Financial Economics: Risk Sharing and Asset Pricing in General Equilibrium©

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1. Introduction

This course in financial economics addresses two main questions.

- **Efficiency:** does trade in financial assets lead to an efficient allocation of economic risks?
- **Asset pricing:** what determines asset prices?

To answer these questions we approach the field of financial economics from the perspective of general equilibrium theory with complete markets. (Incomplete markets are much harder to analyze and there are much fewer general results.)

Results in this field come at two strikingly different levels of mathematical difficulty. Examples include several of the classical results in financial economics:

- *Almost* all results require only very basic math. Examples include
  - the first welfare theorem with financial markets
  - the fundamental equations of asset pricing in two-period and multi-period setups
  - the shareholder unanimity theorem
  - the Modigliani-Miller theorem
  - the CAPM
  - the efficient markets hypothesis
  - fundamental value.

- Some results, which assert the *existence* of something, require the use of separating hyperplane or fixed point theorems:
  - *existence* of equilibrium
  - the 2nd welfare theorem (for any Pareto-optimal allocation, *there exist* endowments and prices such that the allocation and the prices are an equilibrium)
  - the fundamental theorem of asset pricing (in the absence of arbitrage opportunities, *there exist* state prices that can be used to value assets).

These slides focus on the former set of results. A second set of slides “Financial Economics: Risk Sharing and Asset Pricing in General Equilibrium II”© summarize the latter set of results and the necessary math.

The classical theorems in financial economics that we prove do not work very well in practice.

As explained in boxes like this one,

- making risks tradable does not necessarily improve the allocation of risk,
- the fundamental equations of asset pricing fail to explain the equity premium,
- financial structure matters,
- the CAPM does not work well, and
- asset prices do not accurately reflect fundamental value.

That does not mean that the theorems bear no practical significance. They are the perfect-markets perfect-information reference point for assessing inefficiencies.

- Risk sharing with trade in new financial products may work less efficiently *than suggested by the perfect-markets model,*
- the equity premium *compared to the prediction of the fundamental equations of asset pricing* is a puzzle,
- firms' financial structure is not, *as the perfect-markets model suggests,* irrelevant,
- there are asset pricing anomalies, i.e., deviations from the asset returns *predicted by the CAPM,* and
- asset prices do not accurately reflect the fundamental value that is *implied by the fundamental equations of asset pricing.*

EFFICIENT RISK SHARING

2. Two-period two-state model

Endowment economy with

- I consumers $i = 1, 2, \ldots, I$,
- one (physical) good,
- two periods, t and $t + 1$,
- two states $s = 1, 2$ with probabilities π_s ($\pi_s > 0$, $\sum_{s=1}^{2} \pi_s = 1$).

Convention:

- Subscript 0: variable refers to date t
- Subscript s: variable refers to state s at date $t+1$.

Endowments: y_0^i, y_1^i, y_2^i.
Preferences: utility functions $U^i(c_0^i, c_1^i, c_2^i)$ ($U^i : \mathbb{R}_+^3 \rightarrow \mathbb{R}$). The only assumption we place on the U^i's is that they are strictly increasing.

For asset pricing, we assume a time-separable expected utility function:

$$U^i(c_0^i, c_1^i, c_2^i) = u^i(c_0^i) + \beta^i \sum_{s=1}^{2} \pi_s u^i(c_s^i).$$

with

- positive discounting ($0 < \beta^i < 1$),
- $u^i : \mathbb{R}_+ \to \mathbb{R}$ twice differentiable,
- $(u^i)'(c) > 0 > (u^i)''(c)$ for all $c > 0$, and
- $(u^i)'(0) = \infty$.

But that's inessential for much of the equilibrium analysis.

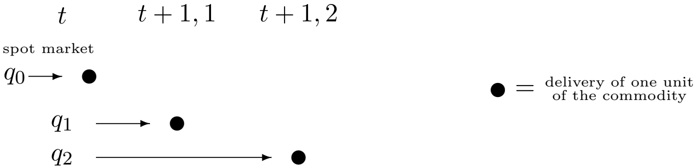
The model is simple to the point of triviality. That doesn't mean that it's easy to analyze, however!
It offers deep insights into the nature of general equilibrium, its welfare properties, and efficient risk bearing.

Literature:

- Mas-Colell, Andreu, Michael D. Whinston, and Jerry R. Green, *Microeconomic Theory*, Oxford University Press (1995), Chapters 16, 19.
- Magill, Michael, and Martine Quinzii, *Theory of Incomplete Markets*, MIT (2002), Section 6.

3 Efficient risk sharing: contingent-commodity markets

To begin with, ignore financial markets. There is a spot market with payment and delivery at date 0. Further, there are ***contingent-commodity markets (CCMs)***. In these markets, claims to the delivery of one unit of the consumption good *contingent upon the realization of state s* are traded at price q_s. If s does not materialize, nothing is delivered.



- $\mathbf{q} = (q_0, q_1, q_2)$: price vector
- $\mathbf{y}^i = (y_0^i, y_1^i, y_2^i)$: i's endowment vector
- $\mathbf{c}^i = (c_0^i, c_1^i, c_2^i)$: i's consumption vector.

Given that the U^i's are strictly increasing, we can restrict attention to $\mathbf{q} \in \mathbb{R}^3_{++}$. Without further comment, we let $\mathbf{c}^i \in \mathbb{R}^3_+$ (and analogously when we consider the model with more than two states).

i's budget constraint is:

$$\mathbf{q}(\mathbf{c}^i - \mathbf{y}^i) \leq 0.$$

Convention: the product of two vectors is the scalar product. In equilibrium, due to strictly increasing utility, the budget equations will hold with equality.
All trade takes place at the inital date 0.

Definition: *An allocation* $(\mathbf{c}^i)_{i=1}^I$ *is feasible if*

$$\sum_{i=1}^{I} \mathbf{c}^i \leq \sum_{i=1}^{I} \mathbf{y}^i.$$

An allocation is Pareto-optimal if it is feasible and there is no other feasible allocation $(\mathbf{c}^{i'})_{i=1}^I$ *such that*

$$U^i(\mathbf{c}^{i'}) \geq U^i(\mathbf{c}^i)$$

for all i *with strict inequality for at least one* i.

Definition: *An allocation* $(\mathbf{c}^i)_{i=1}^I$ *and a price vector* $\mathbf{q}$ *are an equilibrium with CCMs (ECCM) if*

- $\mathbf{c}^i$ *maximizes* U^i *subject to the individual's budget constraint for all consumers* i *and*
- *markets clear:*

$$\sum_{i=1}^{I} \mathbf{c}^i = \sum_{i=1}^{I} \mathbf{y}^i.$$

Theorem (1st Welfare Theorem with CCMs): *Let* $((\mathbf{c}^i)_{i=1}^I, \mathbf{q})$ *be an ECCM. Then* $(\mathbf{c}^i)_{i=1}^I$ *is Pareto-optimal.*

Proof: Suppose not. Then there is $(\mathbf{c^{i'}})_{i=1}^I$ which is feasible and Pareto-superior. Pareto-superiority implies

$$\mathbf{qc^{i'}} \geq \mathbf{qc}^i$$

for all i with strict inequality for at least one i. This is because otherwise i could have afforded a better consumption bundle than $\mathbf{c}^i$ (use is made of strictly increasing utility).

