u(t)\|^2_{L^2} + \|f(t)\|^2_{L^2} \right) .$$

So assertion (ii) follows after integration using the a priori estimate (i). □

Using a compactness result of Luckhaus [16] we get with the help of the a priori estimate in Theorem 3.1

$$\int_0^T \int_\Omega \left| u^h(t) - u^h(t-\tau) \right| + \left| \chi^h(t) - \chi^h(t-\tau) \right| \leq C \begin{cases} \tau^{\frac{1}{3}} & , n = 2,3 \\ \tau^{\frac{n}{4n-4}} & , n \geq 4. \end{cases} \tag{19}$$

Hence we can establish the existence of functions

$$\begin{aligned} & u \in \left(u^D + L^2\left(0,T; H_0^{1,2}(\Omega)\right)\right) \cap L^\infty\left(0,T; L^2(\Omega)\right) \\ \text{and} \quad & \chi \in L^1\left(\Omega_T, \{0,1\}\right) \text{ with } \operatorname*{ess\,sup}_{t\in[0,T]} \int_\Omega |\nabla\chi| < \infty \end{aligned}$$

such that for $h \to 0$

$$\begin{aligned} (i) \quad & u^h \rightharpoonup u \text{ in } L^2\left(0,T; H^{1,2}(\Omega)\right), \\ (ii) \quad & u^h \longrightarrow u \text{ in } L^1\left(0,T; L^1(\Omega)\right), \\ (iii) \quad & \chi^h \longrightarrow \chi \text{ in } L^2\left(0,T; L^2(\Omega)\right), \\ (iv) \quad & u^h(t) \longrightarrow u(t) \text{ in } L^2(\Omega) \text{ a.e. in } t \in [0,T], \\ (v) \quad & \chi^h(t) \longrightarrow \chi(t) \text{ in } L^1(\Omega) \text{ a.e. in } t \in [0,T]. \end{aligned}$$

Using the above convergence properties we can pass to the limit in a time integrated version of (16) and obtain $(A3)(i)$, compare also Luckhaus [16]. It remains to show $(A3)(ii)$. In a final step we determine the first variation of the anisotropic interfacial free energy. This will be the main new contribution of this work.

Theorem 3.2 (anisotropic Gibbs-Thomson law for time discrete solutions) *The time discrete solution* (χ^h, u^h) *satisfies the equation:*

$$\begin{aligned} 0 \;=\; & \int_\Omega \left(div\xi \, D\gamma\left(\overline{\nu}^h(t)\right) \cdot \overline{\nu}^h(t) - \overline{\nu}^h(t) \cdot D\xi \, D\gamma\left(\overline{\nu}^h(t)\right)\right) d\left|\nabla \chi^h(t)\right| \\ & - \int_\Omega div\left(u^h(t)\,\xi\right)\chi^h(t) \\ & + \int_\Omega div\left(L_h^0\left(u^h(t) - u^h(t-h) + \chi^h(t) - \chi^h(t-h)\right)\xi\right)\chi^h(t) \end{aligned}$$

for all $\xi \in C^1\left(\overline{\Omega}, \mathbb{R}^n\right)$*, with* $\xi \cdot \nu_{\partial\Omega} = 0$*, where* $\overline{\nu}^h(t) = \frac{\nabla\chi^h(t)}{\left|\nabla\chi^h(t)\right|}$.

Proof: We choose a family of diffeomorphisms $\Phi(\tau, .)$, $\tau \in [-\tau_0, \tau_0]$ of Ω, defined by

$$\Phi_{,\tau}(\tau, x) = \xi(\Phi(\tau, x)) \quad \text{and} \quad \Phi(0, x) = x$$

for $x \in \Omega$ and $\tau \in [-\tau_0, \tau_0]$. Let $\Psi(\tau, .)$ be the inverse function of $\Phi(\tau, .)$. Then the following properties are satisfied for all $y \in \Omega$ and $\tau \in (-\tau_0, \tau_0)$

$$\begin{aligned} (\alpha) \quad & \frac{d}{d\tau}\Psi(\tau, y)\bigg|_{\tau=0} = -\xi(y), \\ (\beta) \quad & \frac{d}{d\tau}\left|\det D\Psi(\tau, y)\right|\bigg|_{\tau=0} = -\mathrm{div}\,\xi(y), \end{aligned}$$

where D is the derivative of Ψ with respect to y.

In order to compute the first variation it is convenient to reformulate the functional $F_{h,t}$. From the definition of the operator $u_{h,t}$, we have the following equations (which hold in a weak form)

$$\begin{aligned} h\, f_i \;=\; & \left(u_{h,t}\left(\chi^h(t, \Phi(\tau, x))\right) + \chi^h(t, \Phi(\tau, x))\right) - (u_{i-1} + \chi_{i-1}) \\ & - h\Delta u_{h,t}\left(\chi^h(t, \Phi(\tau, x))\right), \\ h\, f_i \;=\; & (u_i + \chi_i) - (u_{i-1} + \chi_{i-1}) - h\Delta u_i. \end{aligned}$$

We subtract the second equation from the first one and obtain

$$\begin{aligned} & \left(u_{h,t}\left(\chi^h(t, \Phi(\tau, x))\right) - u_i\right) - h\Delta\left(u_{h,t}\left(\chi^h(t, \Phi(\tau, x))\right) - u_i\right) \\ = \; & -\left(\chi^h(t, \Phi(\tau, x)) - \chi_i\right). \end{aligned} \tag{20}$$

Testing this equation with $u_{h,t}\left(\chi^h(t, \Phi(\tau, x))\right) - u_i \in H_0^{1,2}(\Omega)$ gives

$$\begin{aligned} 0 \;=\; & \int_\Omega \left(u_{h,t}\left(\chi^h(t, \Phi(\tau, x))\right) - u_i\right)^2 + h\left|\nabla\left(u_{h,t}\left(\chi^h(t, \Phi(\tau, x))\right) - u_i\right)\right|^2 \\ & + \int_\Omega \left(\chi^h(t, \Phi(\tau, x)) - \chi_i\right)\left(u_{h,t}\left(\chi^h(t, \Phi(\tau, x))\right) - u_i\right) \end{aligned}$$

and hence

$$\int_\Omega -\frac{1}{2}\, u_{h,t}\left(\chi^h(t,\Phi(\tau,x))\right)^2 = \int_\Omega \frac{1}{2}\, h\left|\nabla\left(u_{h,t}\left(\chi^h(t,\Phi(\tau,x))\right)-u_i\right)\right|^2$$
$$+\int_\Omega \frac{1}{2}\left(\chi^h(t,\Phi(\tau,x))-\chi_i\right)\left(u_{h,t}\left(\chi^h(t,\Phi(\tau,x))\right)-u_i\right)$$
$$+\int_\Omega (-u_{h,t}\left(\chi^h(t,\Phi(\tau,x))\right)\, u_i + \frac{1}{2}u_i^2). \tag{21}$$