So

$$\sum_{i=1}^{I} \mathbf{q}\mathbf{c}^{\mathbf{i}'} > \sum_{i=1}^{I} \mathbf{q}\mathbf{c}^{i}.$$

Feasibility of the allocation $(\mathbf{c}^{\mathbf{i}'})_{i=1}^{I}$ requires $\sum_{i=1}^{I} \mathbf{c}^{\mathbf{i}'} \leq \sum_{i=1}^{I} \mathbf{y}^{i}$, hence

$$\sum_{i=1}^{I} \mathbf{q}\mathbf{c}^{\mathbf{i}'} \leq \sum_{i=1}^{I} \mathbf{q}\mathbf{y}^{i}.$$

The fact that $((\mathbf{c}^i)_{i=1}^I, \mathbf{q})$ is an equilibrium (i.e., $\sum_{i=1}^I \mathbf{c}^i = \sum_{i=1}^I \mathbf{y}^i$) implies

$$\sum_{i=1}^{I} \mathbf{q}\mathbf{c}^i = \sum_{i=1}^{I} \mathbf{q}\mathbf{y}^i.$$

Taking stock, we get a contradiction:

$$\sum_{i=1}^{I} \mathbf{q}\mathbf{y}^i \geq \sum_{i=1}^{I} \mathbf{q}\mathbf{c}^{\mathbf{i}'} > \sum_{i=1}^{I} \mathbf{q}\mathbf{c}^i = \sum_{i=1}^{I} \mathbf{q}\mathbf{y}^i.$$

Q.E.D.

As usual in non-monetary micro models, we can arbitrarily normalize one price (e.g., $q_0 = 1$) without affecting the equilibrium allocation or the equilibrium relative prices.

Theorem (irrelevance of price normalization): *If* $((\mathbf{c}^i)_{i=1}^I, \mathbf{q})$ *is an ECCM, then* $((\mathbf{c}^i)_{i=1}^I, \lambda\mathbf{q})$ *is an ECCM for any* $\lambda > 0$.

Proof: Markets clear by construction.

$$(\lambda\mathbf{q})(\mathbf{c}^i - \mathbf{y}^i) \leq 0$$

if, and only if, $\mathbf{q}(\mathbf{c}^i - \mathbf{y}^i) \leq 0$. That is, the budget set remains the same after multiplying all the prices by a constant, so the optimal $\mathbf{c}^i$ also remains the same. Q.E.D.

Two other important theorems are:

- **Existence of equilibrium:** An ECCM exists.
- **2nd Welfare Theorem:** For any Pareto-optimal allocation $(\mathbf{c}^i)_{i=1}^I$, there exists endowments $(\mathbf{y}^i)_{i=1}^I$ and a price vector $\mathbf{q}$ such that $((\mathbf{c}^i)_{i=1}^I, \mathbf{q})$ is an ECCM

The proofs of these theorems require additional (convexity, continuity, and boundary) assumptions and the use of a fixed point theorem and a separating hyperplane theorem, respectively, and can be found in the slides "Financial Economics: Risk Sharing and Asset Pricing in General Equilibrium II"©.

The analysis is unchanged if one interprets the three commodities as three goods traded in spot markets at a given point in time. Hence, the 1st Welfare Theorem above is in essence a reinterpretation of the standard theorem for non-contingent commodity markets. It shows that the standard analysis of trade at a point in time without uncertainty carries over to intertemporal trade in the presence of risk. This holds true, more generally, for all results of standard general equilibrium theory.

A mere *reinterpretation* of the concept of a “good” is sufficient to turn the standard static general equilibrium without uncertainty into a model with time and risk. The problem with this approach is that in reality futures markets exist for very few commodities (energy, oil, gas, metals, agricultural products) and even in these markets delivery is not state-contingent.

https://www.cnbc.com/futures-and-commodities/

In what follows we give up the concept of contingent commodities and replace CCMs with financial markets.

Literature:

- Mas-Colell, Andreu, Michael D. Whinston, and Jerry R. Green, *Microeconomic Theory*, Oxford University Press (1995), Chapter 19.
- Magill, Michael, and Martine Quinzii, *Theory of Incomplete Markets*, MIT (2002), Section 7.

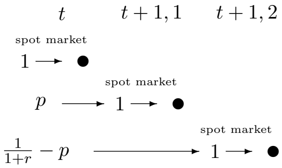
4 Efficient risk sharing: finance economy

The CCM economy achieves Pareto-optimal risk sharing, as CCMs allow state-contingent consumption plans.
We now assume that the only goods markets are spot markets: the price is paid on delivery. The spot price is normalized to unity. There are no CCMs.
Financial markets (FMs) serve the same purpose as CCMs (indirectly): they allow the state-contingent transfer of *purchasing power* which can then be used to buy goods.
Not all trade takes place at date 0: people buy goods using the proceeds of their financial claims at dat 1.

There are a risky and a safe asset, also called asset 1 and asset 2, respectively.

- The risky asset trades at prices p at t and pays a dividend $a_1 = 1$ in state 1 and $a_2 = 0$ in state 2 at date $t + 1$.
- The safe asset trades at price $1/(1 + r)$ and pays one unit of income in either state at date $t + 1$.

By buying one unit of the safe asset and (short-) selling one unit of the risky asset, one gets nothing in state 1 and one unit of income in state 2. The financial market is ***complete***, in that it is possible to buy purchasing power for the different states.



Let z_1^i denote consumer i's investment in the risky asset and z_2^i his purchases of the safe asset and $\mathbf{z}^i = (z_1^i, z_2^i)$. His budget constraints are:

$$c_0^i \leq y_0^i - pz_1^i - \frac{1}{1+r}z_2^i$$

$$c_1^i \leq y_1^i + z_1^i + z_2^i$$
$$c_2^i \leq y_2^i + z_2^i.$$

Definition: *An allocation* $(\mathbf{c}^i)_{i=1}^I$*, asset holdings* $(\mathbf{z}^i)_{i=1}^I$*, and a price vector* (p, r) *are an* ***equilibrium with FMs (EFM)*** *if*

- $\mathbf{c}^i$ *and* $\mathbf{z}^i$ *maximize* U^i *subject to the individual's budget constraints for all consumers* i *and*
- *markets clear:*

$$\sum_{i=1}^I \mathbf{c}^i = \sum_{i=1}^I \mathbf{y}^i$$

and

$$\sum_{i=1}^I \mathbf{z}^i = \mathbf{0}.$$

Consider an ECCM. Without loss of generality (as the absolute price level is indeterminate), set $q_0 = 1$.

- By buying one unit of the risky asset, one gets the purchasing power needed to buy one unit of the good in state 1. If

$$p = q_1,$$

this costs the same amount of money as one unit of contingent commodity 1 in the ECCM.

- By buying one unit of the safe asset and short-selling the risky asset, one gets the purchasing power needed to buy one unit of the good in state 2. If

$$\frac{1}{1+r} = q_1 + q_2,$$

this costs the same amount of money as one unit of contingent commodity 2 in the ECCM.

Theorem (ECCM and EFM): *Let* $(\mathbf{c}^{i*})_{i=1}^{I}$ *and* $\mathbf{q}$ *with* $q_0 = 1$ *be an ECCM. Let*

$$p = q_1$$

$$\frac{1}{1+r} = q_1 + q_2.$$

Let

$$z_1^i = \left(c_1^{i*} - y_1^i\right) - \left(c_2^{i*} - y_2^i\right)$$

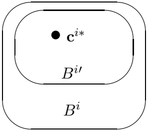
$$z_2^i = c_2^{i*} - y_2^i.$$

Then $((\mathbf{c}^{i*}, \mathbf{z}^i)_{i=1}^{I}, (p, r))$ *is an EFM.*

Proof: To prove the theorem we have to show that $\mathbf{c}^{i*}$ and $\mathbf{z}^i$ maximize i's utility and markets clear.
Let B^i denote the set of affordable consumption vectors $\mathbf{c}^i$ for consumer i with CCMs given equilibrium CCM prices $\mathbf{q}$, and let $B^{i\prime}$ denote the set of affordable consumption vectors $\mathbf{c}^i$ in the finance economy given the asset prices in the theorem.
To prove that $\mathbf{c}^{i*}$ maximizes utility in the finance economy, it suffices to show that

$$\mathbf{c}^{i*} \in B^{i\prime} \subseteq B^i.$$

That is, $\mathbf{c}^{i*}$ is affordable, and there is no better consumption vector in $B^{i\prime}$, because it would have been chosen in the CCM economy.



We first prove $\mathbf{c}^{i*} \in B^{i\prime}$. The choice of the portfolio $\mathbf{z}^i$ in the theorem implies that i can afford the same date-$t+1$ consumption levels as in the ECCM. The question is whether he can also afford c_0^{i*}. Since $\mathbf{c}^{i*}$ satisfies the CCM budget constraint with equality, we have $\mathbf{q}(\mathbf{c}^{i*} - \mathbf{y}^i) = 0$. Using the asset prices and the equations for i's portfolio in the theorem, this can be rewritten as

$$(c_0^{i*} - y_0^i) + p(z_1^i + z_2^i) + \left(\frac{1}{1+r} - p\right) z_2^i = 0.$$

Rearranging terms shows that the date-t budget constraint is satisfied. This proves $\mathbf{c}^{i*} \in B^{i\prime}$.

To prove $B^{i\prime} \subseteq B^i$, suppose $\mathbf{c}^i \in B^{i\prime}$, i.e., $\mathbf{c}^i$ and some $\mathbf{z}^i$ satisfy the finance economy budget constraints. Using the pricing formulas in the theorem, the date-t budget constraint becomes

$$c_0^i - y_0^i \leq -q_1(z_1^i + z_2^i) - q_2 z_2^i.$$

From the date-$t+1$ budget constraints, $z_1^i + z_2^i \geq c_1^i - y_1^i$ and $z_2^i \geq c_2^i - y_2^i$. So

$$c_0^i - y_0^i \leq -q_1(c_1^i - y_1^i) - q_2(c_2^i - y_2^i),$$

i.e., $\mathbf{q}(\mathbf{c}^i - \mathbf{y}^i) \leq 0$. Thus, $\mathbf{c}^i \in B^{i\prime}$ implies $\mathbf{c}^i \in B^i$. That is, $B^{i\prime} \subseteq B^i$.

It remains for us to show that markets clear. For the goods markets, this holds true by construction. For the asset markets, this follows immediately upon summing the individuals' asset demands in the theorem: $\sum_{i=1}^{I} \mathbf{z}^i = \mathbf{0}$. Q.E.D.

The equilibrium allocation $(\mathbf{c}^{i*})_{i=1}^{I}$ is Pareto optimal, as it is identical to the allocation in the CCM economy. So a 1st Welfare Theorem follows immediately from the theorem that relates an EFM to an ECCM: if an EFM "replicates" an ECCM, then the equilibrium allocation is Pareto optimal.

The theorem does not rule out that there are other EFMs, which do no replicate an ECCM and with a Pareto-nonoptimal allocation.
It is easy to see that such EFMs cannot exist: analogously to the proof of $\mathbf{c}^{i*} \in B^{i\prime}$, one can see that any $\mathbf{c}^i \in B^i$ is also in $B^{i\prime}$. So the budget constraints are the same in the CCM and FM economies and, hence, the set of equilibria are also the same.
For the sake of convenience, without further mention we consider only EFMs which replicate an ECCM in what follows.

The theory says that financial markets help achieve efficient risk sharing. The risk of having to make payments in states with low consumption can be shifted to the agents most willing (least unwilling) to accept it. The most striking counterargument is the role of securitization in the financial crisis 2007-2009: by making credit risk tradable, securitization contributed to the onset of the crisis as well as to its spread across the globe (Brunnermeier, 2009).

https://www.princeton.edu/~markus/research/papers/liquidity_credit_crunch.pdf

Literature:

- Brunnermeier, Markus K. (2009), "Deciphering the Liquidity and Credit Crunch 2007-2008", *Journal of Economic Perspectives* 23, 77-100.
- Mas-Colell, Andreu, Michael D. Whinston, and Jerry R. Green, *Microeconomic Theory*, Oxford University Press (1995), Chapter 19.
- Magill, Michael, and Martine Quinzii, *Theory of Incomplete Markets*, MIT (2002), Section 8.

CONSUMPTION-BASED ASSET PRICING

5 The fundamental equations of asset pricing

The central idea in consumption-based asset pricing is to exploit the necessary optimality conditions which characterize consumer choice in a market equilibrium.
So we consider an EFM and derive its implications for asset prices.
To do so, we now consider the time-separable expected utility function:

$$U^i(c_0^i, c_1^i, c_2^i) = u^i(c_0^i) + \beta^i \sum_{s=1}^{2} \pi_s u^i(c_s^i)$$

with $(u^i)'(c) > 0 > (u^i)''(c)$ for all $c > 0$ and $(u^i)'(0) = \infty$.
Let a denote the payoff of the risky asset (i.e., $a_1 = 1$ or $a_2 = 0$).

Consider individual *i*'s utility maximization problem. The budget constraints hold with equality. Substituting for the consumption levels c_0^i, c_1^i, and c_2^i yields the following problem:

$$\max_{\mathbf{z}^i} : u^i\left(y_0^i - pz_1^i - \frac{1}{1+r}z_2^i\right) + \beta^i E\left[u^i\left(y^i + az_1^i + z_2^i\right)\right].$$

The optimality conditions can be written as:

$$p = E\left[\beta^i \frac{(u^i)'(c^i)}{(u^i)'(c_0^i)} a\right]$$

$$\frac{1}{1+r} = E\left[\beta^i \frac{(u^i)'(c^i)}{(u^i)'(c_0^i)}\right].$$

These equations relate the asset prices to individual *i*'s marginal utility function and his consumption levels. As such, these are not remarkable asset pricing formulas.

The reason why they do yield interesting implications is that they can be used to express asset pricing without reference to individual-specific variables.
To see this, note that individual *i*'s MRS between consumption at t and consumption in state s at $t+1$ is determined by

$$dU^i = (u^i)'(c_0^i)dc_0^i + \pi_s \beta^i (u^i)'(c_s^i)dc_s^i = 0.$$

That is:

$$MRS_s^i = -\frac{dc_0^i}{dc_s^i} = \pi_s \beta^i \frac{(u^i)'(c_s^i)}{(u^i)'(c_0^i)}.$$

Theorem (Equality of MRSs when the allocation is Pareto-optimal): Let $(\mathbf{c}^i)_{i=1}^I$ be a Pareto-optimal allocation. Then for each s,

$$MRS_s^i = MRS_s^{i'}$$

for all i and i'.

Proof: Suppose for $s = 1$ or $s = 2$ the MRSs of individuals i and i' are different. Without loss of generality, let $MRS_s^i > MRS_s^{i'}$. That is, i "has a stronger preference" for consumption in s at $t + 1$ than i'.

Let

$$0 < dc_s^i = -dc_s^{i'}$$

$$dc_0^i = -\frac{MRS_s^i + MRS_s^{i'}}{2} dc_s^i = -dc_0^{i'}.$$

The resulting allocation is feasible. Then

$$\frac{dU^i}{(u^i)'(c_0^i)} = \frac{MRS_s^i - MRS_s^{i'}}{2} dc_s^i > 0$$

$$\frac{dU^{i'}}{(u^{i'})'(c_0^{i'})} = -\frac{MRS_s^i - MRS_s^{i'}}{2} dc_s^{i'} > 0.$$

This contradicts Pareto-optimality. Q.E.D.