Furthermore, we have the following equation:

$$-u_{h,t}\left(\chi^h(t,\Phi(\tau,x))\right)\chi^h(t,\Phi(\tau,x)) = -u_{h,t}\left(\chi^h(t,\Phi(\tau,x))\right)\chi_i + u_i\,\chi_i$$
$$-u_i\,\chi^h(t,\Phi(\tau,t)) - \left(u_{h,t}\left(\chi^h(t,\Phi(\tau,x))\right)-u_i\right)\left(\chi^h(t,\Phi(\tau,x))-\chi_i\right). \tag{22}$$

Using the equations (21) and (22) we can rewrite $F_{h,t}(\chi^h(t,\Phi(\tau,x))$ as follows

$$F_{h,t}\left(\chi^h(t,\Phi(\tau,x))\right) = \underbrace{\int_\Omega \left|\nabla\chi^h(t,\Phi(\tau,x))\right|_\gamma}_{=:(I)} + \underbrace{\int_\Omega \frac{1}{2}\, h\left|\nabla\left(u_{h,t}\left(\chi^h(t,\Phi(\tau,x))\right)-u_i\right)\right|^2}_{=:(II)}$$
$$-\underbrace{\int_\Omega \frac{1}{2}\left(\chi^h(t,\Phi(\tau,x))-\chi_i\right)\left(u_{h,t}\left(\chi^h(t,\Phi(\tau,x))\right)-u_i\right)}_{=:(III)}$$
$$+\underbrace{\int_\Omega \frac{1}{2}\, u_i^2 + u_i\,\chi_i - u_i\,\chi^h(t,\Phi(\tau,x))}_{=:(IV)} - \underbrace{\int_\Omega u_{h,t}\left(\chi^h(t,\Phi(\tau,x))\right)(u_i-u_{i-1}+\chi_i-\chi_{i-1})}_{=:(V)}.$$

Since $\chi^h(t)$ is a minimum of $F_{h,t}$ we have $0 = \frac{d}{d\tau}F_{h,t}(\chi^h(t,\Phi(\tau,x)))|_{\tau=0}$. In order to compute this derivative the above reformulation of $F_{h,t}$ is more convenient as for example (II) and (III) vanish quadratically for $\tau\to 0$. We now compute the derivative $\frac{d}{d\tau}F_{h,t}(\chi^h(t,\Phi(\tau,x)))$ using the above reformulation of $F_{h,t}(\chi^h(t,\Phi(\tau,x))$. We start with term (I) which leads to the main new technical difficulty arising from the anisotropy γ. First we observe that for arbitrary $g\in C_0^1(\Omega,\mathbb{R}^n)$ and $f\in BV(\Omega,\{0,1\})$ it holds:

$$\int_\Omega \operatorname{div}\left(g\left(\Phi(\tau,x)\right)\right) f\left(\Phi(\tau,x)\right)dx = \int_\Omega g_i(y)\, H_{ij}(\tau,y)\,\nu_{f,i}\, d|\nabla f|,$$

where $\nu_f = \frac{\nabla f}{|\nabla f|}\in L^1\left(|\nabla f|\right)$ and $H_{ij}(\tau,y) = \partial_i\Phi_j(\tau,\Psi(\tau,y))\left|\det D\Psi(\tau,y)\right|$ (see Giusti [12]).

This implies:

$$\begin{aligned}
&\int_\Omega \left|\nabla\chi^h(t,\Phi(\tau,x))\right|_\gamma \\
= \ &\sup\left\{-\int_\Omega \operatorname{div}\widetilde{g}(x)\,\chi^h(t,\Phi(\tau,x))dx \mid \widetilde{g}\in C_0^1(\Omega,\mathbb{R}^n),\ \gamma^0(\widetilde{g}(x))\le 1,\ x\in\Omega\right\} \\
= \ &\sup\left\{-\int_\Omega \operatorname{div} g\left(\Phi(\tau,x)\right)\ \chi^h(t,\Phi(\tau,x))dx \mid g\in C_0^1(\Omega,\mathbb{R}^n),\ \gamma^0(g(x))\le 1,\ x\in\Omega\right\} \\
= \ &\sup\left\{\int_\Omega g_i(y)\,H_{ij}(\tau,y)\,\overline{\nu}_j^h(t)\,d\left|\nabla\chi^h(t)\right| \mid g\in C_0^1(\Omega,\mathbb{R}^n),\ \gamma^0(g(x))\le 1,\ x\in\Omega\right\} \\
\overset{(*)}{=} \ &\int_\Omega \gamma\left(H(\tau,y)\,\overline{\nu}^h(t)\right)d\left|\nabla\chi^h(t)\right|.
\end{aligned}$$

where the last equality still has to be verified.
$(*)$: "$\le$"
Since

$$g(y)\cdot H(\tau,y)\,\overline{\nu}^h(t)\le \gamma^0(g(y))\,\gamma\left(H(\tau,y)\,\overline{\nu}^h(t)\right),$$

we derive:

$$\begin{aligned}
&\sup\left\{\int_\Omega g(y)\cdot H(\tau,y)\,\overline{\nu}^h(t)\,d\left|\nabla\chi^h(t)\right| \mid g\in C_0^1(\Omega,\mathbb{R}^n),\ \gamma^0(g(x))\le 1,\ x\in\Omega\right\} \\
&\qquad\le \int_\Omega \gamma\left(H(\tau,y)\,\overline{\nu}^h(t)\right)d\left|\nabla\chi^h(t)\right|.
\end{aligned}$$

"$\ge$"
There exist functions $\varphi_k\in C_0^1(\Omega,\mathbb{R}^n)$ such that

$$\begin{aligned}
&\varphi_k\longrightarrow\overline{\nu}^h(t) \quad \text{in } L^1\left(\left|\nabla\chi^h\right|\right) \\
\text{and}\quad &\varphi_k\longrightarrow\overline{\nu}^h(t) \quad \left|\nabla\chi^h\right|-\text{a.e.}.
\end{aligned}$$

Since $H(0,.)=1$ and $\left|\overline{\nu}^h(t)\right|=1$ $\left|\nabla\chi^h\right|$-a.e. we can choose τ small enough such that

$$\frac{1}{2}\le\left|H(\tau,y)\overline{\nu}^h(t)\right|\le 2 \quad \left|\nabla\chi^h\right|-\text{a.e.}.$$

Next we take a function $\eta:\mathbb{R}\to\mathbb{R}$ with $\eta\in C_0^\infty(\mathbb{R})$ and $|\eta|\le 1$ such that

$$\begin{aligned}
&\eta=1 \quad \text{in } [1/4;4] \\
\text{and}\quad &\eta=0 \quad \text{in } \mathbb{R}\setminus[1/8;8],
\end{aligned}$$

and we define $F:\mathbb{R}^n\to\mathbb{R}^n$ by

$$\begin{aligned}
&F(p):=\eta(|p|)D\gamma(p) \quad \text{for } p\in\mathbb{R}^n\setminus\{0\} \\
\text{and}\quad &F(0):=0.
\end{aligned}$$

We see that $F \in C_0^1(\mathbb{R}^n;\mathbb{R}^n)$. Furthermore, we approximate $H(\tau,.)$ uniformly by $H_k(.) \in C^1(\mathbb{R}^n;\mathbb{R}^{n\times n})$. Then it holds

$$\begin{aligned}
&(i) \quad F\circ(H_k\varphi_k) \in C_0^1(\Omega;\mathbb{R}^n),\\
&(ii) \quad \gamma^0(F\circ(H_k\varphi_k)) = \underbrace{\eta(|H_k\varphi_k|)}_{\le 1}\underbrace{\gamma^0(D\gamma(H_k\varphi_k))}_{=1} \le 1,\\
&\qquad\ \ \gamma^0(F\circ(H_k\varphi_k)) = 0 \quad \text{for } H_k\varphi_k = 0,\\
&(iii) \quad F\circ(H_k(y)\varphi_k(y)) \overset{k\to\infty}{\longrightarrow} F\circ(H(\tau,y)\overline{\nu}^h(y)) \quad \left|\nabla\chi^h\right| - \text{a.e.},
\end{aligned}$$

where we have used Lemma 2.2 in (ii). As F is bounded, it follows

$$\int_\Omega F\circ(H_k(y)\varphi_k(y))\cdot H(\tau,y)\overline{\nu}^h\,d\left|\nabla\chi^h(t)\right| \to \int_\Omega F\circ(H(\tau,y)\overline{\nu}^h)\cdot H(\tau,y)\overline{\nu}^h\,d\left|\nabla\chi^h(t)\right|.$$

Because of $\frac{1}{2} \le \left|H(\tau,y)\overline{\nu}^h(y)\right| \le 2$, we get $F\circ(H(\tau,y)\overline{\nu}^h(y)) = D\gamma(H(\tau,y)\overline{\nu}^h(y))$ $\left|\nabla\chi^h(\tau,y)\right|$-a.e.. Hence, by Lemma 2.1 we have

$$\begin{aligned}
\sup &\left\{\int_\Omega g(y)\cdot H(\tau,y)\,\overline{\nu}^h(t)\,d\big|\nabla\chi^h(t)\big| \mid g\in C_0^1(\Omega,\mathbb{R}^n),\ \gamma^0(g(x))\le 1,\ x\in\Omega\right\}\\
&\ge \int_\Omega D\gamma\left(H(\tau,y)\,\overline{\nu}^h(t)\right)\cdot\left(H(\tau,y)\,\overline{\nu}^h(t)\right)d\big|\nabla\chi^h(t)\big|\\
&= \int_\Omega \gamma\left(H(\tau,y)\,\overline{\nu}^h(t)\right)d\big|\nabla\chi^h(t)\big|.
\end{aligned}$$

Hence, $(*)$ is established.
We are now in a position to compute the derivative of (I).

$$\begin{aligned}
\frac{d}{d\tau}\int_\Omega \left|\nabla\chi^h(t,\Phi(\tau,x))\right|_\gamma\Big|_{\tau=0} &= \frac{d}{d\tau}\int_\Omega \gamma\left(H(\tau,y)\,\overline{\nu}^h(t)\right)d\big|\nabla\chi^h(t)\big|\Big|_{\tau=0}\\
&= \int_\Omega \frac{d}{d\tau}\gamma\left(H(\tau,y)\,\overline{\nu}^h(t)\right)\Big|_{\tau=0} d\big|\nabla\chi^h(t)\big|\\
&= \int_\Omega D\gamma\left(H(0,y)\,\overline{\nu}^h(t)\right)\frac{d}{d\tau}H(\tau,y)\Big|_{\tau=0}\overline{\nu}^h(t)\,d\big|\nabla\chi^h(t)\big|\\
&= \int_\Omega D\gamma\left(\overline{\nu}^h(t)\right)\frac{d}{d\tau}H(\tau,y)\Big|_{\tau=0}\overline{\nu}^h(t)\,d\big|\nabla\chi^h(t)\big|,
\end{aligned}$$

where we have used $H(0,y) = 1$. For the purpose of simplifying the last expression, we use that for all $i,j = 1,\ldots,n$

$$\frac{d}{d\tau}H_{ij}(\tau,y)\Big|_{\tau=0} = \operatorname{div}\xi(y)\,\delta_{ij} + \partial_i\xi_j(y)\,,$$

see Giusti [12]. This implies:

$$\begin{aligned}
\frac{d}{d\tau} &\int_\Omega \left|\nabla\chi^h(t,\Phi(\tau,x))\right|_\gamma|_{\tau=0} \\
&= \int_\Omega \left(D\gamma\left(\overline{\nu}^h(t)\right)\cdot\overline{\nu}^h(t)\ \left(-\operatorname{div}\xi\right) + D\gamma\left(\overline{\nu}^h(t)\right)\cdot D\xi^\top\overline{\nu}^h(t)\right) d\left|\nabla\chi^h(t)\right| \\
&= -\int_\Omega \left(D\gamma\left(\overline{\nu}^h(t)\right)\cdot\overline{\nu}^h(t)\operatorname{div}\xi - \overline{\nu}^h(t)\cdot D\xi\, D\gamma\left(\overline{\nu}^h(t)\right)\right) d\left|\nabla\chi^h(t)\right|.
\end{aligned}$$

We now consider the terms (II) and (III). We define $F_i \in H^{-1,2}(\Omega)$ by

$$\langle F_i,\mu\rangle := \int_\Omega \chi_i \operatorname{div}(\mu\,\xi) \quad \text{for all } \mu\in H_0^{1,2}(\Omega).$$

Let $v_i \in H_0^{1,2}(\Omega)$ be the unique weak solution of the problem

$$\begin{aligned}
v_i - h\Delta v_i &= F_i \quad \text{in } \Omega,\\
v_i &= 0 \quad \text{on } \partial\Omega,
\end{aligned}$$

where the existence and uniqueness follows from the Lax-Milgram theorem. The definition of v_i and the equation (20) yield for all $\mu\in H_0^{1,2}(\Omega)$:

$$\begin{aligned}
\lim_{\tau\searrow 0} &\left|\int_\Omega \left(\tfrac{u_{h,t}\left(\chi^h(t,\Phi(\tau,.))\right)-u_i}{\tau} - v_i\right)\mu + h\nabla\left(\tfrac{u_{h,t}\left(\chi^h(t,\Phi(\tau,.))\right)-u_i}{\tau} - v_i\right)\cdot\nabla\mu\right| \\
&= \lim_{\tau\searrow 0}\left|\int_\Omega -\tfrac{\chi^h(t,\Phi(\tau,.))-u_i}{\tau}\,\mu - \chi_i\operatorname{div}(\mu\,\xi)\right| \\
&= \left|\frac{d}{d\tau}\int_\Omega \chi^h(t,\Phi(\tau,x))\,\mu\Big|_{\tau=0} + \int_\Omega \chi_i\operatorname{div}(\mu\,\xi)\right| \\
&= \left|\frac{d}{d\tau}\int_\Omega \chi^h(t,y)\,\mu(\Psi(\tau,y))\left|\det D\Psi(\tau,y)\right|\Big|_{\tau=0} + \int_\Omega \chi_i\operatorname{div}(\mu\,\xi)\right| \\
&= \left|\int_\Omega \chi^h(t,y)\,\nabla\mu(\Psi(0,y))\cdot\frac{d}{d\tau}\Psi(\tau,y)\Big|_{\tau=0}\left|\det D\Psi(\tau,y)\right|\right. \\
&\quad\left. + \int_\Omega \chi^h(t,y)\,\mu(\Psi(0,y))\,\frac{d}{d\tau}\left|\det D\Psi(\tau,y)\right|\Big|_{\tau=0} + \int_\Omega \chi_i\operatorname{div}(\mu\,\xi)\right| \\
&= \left|\int_\Omega \chi^h(t,y)\left(\nabla\mu(y)\cdot(-\xi(y)) + \mu(y)\left(-\operatorname{div}\,\xi(y)\right)\right) + \int_\Omega \chi_i\operatorname{div}(\mu\,\xi)\right| \\
&= \left|-\int_\Omega \chi_i\operatorname{div}(\mu\,\xi) + \int_\Omega \chi_i\operatorname{div}(\mu\,\xi)\right| = 0,
\end{aligned}$$

where we have used the transformation $x=\Psi(\tau,y)$ and the properties (α) and (β) of Ψ. This means for fixed h:

$$\frac{u_{h,t}\left(\chi^h(t,\Phi(\tau,x))\right)-u_i}{\tau} \rightharpoonup v_i \text{ in } H_0^{1,2}(\Omega)$$

and especially:

$$\left\| \frac{u_{h,t}\left(\chi^h(t,\Phi(\tau,x))\right) - u_i}{\tau} \right\|_{H^{1,2}(\Omega)} \leq C.$$

Because of that, we can calculate $\frac{d}{d\tau}(II)\big|_{\tau=0}$ and $\frac{d}{d\tau}(III)\big|_{\tau=0}$. Notice that $w_{h,t}\left(\chi^h(t,\Phi(0,x))\right) = w_i$ and $\chi^h(t,\Phi(0,x)) = \chi_i$.

$$\frac{d}{d\tau}(II)\bigg|_{\tau=0} = \lim_{\tau\searrow 0} \tau \int_\Omega \frac{1}{2}\, h \left| \frac{\nabla\left(u_{h,t}\left(\chi^h(t,\Phi(\tau,x))\right) - u_i\right)}{\tau}\right|^2 \leq \lim_{\tau\searrow 0} \tau\, C = 0.$$

We use a similar argument for (III).

$$\begin{aligned}
\frac{d}{d\tau}(III)\bigg|_{\tau=0} &= \lim_{\tau\searrow 0} \frac{1}{\tau} \int_\Omega \frac{1}{2} \left(\chi^h(t,\Phi(\tau,x)) - \chi_i\right)\left(u_{h,t}\left(\chi^h(t,\Phi(\tau,x))\right) - u_i\right) \\
&= \lim_{\tau\searrow 0} \frac{\tau}{2} \int_\Omega \left| \tfrac{u_{h,t}\left(\chi^h(t,\Phi(\tau,x))\right) - u_i}{\tau}\right|^2 + h \left| \tfrac{\nabla\left(u_{h,t}\left(\chi^h(t,\Phi(\tau,x))\right) - u_i\right)}{\tau}\right|^2 \\
&\leq \lim_{\tau\searrow 0} \tau\, C = 0,
\end{aligned}$$

where we have used equation (20).
Since in the term (IV) only the last summand depends on τ, we conclude with the transformation $x = \Psi(\tau,y)$ and the properties (α) and (β) of Ψ:

$$\begin{aligned}
\frac{d}{d\tau}(IV)\big|_{\tau=0} &= \frac{d}{d\tau} \int_\Omega -u_i\, \chi^h(t,\Phi(\tau,x))\bigg|_{\tau=0} \\
&= \frac{d}{d\tau} \int_\Omega -u^h(t,\Psi(\tau,y))\, \chi^h(t,y) |\det D\Psi(\tau,y)|\bigg|_{\tau=0} \\
&= -\int_\Omega \nabla u^h(t,\Psi(0,y)) \cdot \frac{d}{d\tau}\Psi(\tau,y)\bigg|_{\tau=0} \chi^h(t,y) |\det D\Psi(\tau,y)| \\
&\quad - \int_\Omega u^h(t,\Psi(0,y))\, \chi^h(t,y)\, \frac{d}{d\tau}|\det D\Psi(\tau,y)|\bigg|_{\tau=0} \\
&= -\int_\Omega \left(\nabla u^h(t,y)\cdot(-\xi(y)) + u^h(t,y)\,(-\operatorname{div}\xi(y))\right) \chi^h(t,y) \\
&= \int_\Omega \operatorname{div}\left(u^h(t,y)\,\xi(y)\right) \chi^h(t,y).
\end{aligned}$$

In order to calculate the derivative of the term (V), we use a transformation. Let $g \in L^2(\Omega)$ be arbitrary. We test the equation (20) with $L_h^0(g)$ and the equation for $L_h^0(g)$ with $\left(w_{h,t}\left(\chi^h(t,\Phi(\tau,x))\right) - w_i\right)$. Subtracting the resulting equations we obtain

$$\int_\Omega \left(\chi^h(t,\Phi(\tau,x)) - \chi_i\right) L_h^0(g) = \int_\Omega \left(u_{h,t}\left(\chi^h(t,\Phi(\tau,x))\right) - u_i\right) g. \tag{23}$$