Theorem (The fundamental equations of asset pricing): Let the utility function be given by

$$U^i(c_0^i, c_1^i, c_2^i) = u^i(c_0^i) + \beta^i \sum_{s=1}^{2} \pi_s u^i(c_s^i),$$

where u^i is twice differentiable, with $(u^i)'(c) > 0 > (u^i)''(c)$ for all $c > 0$ and $(u^i)'(0) = \infty$. Then, at an EFM with a Pareto optimal allocation there exists a stochastic discount factor (SDF) M such that

$$p = E(Ma)$$

$$\frac{1}{1+r} = E(M).$$

Proof: Since the MRSs

$$MRS_s^i = \pi_s \beta^i \frac{(u^i)'(c_s^i)}{(u^i)'(c_0^i)}$$

are uniform across individuals, so is

$$M_s \equiv \frac{MRS_s^i}{\pi_s} = \beta^i \frac{(u^i)'(c_s^i)}{(u^i)'(c_0^i)}$$

The assertion then follows immediately from the necessary optimality conditions for the individuals' utility maximization problems. Q.E.D.

An alternative proof of the fact that the MRS are uniform makes use of

$$p = E(Ma) = \pi_1 M_1$$

(since $a_1 = 1$ and $a_2 = 0$) and

$$\frac{1}{1+r} - p = E(M) - E(Ma) = \pi_2 M_2.$$

The individual MRS are identical (and, hence, the individual SDFs are identical) because they are adjusted to the same prices (viz., the prices of purchasing power in the respective states). (This version of the proof will be generalized in the model with more than two states.)

Future returns are discounted weakly (M_s is high) if consumption is low in state s. Future returns are discounted heavily (M_s is low) if consumption is high in s. That is, the SDF takes care that returns are valued according to whether they provide a hedge against low consumption.

Notice that what makes this an asset pricing theory is the underlying general equilibrium model, which ensures that the MRSs are uniform, so that a single SDF exists which can be used to discount payoffs.

Notice the difference between the stochastic discount factor in

$$p = E(SDF \cdot a)$$

and the common, non-stochastic, discount factor DF in

$$p = DF \cdot E(a),$$

where

$$DF = \frac{1}{1 + r + \text{risk premium}}.$$

The next section brings the two ways of discounting closer together.

The biggest challenge for consumption-based asset pricing is the equity premium puzzle. The return on U.S. stocks from 1889 to 1978 was about 7%, the return on bonds below 1%, so there is a 6 percentage points equity premium. Why is this a puzzle? If one assumes constant relative risk aversion and that consumption growth and dividend growth are i.i.d. and lognormal and calibrates the fundamental asset pricing equation to the data, then the implied coefficient of relative risk aversion is 47.6. Empirically, the coefficient seems to be around 2-3. So a highly unrealistic degree of risk aversion is required in order to make consumption-based asset pricing consistent with stock market data (Mehra, 2003).

www.nber.org/papers/w9512.pdf
https://www.tandfonline.com/doi/abs/10.2469/faj.v59.n1.2503

Literature:

- Cochrane, John F., *Asset Pricing*, 2nd Edition, Princeton University Press (2005), Chapter 1.
- Mehra, Rajnish (2003), “The Equity Premium: Why is it a Puzzle?”, *Financial Analysts Journal* 59, 54-69.

6 Applications

The fundamental equations of asset pricing look quite different than standard asset pricing formulas. This section shows that they can be rearranged such that they become more similar.

For now, the asset pricing formulas we derive apply only to the risky asset with payoff 1 in state 1 and 0 in state 2. Later on, we will see that they generalize to assets with arbitrary payoff vectors and to a multi-period model. In that multi-period model, a risky asset has a sequence of dividends $a_t, a_{t+1}, \ldots$, and having bought it at t at price p_t, it can be resold at date $t+1$ at price p_{t+1}. So we add a time subscript to the expectations operator, asset payoffs, the asset price, the interest rate, and the SDF and replace a_{t+1} with $p_{t+1} + a_{t+1}$, where it is understood that $p_{t+1} = 0$ in the two-period setup studied so far.

Application 1: Covariance Define $\sigma_{M,p+a}$ as the covariance between the SDF and the risky asset's return:

$$\sigma_{M,p+a} \equiv E_t[M_{t+1}(p_{t+1} + a_{t+1})] - E_t(M_{t+1})E_t(p_{t+1} + a_{t+1}).$$

Then

$$\begin{aligned} p_t &= E_t(M_{t+1})E_t(p_{t+1} + a_{t+1}) + \sigma_{M,p+a} \\ &= \frac{E_t(p_{t+1} + a_{t+1})}{1 + r_{t+1}} + \sigma_{M,p+a}. \end{aligned}$$

That is, the asset price is the expected payoff discounted at the riskless rate plus a risk adjustment, and the risk adjustment is simply the covariance of its return with the SDF. The risky asset is expensive if the SDF is high (i.e., consumption is low) in state 1, in which it pays off.

Application 2: Systematic risk Let

$$R_{t+1} = \frac{p_{t+1} + a_{t+1}}{p_t} - 1$$

denote the rate of return on the risky asset. Then, using $\sigma_{M,p+a} = p_t\, \sigma_{M,R}$,

$$E_t(R_{t+1}) - r_{t+1} = -(1 + r_{t+1})\sigma_{M,R}.$$

The risky asset's risk premium is proportional to the covariance between its returns and the SDF (its ***systematic risk***).

Application 3: Idiosyncratic risk If the asset's payoff is uncorrelated with the SDF, then the asset does not pay a risk premium, irrespective of how volatile its returns are:

$$E_t(R_{t+1}) - r_{t+1} = 0 \text{ if } \sigma_{M,R} = 0.$$

Application 4: beta Another way to write the fundamental pricing equation for the risky asset is

$$E_t(R_{t+1}) - r_{t+1} = -\beta \frac{\sigma_M^2}{E_t(M_{t+1})},$$

where

$$\beta = \frac{\sigma_{M,R}}{\sigma_M^2}.$$

That is, (analogously to the CAPM) the asset's risk premium is the product of a non-asset specific "market" term and its beta.

Application 5: Random walk Contrary to what has been assumed so far, assume the agents are risk-neutral: $u^i(c^i) = c^i$. Consider a "short" time interval, in which discounting doesn't play a role ($\beta^i = 1$) and the asset doesn't pay a dividend ($a_{t+1} = 0$). The fundamental equation for the risky asset becomes:

$$E_t(p_{t+1} - p_t) = 0.$$

The asset price is a random walk (which will become meaningful in the multi-period model).

Literature:

- Cochrane, John F., *Asset Pricing*, 2nd Edition, Princeton University Press (2005), Chapter 1.

COMPLETE MARKETS

7 Complete markets: Efficient risk sharing

The restriction to $S = 2$ states is immaterial for most of the analysis. Let $S \geq 2$. State s occurs with probability π_s ($\pi_s > 0$, $\sum_{s=1}^{S} \pi_s = 1$). Define

- $\mathbf{q} = (q_0, q_1, \ldots, q_S)$: price vector
- $\mathbf{y}^i = (y_0^i, y_1^i, \ldots, y_S^i)$: i's endowment vector
- $\mathbf{c}^i = (c_0^i, c_1^i, \ldots, c_S^i)$: i's consumption vector.

U^i is defined over $\mathbf{c}^i$ ($U^i : \mathbb{R}_+^{S+1} \to \mathbb{R}$).
The whole analysis of the CCM economy goes through without any modification.

The financial market in the two-state economy in Section 4 is complete in that it is possible to transfer income to a specific state s $(= 1, 2)$ by forming a suitable portfolio.
Assume that there are financial products, called ***Arrow securities (ASs)***, which do so directly: for each state s $(= 1, \ldots, S)$, there is an AS that costs $\tilde{p}_s$ at t and entitles the owner to the payment of one unit of income in s at $t + 1$ (and nothing in any other state).