We can now compute

$$
\begin{aligned}
\frac{d}{d\tau}(V)|_{\tau=0} = \lim_{\tau \searrow 0} \int_\Omega \frac{u_{h,t}\left(\chi^h(t,\Phi(\tau,x))\right) - u_i}{\tau} \; (u_i - u_{i-1} + \chi_i - \chi_{i-1}) \\
= \lim_{\tau \searrow 0} \int_\Omega \frac{\chi^h(t,\Phi(\tau,x)) - \chi_i}{\tau} \, L_h^0(u_i - u_{i-1} + \chi_i - \chi_{i-1}) \\
= \frac{d}{d\tau} \int_\Omega \chi^h(t,\Phi(\tau,x)) \, L_h^0(u_i - u_{i-1} + \chi_i - \chi_{i-1})\Big|_{\tau=0} \\
= \frac{d}{d\tau} \int_\Omega \chi^h(t,y) \, L_h^0(u_i - u_{i-1} + \chi_i - \chi_{i-1})(\Psi(\tau,y)) |\det D\Psi(\tau,y)|\Big|_{\tau=0} \\
= \int_\Omega \chi^h(t,y) \, \nabla L_h^0(u_i - u_{i-1} + \chi_i - \chi_{i-1})(\Psi(0,y)) \cdot \frac{d}{d\tau}\Psi(\tau,y)\Big|_{\tau=0} |\det D\Psi(\tau,y)| \\
+ \int_\Omega \chi^h(t,y) \, L_h^0(u_i - u_{i-1} + \chi_i - \chi_{i-1})(\Psi(0,y)) \frac{d}{d\tau} |\det D\Psi(0,y)|\Big|_{\tau=0} \\
= \int_\Omega \chi^h(t,y) \, \nabla L_h^0(u_i - u_{i-1} + \chi_i - \chi_{i-1})(y) \cdot (-\xi(y)) \\
+ \int_\Omega \chi^h(t,y) \, L_h^0(u_i - u_{i-1} + \chi_i - \chi_{i-1})(y) \, (-\mathrm{div}\, \xi(y)) \\
= - \int_\Omega \mathrm{div} \left(L_h^0(u_i - u_{i-1} + \chi_i - \chi_{i-1})(y) \, \xi(y) \right) \chi^h(t,y),
\end{aligned}
$$

where we have used the transformation $x = \Psi(\tau, y)$ and the properties (α) and (β). Altogether the assertion of the theorem follows. □

4 Convergence of the time discrete solutions

Finally, we want pass to the limit in the anisotropic Gibbs-Thomson law for time discrete solutions. For that we need the following lemma.

Lemma 4.1 *It holds for almost every $t \in [0,T]$:*

$$\int_\Omega \left|\nabla \chi^h(t)\right|_\gamma \longrightarrow \int_\Omega |\nabla \chi(t)|_\gamma \qquad \text{for } h \to 0.$$

Proof: Since

$$\chi^h(t) \longrightarrow \chi(t) \text{ in } L^1(\Omega) \qquad \text{for almost every } t,$$

we obtain by Lemma 3.1

$$\int_0^T \int_\Omega |\nabla \chi(t)|_\gamma dt \le \liminf_{h \to 0} \int_0^T \int_\Omega \left|\nabla \chi^h(t)\right|_\gamma dt.$$

Further, we have by definition of $\xi^h(t)$:

$$F_{h,t}(\chi^h(t)) \leq F_{h,t}(\chi(t)) \quad \forall \chi \in BV\left(\Omega; \{0,1\}\right).$$

So we can conclude:

$$\begin{aligned}
&\int_\Omega \left|\nabla \chi^h(t)\right|_\gamma - \int_\Omega \frac{1}{2}\left(u_{h,t}(\chi^h(t))\right)^2 - \int_\Omega u_{h,t}(\chi^h(t))\,\chi^h(t) \\
&\qquad + \int_\Omega u_{h,t}(\chi^h(t))\left(u^h(t-h) + \chi^h(t-h)\right) \\
&\leq \int_\Omega |\nabla \chi(t)|_\gamma - \int_\Omega \frac{1}{2}\left(u_{h,t}(\chi(t))\right)^2 - \int_\Omega u_{h,t}(\chi(t))\,\chi(t) \\
&\qquad + \int_\Omega u_{h,t}(\chi(t))\left(u^h(t-h) + \chi^h(t-h)\right).
\end{aligned}$$

Passing to the limit in the integrated version of this inequality gives

$$\limsup_{h\to 0} \int_0^T \int_\Omega \left|\nabla \chi^h(t)\right|_\gamma dt \leq \int_0^T \int_\Omega |\nabla \chi(t)|_\gamma dt,$$

where we have used the convergence

$$\left\|u_{h,t}(\chi(t)) - u_{h,t}(\chi^h(t))\right\|_{L^2(\Omega)} \leq \left\|\chi(t) - \chi^h(t)\right\|_{L^2(\Omega)} \longrightarrow 0,$$

which holds by inequality (15) and since $u^h(t) \to u(t)$ for almost all $t \in (0,T)$. The lemma thus follows. □

Now we can show that the time discrete solutions converge to a solution of the continuous problem.

Theorem 4.1 *The following convergences are satisfied:*

$$\begin{aligned}
&(i) \quad \int_0^T \int_\Omega div\,\xi D\gamma\left(\overline{\nu}^h\right) \cdot \overline{\nu}^h d\left|\nabla \chi^h\right| dt \longrightarrow \int_0^T \int_\Omega div\,\xi D\gamma\left(\overline{\nu}\right) \cdot \overline{\nu} d|\nabla \chi|\, dt, \\
&(ii) \quad \int_0^T \int_\Omega \overline{\nu}^h \cdot D\xi D\gamma\left(\overline{\nu}^h\right)\, d\left|\nabla \chi^h\right| dt \longrightarrow \int_0^T \int_\Omega \overline{\nu} \cdot D\xi D\gamma\left(\overline{\nu}\right)\, d|\nabla \chi|\, dt, \\
&(iii) \quad \int_0^T \int_\Omega div\left(u^h(t)\,\xi(t)\right)\chi^h(t) \longrightarrow \int_0^T \int_\Omega div\left(u(t)\,\xi(t)\right)\chi(t), \\
&(iv) \quad \int_0^T \int_\Omega div\left(L_h^0\left(u^h(t) - u^h(t-h) + \chi^h(t) - \chi^h(t-h)\right)\xi\right)\chi^h(t) \longrightarrow 0,
\end{aligned}$$

where $\overline{\nu}^h(t) = \frac{\nabla \chi^h(t)}{\left|\nabla \chi^h(t)\right|}$, $\overline{\nu}(t) = \frac{\nabla \chi(t)}{|\nabla \chi(t)|}$, $\xi \in C^1\left(\overline{\Omega}_T, \mathbb{R}^n\right)$ *with* $\xi \cdot \nu_{\partial\Omega} = 0$.

Proof: (i): Claim 1:
For all $\epsilon > 0$ there exist $g_\epsilon \in C_0^1(\Omega, \mathbb{R}^n)$ with $\gamma^0(g_\epsilon) \le 1$ and $h_0(\epsilon) \in \mathbb{R}$ such that

$$\int_\Omega \left(\gamma\left(\overline{\nu}^h(t)\right) - g_\epsilon \cdot \overline{\nu}^h(t)\right) d\left|\nabla\chi^h(t)\right| \le \epsilon \quad \text{for all } h \le h_0(\epsilon) \text{ and for a.e. } t\ .$$

Proof of Claim 1:
By definition of $\int_\Omega |\nabla\chi(t)|_\gamma$ there exist functions $g_\epsilon \in C_0^1(\Omega, \mathbb{R}^n)$ with $\gamma^0(g_\epsilon) \le 1$ such that

$$\begin{aligned} \int_\Omega g_\epsilon \cdot \overline{\nu}(t)\, d|\nabla\chi(t)| &= -\int_\Omega \operatorname{div} g_\epsilon\, \chi(t)\, dx \\ \stackrel{\epsilon\to 0}{\longrightarrow} \quad \int_\Omega |\nabla\chi(t)|_\gamma &= \int_\Omega \gamma\left(\overline{\nu}(t)\right)\, d|\nabla\chi(t)|. \end{aligned}$$