As in Section 4, the goods prices are set equal to unity in both periods. Consumer i's budget constraints can be written as:

$$c_0^i - y_0^i \leq -\sum_{s=1}^{S} \tilde{p}_s \tilde{z}_s^i$$

$$c_s^i - y_s^i \leq \tilde{z}_s^i, \quad s = 1, \ldots, S.$$

Let $\tilde{\mathbf{p}} = (\tilde{p}_1, \ldots, \tilde{p}_S)$ denote the vector of AS prices and $\tilde{\mathbf{z}}^i = (\tilde{z}_1^i, \ldots, \tilde{z}_S^i)$ i's AS holdings.
In our one-good model, AS markets are almost the same as CCMs: they allow the purchase of the purchasing power required to purchase the good in one single state.

Definition: *An allocation* $(\mathbf{c}^i)_{i=1}^I$, *AS holdings* $(\tilde{\mathbf{z}}^i)_{i=1}^I$, *and a vector of AS prices* $\tilde{\mathbf{p}}$ *are an* ***equilibrium with a complete set of ASs (ECAS)*** *if*

- $(\mathbf{c}^i, \tilde{\mathbf{z}}^i)$ *maximizes* U^i *subject to the individual's budget constraints for all consumers* i *and*
- *markets clear:*

$$\sum_{i=1}^{I} \mathbf{c}^i = \sum_{i=1}^{I} \mathbf{y}^i$$

$$\sum_{i=1}^{I} \tilde{\mathbf{z}}^i = \mathbf{0}.$$

Consider an ECCM with $q_0 = 1$ (which entails no loss of generality because of the irrelevance of price normalization).

Theorem (ECAS and ECCM): *Let* $((\mathbf{c}^{i*})_{i=1}^{I}, \mathbf{q})$ *be an ECCM,*

$$\tilde{p}_s = q_s, \quad s = 1, \ldots, S,$$

and

$$\tilde{z}_s^i = c_s^{i*} - y_s^i, \quad s = 1, \ldots, S.$$

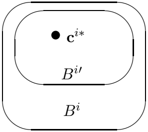
Then $((\mathbf{c}^{i*}, \tilde{\mathbf{z}}^i)_{i=1}^{I}, \tilde{\mathbf{p}})$ *is an ECAS.*

The former condition says that the cost of buying the purchasing power needed to buy one unit of the good in the spot market if state s occurs is the same as the cost of buying the delivery in the CCM. The second condition says that the portfolio finances the ECCM consumption vector.

Proof: The proof works exactly like the proof of the 1st Welfare Theorem for the two-state finance economy.
We have to show that $\mathbf{c}^{i*}$ maximizes i's utility and markets clear.
To prove that $\mathbf{c}^{i*}$ maximizes utility given ECAS prices, it suffices to show that

$$\mathbf{c}^{i*} \in B^{i\prime} \subseteq B^{i},$$

where B^i is the set of affordable consumption vectors $\mathbf{c}^i$ for consumer i with CCMs given equilibrium CCM prices $\mathbf{q}$ and $B^{i\prime}$ is the set of affordable consumption vectors $\mathbf{c}^i$ with a complete set of ASs given the ECAS prices. That is, $\mathbf{c}^{i*}$ is affordable, and there is no better consumption vector in $B^{i\prime}$, because it would have been chosen in the CCM economy.



To prove $\mathbf{c}^{i*} \in B^{i\prime}$, notice that the fact that i chooses $\mathbf{c}^{i*}$ at the ECCM implies $\mathbf{q}(\mathbf{c}^{i*} - \mathbf{y}^{i}) = 0$. Substituting for $\mathbf{q}$ from the former condition of the theorem (and $q_0 = 1$) yields

$$c_0^{i*} - y_0^i + \sum_{s=1}^{S} \tilde{p}_s (c_s^{i*} - y_s^i) = 0.$$

As stipulated by the theorem, let consumer i choose the portfolio

$$\tilde{z}_s^i = c_s^{i*} - y_s^i, \quad s = 1, \ldots, S.$$

From the preceding equation,

$$c_0^{i*} - y_0^i = -\sum_{s=1}^{S} \tilde{p}_s \tilde{z}_s^i.$$

The latter two equations show that $\mathbf{c}^{i*} \in B^{i\prime}$.

To prove $B^{i\prime} \subseteq B^i$, suppose $\mathbf{c}^i \in B^{i\prime}$, i.e., $\mathbf{c}^i$ and some portfolio $\tilde{\mathbf{z}}^i$ satisfy the $S+1$ budget constraints. Substituting for $\tilde{z}^i_s$ from the latter S constraints into the period-t budget constraint yields

$$c^i_0 - y^i_0 \leq -\sum_{s=1}^{S} \tilde{p}_s (c^i_s - y^i_s).$$

The former condition of the theorem and $q_0 = 1$ yield

$$\mathbf{q}(\mathbf{c}^i - \mathbf{y}^i) \leq 0.$$

So $\mathbf{c}^i \in B^i$. The fact that $\mathbf{c}^i \in B^{i\prime} \Rightarrow \mathbf{c}^i \in B^i$ proves $B^{i\prime} \subseteq B^i$.

Goods market clearing in the CCM economy implies goods market clearing in the economy with ASs:

$$\sum_{i=1}^{I} \mathbf{c}^{i*} = \sum_{i=1}^{I} \mathbf{y}^{i}.$$

From the second condition of the theorem,

$$\sum_{i=1}^{I} \tilde{z}_s^i = \underbrace{\sum_{i=1}^{I} (c_s^{i*} - y_s^i)}_{=0} = 0, \quad s = 1, \ldots, S.$$

So the AS markets also clear:

$$\sum_{i=1}^{I} \tilde{\mathbf{z}}^i = \mathbf{0}.$$

Q.E.D.

Next, we consider a less peculiar asset structure. There are K assets $k = 1, \ldots, K$ with prices p_k in t and payoffs a_{sk} in state s at $t + 1$. The payoffs are summarized in the payoff matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1K} \\ \vdots & \ddots & \vdots \\ a_{S1} & \cdots & a_{SK} \end{pmatrix}.$$

The k-th column gives asset k's payoffs in the S states, the s-th row gives the K assets' payoffs in state s.
The set of assets is exogenously given – there is no "financial engineering".

Let $\mathbf{z} = (z_1, \ldots, z_K)$ denote a portfolio.

$$\mathbf{Az} = \begin{pmatrix} a_{11} & \ldots & a_{1K} \\ \vdots & \ddots & \vdots \\ a_{S1} & \ldots & a_{SK} \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_K \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{K} a_{1k} z_k \\ \vdots \\ \sum_{k=1}^{K} a_{Sk} z_k \end{pmatrix}$$

gives the payoffs generated the portfolio in the S states.

Definition: *The financial market is **complete** if* **A** *contains S linearly independent payoff vectors.*

Market completeness implies that for all $\mathbf{x} \in \mathbb{R}^S$, there is $\mathbf{z}$ such that

$$\mathbf{Az} = \mathbf{x}.$$

Let the assets be ordered such that the first S payoff vectors ($k = 1, \ldots, S$) are independent. Then for all $\mathbf{x} \in \mathbb{R}^S$, there is $\mathbf{z} = (z_1, \ldots, z_S, 0, \ldots, 0)$ such that $\mathbf{Az} = \mathbf{x}$.

In theory, the condition for market completeness is not too demanding: it can be shown that it is satisfied if there is one asset with a different payoff in each state s and it is possible to write put or call options on this asset.

Giving up the matrix notation, the completeness condition can be written as

$$\sum_{k=1}^{K} a_{sk} z_k = x_s, \quad s = 1, \dots, S.$$

Consumer *i*'s budget constraints become:

$$c_0^i - y_0^i \leq -\sum_{k=1}^{K} p_k z_k^i$$

$$c_s^i - y_s^i \leq \sum_{k=1}^{K} a_{sk} z_k^i, \quad s = 1, \ldots, S.$$

Let $\mathbf{p} = (p_1, \ldots, p_K)$ denote the asset price vector.