It will turn out that the g_ϵ are smooth approximations of the Cahn-Hoffmann ξ-vectors $D\gamma(\overline{\nu})$. For fixed $\epsilon > 0$ we choose $g_\epsilon \in C_0^1(\Omega, \mathbb{R}^n)$ with $\gamma(g_\epsilon) \le 1$ such that

$$\int_\Omega \left(\gamma\left(\overline{\nu}(t)\right) - g_\epsilon \cdot \overline{\nu}(t)\right) d|\nabla\chi(t)| \le \frac{1}{3}\epsilon\ .$$

Since $\chi^h \to \chi$ in $L^2(\Omega)$ for almost every t, we obtain

$$\begin{aligned} \int_\Omega g_\epsilon \cdot \overline{\nu}^h(t)\, d\left|\nabla\chi^h(t)\right| &= -\int_\Omega \operatorname{div} g_\epsilon\, \chi^h(t)\, dx \\ \stackrel{h\to 0}{\longrightarrow} \quad -\int_\Omega \operatorname{div} g_\epsilon\, \chi(t)\, dx &= \int_\Omega g_\epsilon \cdot \overline{\nu}(t)\, d|\nabla\chi(t)| \text{ for a.e. } t\ . \end{aligned}$$

Choose $\widetilde{h}_0(\epsilon) > 0$ such that for all $h \le \widetilde{h}_0(\epsilon)$ and for almost every t

$$\left|\int_\Omega g_\epsilon \cdot \overline{\nu}^h(t)\, d\left|\nabla\chi^h(t)\right| - \int_\Omega g_\epsilon \cdot \overline{\nu}(t)\, d|\nabla\chi(t)|\right| \le \frac{1}{3}\epsilon\ .$$

From Lemma 4.1, we can choose $\overline{h}_0(\epsilon) > 0$ such that for all $h \le \overline{h}_0(\epsilon)$

$$\left|\int_\Omega \gamma\left(\overline{\nu}^h(t)\right) d\left|\nabla\chi^h(t)\right| - \int_\Omega \gamma\left(\overline{\nu}(t)\right) d|\nabla\chi(t)|\right| \le \frac{1}{3}\epsilon\ .$$

Define $h_0(\epsilon) := \min\left\{\widetilde{h}_0(\epsilon), \overline{h}_0(\epsilon)\right\}$. So we have for all $h \le h_0(\epsilon)$ and almost every t

$$\begin{aligned} \int_\Omega \left(\gamma\left(\overline{\nu}^h\right) - g_\epsilon \cdot \overline{\nu}^h\right) d\left|\nabla\chi^h\right| \ &\le\ \int_\Omega \gamma\left(\overline{\nu}^h\right)\, d\left|\nabla\chi^h\right| - \int_\Omega \gamma\left(\overline{\nu}\right)\, d|\nabla\chi| \\ + \int_\Omega \left(\gamma\left(\overline{\nu}\right) - g_\epsilon \cdot \overline{\nu}\right) d|\nabla\chi| \ &+\ \int_\Omega g_\epsilon \cdot \overline{\nu}\, d|\nabla\chi| - \int_\Omega g_\epsilon \cdot \overline{\nu}^h\, d\left|\nabla\chi^h\right| \\ &\le\ \frac{1}{3}\epsilon + \frac{1}{3}\epsilon + \frac{1}{3}\epsilon = \epsilon \end{aligned}$$

as required.
Now we can show assertion (i). In the following we use $\chi^h(t) \to \chi(t)$ in $L^2(\Omega)$ for almost all t, the fact $g_\epsilon \cdot \overline{\nu}^h \leq \gamma^0(g_\epsilon)\gamma(\overline{\nu}^h) \leq \gamma(\overline{\nu}^h)$ and the corresponding inequality for $\overline{\nu}$ and obtain

$$\begin{aligned}
&\Big| \int_\Omega \operatorname{div} \xi \, D\gamma(\overline{\nu}^h) \cdot \overline{\nu}^h \, d|\nabla \chi^h| - \int_\Omega \operatorname{div} \xi \, D\gamma(\overline{\nu}) \cdot \overline{\nu} \, d|\nabla \chi| \Big| \\
&\quad = \Big| \int_\Omega \operatorname{div} \xi \, \gamma(\overline{\nu}^h) \, d|\nabla \chi^h| - \int_\Omega \operatorname{div} \xi \, \gamma(\overline{\nu}) \, d|\nabla \chi| \Big| \\
&\quad \leq \int_\Omega \underbrace{|\operatorname{div} \xi|}_{\leq C} (\gamma(\overline{\nu}^h) - g_\epsilon \cdot \overline{\nu}^h) \, d|\nabla \chi^h| \\
&\quad + \Big| \int_\Omega \operatorname{div} \xi \, g_\epsilon \cdot \overline{\nu}^h \, d|\nabla \chi^h| - \int_\Omega \operatorname{div} \xi \, g_\epsilon \cdot \overline{\nu} \, d|\nabla \chi| \Big| + \int_\Omega |\operatorname{div} \xi| |g_\epsilon \cdot \overline{\nu} - \gamma(\overline{\nu})| \, d|\nabla \chi| \\
&\quad \leq C \int_\Omega (\gamma(\overline{\nu}^h) - g_\epsilon \cdot \overline{\nu}^h) d|\nabla \chi^h| + \Big| \int_\Omega \operatorname{div}(\operatorname{div} \xi \, g_\epsilon)(\chi^h - \chi) dx \Big| \\
&\quad + C \int_\Omega (\gamma(\overline{\nu}) - g_\epsilon \cdot \overline{\nu}) d|\nabla \chi| \longrightarrow C\epsilon \quad \text{for a.e. } t,
\end{aligned}$$

where we also used the structure theorem for BV-functions, see [8]. Since ϵ was arbitrary we obtain for almost every t:

$$\Big| \int_\Omega \operatorname{div} \xi \, D\gamma\left(\overline{\nu}^h\right) \cdot \overline{\nu}^h \, d|\nabla \chi^h| - \int_\Omega \operatorname{div} \xi \, D\gamma\left(\overline{\nu}\right) \cdot \overline{\nu} \, d|\nabla \chi| \Big| \longrightarrow 0 \, .$$

Moreover, the energy estimate leads to

$$\operatorname*{ess\,sup}_{0 \leq t \leq T} \int_\Omega D\gamma\left(\overline{\nu}^h(t)\right) \cdot \overline{\nu}^h(t) \, d\big|\nabla \chi^h(t)\big| = \operatorname*{ess\,sup}_{0 \leq t \leq T} \int_\Omega \gamma\left(\overline{\nu}^h(t)\right) \, d\big|\nabla \chi^h(t)\big| \leq C.$$