Definition: *An allocation* $(\mathbf{c}^i)_{i=1}^I$*, asset holdings* $(\mathbf{z}^i)_{i=1}^I$*, and a vector of asset prices* $\mathbf{p}$ *are an* ***equilibrium with complete financial markets (ECFM)*** *if the financial market is complete,*

- $(\mathbf{c}^i, \mathbf{z}^i)$ *maximizes* U^i *subject to the individual's budget constraints for all consumers* i *and*
- *markets clear:*

$$\sum_{i=1}^{I} \mathbf{c}^i = \sum_{i=1}^{I} \mathbf{y}^i$$

$$\sum_{i=1}^{I} \mathbf{z}^i = \mathbf{0}.$$

Theorem (ECFM and ECAS): *Let* $((\mathbf{c}^{i*}, \tilde{\mathbf{z}}^{i*})_{i=1}^{I}, \tilde{\mathbf{p}})$ *be an ECAS. Let*

$$p_k = \sum_{s=1}^{S} \tilde{p}_s a_{sk}, \quad k = 1, \ldots, K.$$

Suppose the financial market is complete, and for all $i = 1, \ldots, I$*, let* $\mathbf{z}^{i*} = (z_1^{i\,*}, \ldots, z_S^{i\,*}, 0, \ldots, 0)$ *denote a solution to*

$$\sum_{k=1}^{S} a_{sk} z_k^{i*} = \tilde{z}_s^{i*}, \quad s = 1, \ldots, S.$$

Then $((\mathbf{c}^{i*}, \mathbf{z}^{i*})_{i=1}^{I}, \mathbf{p})$ *is an ECFM.*

Proof: Let $B^{i\prime}$ denote the set of affordable consumption vectors $\mathbf{c}^i$ for i given ECAS prices $\tilde{\mathbf{p}}$, and let $B^{i\prime\prime}$ denote the set of affordable consumption vectors $\mathbf{c}^i$ for i with complete financial markets. We show that

$$\mathbf{c}^{i*} \in B^{i\prime\prime} \subseteq B^{i\prime}.$$

That is, i can afford $\mathbf{c}^{i*}$, and there cannot be a consumption vector he likes better, because he would have chosen it in the ECAS.

The fact that i chooses $(\mathbf{c}^{i*}, \tilde{\mathbf{z}}^{i*})$ in the ECAS implies that it satisfies his budget constraints for the AS economy with equality:

$$c_0^{i*} - y_0^i = -\sum_{s=1}^{S} \tilde{p}_s \tilde{z}_s^{i*}$$

$$c_s^{i*} - y_s^i = \tilde{z}_s^{i*}, \quad s = 1, \ldots, S.$$

The existence of the portfolio $\mathbf{z}^{i*}$ in the theorem is implied by market completeness. This portfolio reproduces the payoffs of the ASs portfolio.

Substituting the formula for $\tilde{z}_s^{i*}$ in the theorem into these constraints and using the pricing formula in the theorem and $z_k^{i\,*} = 0$ for $k = S+1, \ldots, K$ in the first constraint proves that the budget equations for the economy with CFMs are satisfied. This proves $\mathbf{c}^{i*} \in B^{i\prime\prime}$.

Suppose $\mathbf{c}^i \in B^{i\prime\prime}$. That is, there is a portfolio $\mathbf{z}^i$ (not necessarily with zero holdings of assets $k = S+1, \ldots, K$) such that $(\mathbf{c}^i, \mathbf{z}^i)$ satisfies

$$c_0^i - y_0^i \leq -\sum_{k=1}^{K} p_k z_k^i$$

$$c_s^i - y_s^i \leq \sum_{k=1}^{K} a_{sk} z_k^i, \quad s = 1, \ldots, S.$$

Let $\tilde{z}_s^i = \sum_{k=1}^{K} a_{sk} z_k^i$. Using this and the pricing formula in the theorem, it follows that $(\mathbf{c}^i, \tilde{\mathbf{z}}^i)$ satisfies the budget equations for the economy with ASs. That is, $\mathbf{c}^i \in B^{i\prime}$. This proves $B^{i\prime\prime} \subseteq B^{i\prime}$.

Obviously, the goods markets clear.
As for the asset markets, let $z_k^* = \sum_{i=1}^{I} z_k^{i*}$. Asset market clearing in the economy with ASs implies

$$0 = \sum_{i=1}^{I} \tilde{z}_s^{i*} = \sum_{i=1}^{I} \sum_{k=1}^{S} a_{sk} z_k^{i*} = \sum_{k=1}^{S} a_{sk} z_k^*, \quad s = 1, \ldots, S.$$

So

$$\begin{pmatrix} a_{11} & \cdots & a_{1S} \\ \vdots & \ddots & \vdots \\ a_{S1} & \cdots & a_{SS} \end{pmatrix} \begin{pmatrix} z_1^* \\ \vdots \\ z_S^* \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{S} a_{1k} z_k^* \\ \vdots \\ \sum_{k=1}^{S} a_{Sk} z_k^* \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

From linear independence of the S payoff vectors in the matrix, it follows that the only solution to this system of equations is $(z_1^*, \ldots, z_S^*) = (0, \ldots, 0)$. ${z_k^i}^* = 0$ for $k = S+1, \ldots, K$ implies that the other asset markets also clear. Q.E.D.

8 Complete markets: Asset pricing

Using the utility function

$$U^i = u^i(c_0^i) + \beta^i \sum_{s=1}^{S} \pi_s u^i(c_s^i),$$

we can also generalize the fundamental asset pricing equations to the S-states case.

To do so, start with the economy with ASs:

$$U^i = u^i\left(y_0^i - \sum_{s=1}^{S} \tilde{p}_s \tilde{z}_s^i\right) + \beta^i \sum_{s=1}^{S} \pi_s u^i(y_s^i + \tilde{z}_s^i).$$

Necessary optimality conditions:

$$-(u^i)'(c_0^i)\tilde{p}_s + \beta^i \pi_s (u^i)'(c_s^i) = 0$$

or

$$\tilde{p}_s = \pi_s \beta^i \frac{(u^i)'(c_s^i)}{(u^i)'(c_0^i)} = \pi_s M_s,$$

where the SDF M is defined as before. As in the model with only a safe and one risky asset, the fact that M_s is uniform across individuals i for all s follows from the fact that consumers adjust their MRS to asset prices.

Let $\tilde{a}_s$ denote the random payoff of AS s, i.e., 1 in state s and 0 in all other states. Then

$$\tilde{p}_s = \sum_{s' \neq s} \pi_{s'} M_{s'} \cdot 0 + \pi_s M_s \cdot 1$$

or simply

$$\tilde{p}_s = E_t(M \tilde{a}_s).$$

This generalizes the fundamental pricing formula for a risky asset.

Now consider the economy with a complete set of FMs. The fact that the consumption vectors $\mathbf{c}^{i*}$ are the same in an ECFM as in an ECAS implies that the uniform SDF $M_s = \beta^i (u^i)'(c_s^{i*})/(u^i)'(c_0^{i*})$ is also the same. From the pricing rule $p_k = \sum_{s=1}^S \tilde{p}_s a_{sk}$, we get

$$p_k = \sum_{s=1}^{S} \pi_s M_s a_{sk}$$

or, letting a_k denote the random payoff of asset k:

Theorem (The fundamental equations of asset pricing):
Let the utility function be given by

$$U^i = u^i(c_0^i) + \beta^i \sum_{s=1}^{S} \pi_s u^i(c_s^i),$$

where u^i is twice differentiable, with $(u^i)'(c) > 0 > (u^i)''(c)$ for all $c > 0$ and $(u^i)'(0) = \infty$. Then at an ECFM prices obey

$$p_k = E_t\,(M a_k)$$

and, in particular, $p_k = E_t(M)$ for a safe asset with $a_{sk} = 1$ for all $s = 1, \ldots, S$.

It follows that the applications in Section 6 go through without any modification in the economy with more than two states and assets with arbitrary payoff vectors.

Define

$$q_s \equiv (1+r)M_s\pi_s,$$

so that $\sum_{s=1}^{S} q_s = 1$. Then the pricing equation can be rewritten as

$$p_k = \frac{\sum_{s=1}^{S} q_s a_{sk}}{1+r}.$$

That is, the q_s's can be interpreted as risk-neutral probabilities: the price of the asset is its discounted expected value, where the q_s's are used to calculate the expectation.

An important property of the ECAS and the ECFM is that there are no arbitrage opportunities, i.e., it is not possible to form a portfolio that

- has non-positive cost and
- has a non-negative payoff in all states

with one “inequality” strict.
This is obvious for the economy with ASs. A portfolio that has a positive payoff in some state costs something. A portfolio with negative cost has a short position in some AS and, hence, a negative payoff in some state.

Turning to the economy with CFMs, we formalize the definition above:

Definition: *Asset prices* $\mathbf{p}$ *are* ***arbitrage-free*** *if there is no portfolio* $\mathbf{z} \in \mathbb{R}^K$ *such that* $\mathbf{pz} \leq 0$*,* $\mathbf{Az} \geq \mathbf{0}$*, and either* $\mathbf{Az} \neq \mathbf{0}$ *or* $\mathbf{pz} < 0$*.*

Theorem (Arbitrage-freeness of the ECFM): *ECFM prices are arbitrage-free.*

Proof: From the first two conditions and the pricing formula for p_k,

$$\sum_{k=1}^{K} a_{sk} z_k \geq 0, \quad s = 1, \ldots, S,$$

and

$$\sum_{s=1}^{S} \tilde{p}_s \sum_{k=1}^{K} a_{sk} z_k \leq 0.$$

If $\mathbf{Az} \neq \mathbf{0}$, then one of the former equalities is strict, which contradicts the latter inequality.
If $\mathbf{pz} < 0$, then $\sum_{k=1}^{K} a_{sk} z_k$ must be negative for some s, which contradicts the former set of inequalities. Q.E.D.

Another important theorem is:

The fundamental theorem of asset pricing: If asset prices $\mathbf{p} > \mathbf{0}$ are arbitrage-free, then there exist non-negative state prices $\tilde{\mathbf{p}} = (\tilde{p}_1, \ldots, \tilde{p}_S)$ such that

$$p_k = \sum_{s=1}^{S} \tilde{p}_s a_{sk}, \quad k = 1, \ldots, K.$$

The proof requires the use of a separating hyperplane theorem and can be found in the slides “Financial Economics: Risk Sharing and Asset Pricing in General Equilibrium II”©.

Literature:

- Magill, Michael, and Martine Quinzii, *Theory of Incomplete Markets*, MIT (2002), Section 10.

FIRMS

9 Stock market economy

The aim of this section is to show that the results for the exchange economy carry over to a model with firms. This also allows addressing questions about shareholder aims and capital structure.
For simplicity, we don't consider production decisions, but take firms' outputs as exogenously given.

There are J firms $j = 1, \ldots, J$ with exogenous outputs $\tilde{y}_s^j$ in state s in period $t+1$.
The firms are 100 percent owned by the consumers i and (for now) don't issue debt. $x\%$ of firm j's shares entitle the holder to $x\%$ of the firm's revenue $\tilde{y}_s^j$ in any date-$t+1$ state s.
Consumer i is endowed with a share $\bar{\theta}^{ij}$ in firm j, where $\bar{\theta}^{ij} \geq 0$ and $\sum_{i=1}^{I} \bar{\theta}^{ij} = 1$ for all j.
In period t, consumers have endowments y_0^i, and there's no production.

There are markets for the shares and for a complete set of ASs. The value of firm j as of time t is denoted v^j and the vector of firm values as $\mathbf{v} = (v^1, \ldots, v^J)$. i's post trade share holdings are denoted θ^{ij}. We also write $\boldsymbol{\theta}^i = (\theta^{i1}, \ldots, \theta^{iJ})$, $\bar{\boldsymbol{\theta}}^i = (\bar{\theta}^{i1}, \ldots, \bar{\theta}^{iJ})$, and

$$\mathbf{y} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_S \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{I} y_0^i \\ \sum_{j=1}^{J} \tilde{y}_1^j \\ \vdots \\ \sum_{j=1}^{J} \tilde{y}_S^j \end{pmatrix}.$$

We call this a ***stock market economy (SME)*** and analyze it by comparing it to the exchange economy with ASs.

The SME budget constraints read:

$$c_0^i - y_0^i \leq -\sum_{s=1}^{S} \tilde{p}_s \tilde{z}_s^i - \sum_{j=1}^{J} (\theta^{ij} - \bar{\theta}^{ij}) v^j$$

$$c_s^i \leq \tilde{z}_s^i + \sum_{j=1}^{J} \theta^{ij} \tilde{y}_s^j, \quad s = 1, \ldots, S.$$

Definition: *An allocation* $(\mathbf{c}^i)_{i=1}^I$*, AS holdings* $(\tilde{\mathbf{z}}^i)_{i=1}^I$*, shareholdings* $(\theta^i)_{i=1}^I$*, a vector of AS prices* $\tilde{\mathbf{p}}$*, and a vector of firm values* $\mathbf{v}$ *are an* ***equilibrium of the stock market economy (ESME)*** *if*

- $(\mathbf{c}^i, \tilde{\mathbf{z}}^i, \theta^i)$ *maximizes* U^i *subject to the individual's budget constraints for all consumers i and*
- *markets clear:*

$$\sum_{i=1}^{I} \mathbf{c}^i = \mathbf{y}$$

$$\sum_{i=1}^{I} \tilde{\mathbf{z}}^i = \mathbf{0}$$

$$\sum_{i=1}^{I} \theta^i = \mathbf{1}.$$

Theorem (ESME and ECAS): *Let* $(\mathbf{c}^{i*}, \tilde{\mathbf{z}}^{i*})_{i=1}^{I}$ *and* $\tilde{\mathbf{p}}$ *be an ECAS of the exchange economy with endowments*

$$y_s^i = \sum_{j=1}^{J} \bar{\theta}^{ij} \tilde{y}_s^j, \quad s = 1, \ldots, S,\ i = 1, \ldots, I,$$

and let

$$v^j = \sum_{s=1}^{S} \tilde{p}_s \tilde{y}_s^j, \quad j = 1, \ldots, J.$$

Then $(\mathbf{c}^{i*}, \tilde{\mathbf{z}}^{i*}, \bar{\theta}^i)_{i=1}^{I}$ *and* $(\tilde{\mathbf{p}}, \mathbf{v})$ *are an ESME.*

That is, given arbitrage-free pricing of the payoffs generated by the firms, the SME behaves like the exchange economy in which individuals are endowed with what their initial shareholdings are worth.

Proof: Let $B^{i'''}$ be the set of consumption vectors attainable in the SME. As usual by now, we show

$$\mathbf{c}^{i*} \in B^{i'''} \subseteq B^{i'}.$$

The fact that i chooses $(\mathbf{c}^{i*}, \tilde{\mathbf{z}}^{i*})$ in the ECAS implies that it satisfies the budget constraints of the exchange economy with a complete set of ASs with equality:

$$c_0^{i*} - y_0^i = -\sum_{s=1}^{S} \tilde{p}_s \tilde{z}_s^{i*}$$

$$c_s^{i*} - y_s^i = \tilde{z}_s^{i*}.$$

Using the definition of y_s^i in the theorem, it follows that $(\mathbf{c}^{i*}, \tilde{\mathbf{z}}^{i*}, \bar{\theta}^i)$ satisfies the SME budget constraints, i.e., $\mathbf{c}^{i*} \in B^{i'''}$.