Then the assertion follows from Lebesgue's dominated convergence theorem, as $|\operatorname{div} \xi(t)| \leq C$.
(ii): Claim 2:
For all $\epsilon > 0$ there exists $g_\epsilon \in C_0^1(\Omega, \mathbb{R}^n)$ with $\gamma(g_\epsilon) \leq 1$ and $h_0(\epsilon) \in \mathbb{R}$ such that

$$\int_\Omega \left| D\gamma\left(\overline{\nu}^h(t)\right) - g_\epsilon \right| \, d\big|\nabla \chi^h(t)\big| \leq C\sqrt{\epsilon}$$

for all $h < h_0(\epsilon)$ and for almost every t.
Proof of Claim 2:
From Claim 1, we can find for $\epsilon > 0$ a function $g_\epsilon \in C_0^1(\Omega, \mathbb{R}^n)$ with $\gamma^0(g_\epsilon) \leq 1$ and $h_0(\epsilon) \in \mathbb{R}$ such that

$$\int_\Omega \left(\gamma\left(\overline{\nu}^h(t)\right) - g_\epsilon \cdot \overline{\nu}^h(t)\right) d\big|\nabla \chi^h(t)\big| \leq \epsilon$$

for all $h \leq h_0(\epsilon)$ and almost every t. From Lemma 2.3, we obtain for all $h \leq h_0(\epsilon)$ and almost every t

$$\begin{aligned}
\int_\Omega |D\gamma\left(\nu^h\right) - g_\epsilon| \, d|\nabla\chi^h| &\leq \left(\int_\Omega |D\gamma\left(\nu^h\right) - g_\epsilon|^2 \, d|\nabla\chi^h|\right)^{1/2} \cdot \left(\int_\Omega d|\nabla\chi^h|\right)^{1/2} \\
&\leq C\left(\int_\Omega \left(\gamma\left(\overline{\nu}^h\right) - g_\epsilon \cdot \overline{\nu}^h\right) d|\nabla\chi^h|\right)^{1/2} \leq C\sqrt{\epsilon}\,,
\end{aligned}$$

where we have used $\int_\Omega d\left|\nabla\chi^h\right| \leq C$ which follows from the a priori estimate. This shows Claim 2.
Obviously, we have the equation:

$$\begin{aligned}
\int_\Omega \overline{\nu}(t) \cdot D\xi \, D\gamma\left(\overline{\nu}\right) \, d|\nabla\chi| &= \int_\Omega \overline{\nu} \cdot D\xi \left(D\gamma\left(\overline{\nu}\right) - g_\epsilon + g_\epsilon\right) \, d|\nabla\chi| \\
&= \int_\Omega \left[\overline{\nu} \cdot D\xi \left(D\gamma\left(\overline{\nu}\right) - g_\epsilon\right) + \overline{\nu} \cdot D\xi \, g_\epsilon\right] \, d|\nabla\chi|
\end{aligned}$$

and an analogous equation for the term

$$\int_\Omega \overline{\nu}^h \cdot D\xi \, D\gamma\left(\overline{\nu}^h\right) \, d\left|\nabla\chi^h\right|.$$

Hence, we can estimate

$$\begin{aligned}
&\Big|\int_\Omega \overline{\nu}^h \cdot D\xi \, D\gamma\left(\overline{\nu}^h\right) \, d|\nabla\chi^h| - \int_\Omega \overline{\nu} \cdot D\xi \, D\gamma\left(\overline{\nu}\right) \, d|\nabla\chi|\Big| \\
&\quad\leq C\int_\Omega |D\gamma\left(\overline{\nu}^h\right) - g_\epsilon| \, d|\nabla\chi^h| + C\int_\Omega |D\gamma\left(\overline{\nu}\right) - g_\epsilon| \, d|\nabla\chi| \\
&\qquad + \Big|\int_\Omega \overline{\nu}^h \cdot D\xi \, g_\epsilon \, d|\nabla\chi^h| - \int_\Omega \overline{\nu} \cdot D\xi \, g_\epsilon \, d|\nabla\chi|\Big| \\
&\quad\leq C\sqrt{\epsilon} + \left|\int_\Omega \operatorname{div}\left(D\xi \, g_\epsilon\right)\left(\chi^h - \chi\right) \, dx\right| \\
&\quad\leq C\sqrt{\epsilon} + \underbrace{\left(\int_\Omega \left(\operatorname{div}\left(D\xi \, g_\epsilon\right)\right)^2 dx\right)^{1/2}}_{\leq C \text{ for fixed } \epsilon} \underbrace{\|\chi^h - \chi\|_{L^2(\Omega)}}_{\longrightarrow 0 \text{ for } h \to 0} \\
&\quad\overset{h\to 0}{\longrightarrow} C\sqrt{\epsilon}\,,
\end{aligned}$$

where we have used Claim 2 and the structure theorem. Hence, we obtain for almost every t:

$$\int_\Omega \overline{\nu}^h \cdot D\xi \, D\gamma\left(\overline{\nu}^h\right) \, d\left|\nabla\chi^h\right| \longrightarrow \int_\Omega \overline{\nu} \cdot D\xi \, D\gamma\left(\overline{\nu}\right) \, d|\nabla\chi|.$$

Now assertion (ii) follows from Lebesgue's dominated convergence theorem.
(iii): We can estimate

$$\begin{aligned}
\Big|\int_{\Omega_T} \operatorname{div}\left(u^h\,\xi\right)\chi^h - \operatorname{div}\left(u\,\xi\right)\chi\Big| &\leq \Big|\int_{\Omega_T}\left(\operatorname{div}\left(u^h\,\xi\right) - \operatorname{div}\left(u\,\xi\right)\right)\chi\Big| \\
+\Big|\int_{\Omega_T} \operatorname{div}\left(u^h\,\xi\right)\left(\chi^h-\chi\right)\Big| &\leq \Big|\int_{\Omega_T}\left(\nabla u^h - \nabla u\right)\cdot\xi\,\chi + \left(u^h-u\right)\operatorname{div}\xi\,\chi\Big| \\
&\quad + \left(\int_{\Omega_T}\left|\operatorname{div}\left(u^h\,\xi\right)\right|^2\right)^{1/2}\|\chi^h-\chi\|_{L^2(\Omega_T)} \longrightarrow 0,
\end{aligned}$$

where we have used $u^h \rightharpoonup u$ in $L^2\left(0,T;H^{1,2}(\Omega)\right)$ and $\chi^h \to \chi$ in $L^2\left(0,T;L^2(\Omega)\right)$.
(iv): In order to prove this assertion, we show that the term $L^0_h(u_i-u_{i-1}+\chi_i-\chi_{i-1})$ converges weakly to 0 in $L^2\left(0,T;H^{1,2}(\Omega)\right)$. For that purpose, we test the equation defining $L^0_h(u_i-u_{i-1}+\chi_i-\chi_{i-1})$ with $L^0_h(u_i-u_{i-1}+\chi_i-\chi_{i-1}) \in H^{1,2}_0(\Omega)$ and divide by h. We then obtain

$$\begin{aligned}
&\int_{\Omega_T}\left|\nabla L^0_h(u_i-u_{i-1}+\chi_i-\chi_{i-1})\right|^2 + \frac{1}{h}\int_{\Omega_T}\left|L^0_h(u_i-u_{i-1}+\chi_i-\chi_{i-1})\right|^2 \\
&= -\int_{\Omega_T}\partial_t^{-h}(u_i+\chi_i)\,L^0_h(u_i-u_{i-1}+\chi_i-\chi_{i-1}) \\
&= -\int_{\Omega_T} f_i\,L^0_h(u_i-u_{i-1}+\chi_i-\chi_{i-1}) + \int_{\Omega_T}\nabla u_i\cdot\nabla L^0_h(u_i-u_{i-1}+\chi_i-\chi_{i-1}),
\end{aligned}$$

where we have used equation (16). Moreover, using that u^h is uniformly bounded in $L^2\left(0,T;H^{1,2}(\Omega)\right)$ and f^h is uniformly bounded in $L^2\left(0,T;L^2(\Omega)\right)$ we obtain that

$$\int_{\Omega_T}\left|\nabla L^0_h(u_i-u_{i-1}+\chi_i-\chi_{i-1})\right|^2 + \frac{1}{h}\int_{\Omega_T}\left|L^0_h(u_i-u_{i-1}+\chi_i-\chi_{i-1})\right|^2 \tag{24}$$

is uniformly bounded. Hence, we can find a function $\varphi \in L^2\left(0,T;H^{1,2}(\Omega)\right)$ such that for $h \to 0$

$$L^0_h(u_i-u_{i-1}+\chi_i-\chi_{i-1}) \longrightarrow \varphi \text{ in } L^2\left(0,T;H^{1,2}(\Omega)\right).$$

Since (24) implies that

$$L^0_h(u_i-u_{i-1}+\chi_i-\chi_{i-1}) \to 0 \quad \text{in } L^2(0,T;L^2(\Omega)),$$

we conclude $\varphi \equiv 0$. Altogether we obtain

$$\begin{aligned}
\int_{\Omega_T}\operatorname{div}\left(L^0_h\left(u_i-u_{i-1}+\chi_i-\chi_{i-1}\right)\xi\right)\chi^h &= \int_{\Omega_T}\nabla L^0_h\left(u_i-u_{i-1}+\chi_i-\chi_{i-1}\right)\cdot\xi\,\chi^h \\
&+ \int_{\Omega_T} L^0_h\left(u_i-u_{i-1}+\chi_i-\chi_{i-1}\right)\operatorname{div}\xi\,\chi^h \longrightarrow 0,
\end{aligned}$$

where we have used $L_h^0(u_i - u_{i-1} + \chi_i - \chi_{i-1}) \rightharpoonup 0$ in $L^2(0,T;H^{1,2}(\Omega))$ and $L_h^0(u_i - u_{i-1} + \chi_i - \chi_{i-1}) \to 0$, $\chi^h \to \chi$ in $L^2(0,T;L^2(\Omega))$. $\square$

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References

[1] Amar, M. and Bellettini, G., *A notion of total variation depending on a metric with discontinuous coefficients*, Ann. Inst. H. Poincaré, Analyse Non-Linéaire 11 (1994), 91–133.

[2] Amar, M. and Bellettini, G., *Approximation by Γ-convergence of a total variation with discontinuous coefficients*, Asymptotic Anal. 10, no. 3 (1995), 225–243.

[3] Ambrosio, L., Fusco, N. and Pallara, D., *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Science Publications 2000.

[4] Barrett, J.W., Garcke, H. and Nürnberg, R., *On stable parametric finite element methods for the Stefan problem and the Mullins-Sekerka problem with applications to dendritic growth*, J. Comput. Phys. 229, no. 18 (2010), 6270-6299.

[5] Bellettini, G. and Paolini, M., *Anisotropic motion by mean curvature in the context of Finsler geometry*, Hokkaido Math. J. 25, no. 3 (1996), 537–566.

[6] Cialese, M., Nagase, Y. and Pisante, G., *The Gibbs-Thomson relation for non homogeneous anisotropic phase transitions*, Advances in Calculus of Variations 3, no. 3 (2010), 321–344.

[7] Dziuk, G., *Discrete anisotropic curve shortening flow*, SIAM J. Numer. Ana. 36, no. 6 (1999), 1808–1830.

[8] Evans, L.C. and Gariepy, R.F., *Measure Theory and Fine Properties of Functions*, Studies in Advanced Mathematics, CRC Press, 1992.

[9] Garcke, H. and Kraus, Ch., *An anisotropic, inhomogeneous, elastically modified Gibbs–Thomson law as singular limit of a diffuse interface model*, Adv. Math. Sci. Appl. (2010)

[10] Garcke, H., Nestler, B. and Stoth, B., *On anisotropic order parameter models for multi-phase systems and their sharp interface limits*, Phys. D 115, no. 1-2 (1998), 87-108.

[11] Giga, Y., *Surface Evolution Equations*, Monographs in Mathematics 99, Birkhäuser, Basel, 2006.

[12] Giusti, E., *Minimal surfaces and functions of bounded variation*, Birkhäuser, Boston, 1984.

[13] Gurtin, M.E., *Multiphase thermomechanics with interfacial structure. 1. Heat conduction and the capillary balance law*, Arch. Rational Mech. Anal. 104, no. 3 (1988), 195–221.

[14] Gurtin, M.E., *Thermomechanics of evolving phase boundaries in the plane*, Oxford Mathematical Monographs, New York 1993, xi+148.

[15] Luckhaus, S., *Solutions for the two-phase Stefan problem with the Gibbs-Thomson law for the melting temperature*, European J. Appl. Math. 1 (1990), 101–111.

[16] Luckhaus, S., *The Stefan problem with Gibbs-Thomson law*, Sezione di Analisi Matematica e Probabilitita, Universita di Pisa, 2.75 (591), (1991).

[17] Luckhaus, S. and Modica, L., *The Gibbs-Thomson relation within the gradient theory of phase transitions*, Arch. Ration. Mech. Anal. 107(1) (1989), 71-83.

[18] Röger, M., *Solutions for the Stefan problem with Gibbs-Thomson law by a local minimisation*, Interfaces and Free Boundaries 6 (2004), 105–133.

[19] Röger, M, *Existence of weak solutions for the Mullins-Sekerka flow*, SIAM J. Math. Anal. 37 (2005), no. 1, 291–301.

[20] Schaubeck, St., *Über das Stefan-Problem mit anisotropem Gibbs-Thomson-Gesetz*, diploma thesis, University Regensburg (2009).

[21] Visintin, A., *Models of Phase Transitions*, Progress in Nonlinear Differential Equations and their Applications 28, Birkhäuser Boston, 1996.