Let $\mathbf{c}^i \in B^{i'''}$. That is, there are $\tilde{\mathbf{z}}^i$ and θ^i such that the SME budget constraints hold. Let $\tilde{\mathbf{z}}^{i\prime}$ be given by

$$\tilde{z}^{i\,\prime}_s = \tilde{z}^i_s + \sum_{j=1}^{J} (\theta^{ij} - \bar{\theta}^{ij}) \tilde{y}^j_s, \quad s = 1, \ldots, S.$$

Then $(\mathbf{c}^i, \tilde{\mathbf{z}}^{i\prime}) \in B^{i\prime}$.

The fact that the period-t budget constraint is satisfied follows from the period-t budget constraint for the SME and the definition of v_t^j in the theorem:

$$\begin{aligned}
c_0^i - y_0^i &\leq -\sum_{s=1}^{S} \tilde{p}_s \tilde{z}_s^i - \sum_{j=1}^{J} (\theta^{ij} - \bar{\theta}^{ij}) v^j \\
&= -\sum_{s=1}^{S} \tilde{p}_s \tilde{z}_s^i - \sum_{j=1}^{J} (\theta^{ij} - \bar{\theta}^{ij}) \sum_{s=1}^{S} \tilde{p}_s \tilde{y}_s^j \\
&= -\sum_{s=1}^{S} \tilde{p}_s \left[\tilde{z}_s^i + \sum_{j=1}^{J} (\theta^{ij} - \bar{\theta}^{ij}) \tilde{y}_s^j \right] \\
&= -\sum_{s=1}^{S} \tilde{p}_s \tilde{z}_s^{i\prime}.
\end{aligned}$$

The period-$t+1$ budget constraints also hold:

$$\begin{aligned} c_s^i &\leq \tilde{z}_s^i + \sum_{j=1}^{J} \theta^{ij} \tilde{y}_s^j \\ &= \tilde{z}_s^{i\,\prime} + \sum_{j=1}^{J} \bar{\theta}^{ij} \tilde{y}_s^j \\ &= \tilde{z}_s^{i\,\prime} + y_s^i \\ c_s^i - y_s^i &\leq \tilde{z}_s^{i\,\prime}. \end{aligned}$$

So $\mathbf{c}^i \in B^{i\prime}$ and $B^{i\prime\prime\prime} \subseteq B^{i\prime}$.
The validity of the market clearing conditions in the SME is obvious. Q.E.D.

So the SME behaves essentially like the exchange economy considered before. So if an ESME replicates an ECAS which in turn replicates a ECCM, then the ESME allocation $\mathbf{c}^*$ is Pareto-optimal, since it coincides with the EECM allocation, which is Pareto-optimal due to the 1st welfare theorem. Households do not trade shares at the ESME in the theorem. There are other ESMEs, with trade in shares, however:

Theorem (ESME with trade in shares): *Let* $((\mathbf{c}^{i*}, \tilde{\mathbf{z}}^{i*}, \bar{\boldsymbol{\theta}}^{i})_{i=1}^{I}, \tilde{\mathbf{p}}, \mathbf{v})$ *be an ESME. Suppose household i changes his stock demands by* $\mathbf{d}\boldsymbol{\theta}^{i} = (d\theta^{i1}, \ldots, d\theta^{iJ})$ *(not necessarily small) and his AS demands by*

$$d\tilde{z}_s^i = -\sum_{j=1}^{J} \tilde{y}_s^j d\theta^{ij},$$

where $\sum_{i=1}^{I} d\theta^{ij} = 0$ *for all j. Then* $((\mathbf{c}^{i*}, \tilde{\mathbf{z}}^{i*} + \tilde{\mathbf{dz}}^{i}, \bar{\boldsymbol{\theta}}^{i} + \mathbf{d}\boldsymbol{\theta}^{i})_{i=1}^{I}, \tilde{\mathbf{p}}, \mathbf{v})$ *is also an ESME.*

Proof: From the budget constraints, these asset reallocations do not affect consumption. So if $(\mathbf{c}^{i*}, \tilde{\mathbf{z}}^{i*}, \bar{\boldsymbol{\theta}}^{i})$ maximizes utility, so does $(\mathbf{c}^{i*}, \tilde{\mathbf{z}}^{i*} + \tilde{\mathbf{dz}}^{i}, \bar{\boldsymbol{\theta}}^{i} + \mathbf{d}\boldsymbol{\theta}^{i})$. Markets clear. Q.E.D.

Irrespective of their subjective risk attitudes, shareholders in the SME are unanimous with regard to what the firms they own shares in should do: maximize shareholder value.

Theorem (Shareholder Unanimity): *An increase in v^j expands the budget set of each individual i with $\bar{\theta}^{ij} > 0$.*

Proof: Substituting for $\tilde{z}_s^i$ from the period-$t+1$ budget constraints into the period-t budget constraint yields

$$
\begin{aligned}
c_0^i - y_0^i &\leq -\sum_{s=1}^{S} \tilde{p}_s \left(c_s^i - \sum_{j=1}^{J} \theta^{ij} \tilde{y}_s^j \right) - \sum_{j=1}^{J} (\theta^{ij} - \bar{\theta}^{ij}) v^j \\
&= -\sum_{s=1}^{S} \tilde{p}_s c_s^i + \sum_{j=1}^{J} \theta^{ij} \underbrace{\sum_{s=1}^{S} \tilde{p}_s \tilde{y}_s^j}_{=v^j} - \sum_{j=1}^{J} (\theta^{ij} - \bar{\theta}^{ij}) v^j \\
&= -\sum_{s=1}^{S} \tilde{p}_s c_s^i + \sum_{j=1}^{J} \bar{\theta}^{ij} v^j .
\end{aligned}
$$

So if v^j rises, the budget sets of the initial shareholders (individuals i with $\bar{\theta}^{ij} > 0$) expand. Q.E.D.

In the model as it stands, the result that firms should maximize shareholder value is of little significance because firms do not make any decisions that affect value.

But suppose for each firm j there is a set of different output vectors $(\tilde{y}^j_1, \ldots, \tilde{y}^j_S)$ from which it has to choose one. Then the unanimity theorem states that, in the interest of its initial shareholders, it should maximize shareholder value.

The shareholder value maximization doctrine is one of the most disputed guidelines derived from economic theory. It is the basis of the dichotomy that firms should make money, constrained by rules made by governments. Famously advocated by Milton Friedman ("conduct the business in accordance with [shareholders'] desires, which generally will be to make as much money as possible while conforming to the basic rules of the society"), it has been attacked for various economic and ideological reasons. The basic idea behind it is certainly harder to attack than the doctrine interpreted as a strict guideline that allows for no deviations.

https://review.chicagobooth.edu/economics/2017/article/it-s-time-rethink-milton-friedman-s-shareholder-value-argument

Literature:

- Schechter, Asher (2017), “Where Friedman was wrong”, *ChicagoBoothReview*, Dec. 7, 2017.

10 The Modigliani-Miller theorem

The Modigliani-Miller (MM) theorem states that the division of a firm's cash flow between different kinds of financial claims (i.e., its capital structure) is irrelevant for its value.

This section proves a much more general version of the MM theorem (due to Stiglitz, 1969), which proves that firms' capital structure is irrelevant for real economic activity in general – "finance is a veil".

In doing so, we neglect the issue of investment finance, i.e., how the choice of different financial instruments affects investment activity (as is well known, capital structure is not irrelevant in this regard in the presence of information asymmetries).

Without uncertainty, the value of a firm's payments to stockholders ($\tilde{y}^j - b^j$) and to creditors (b^j) at $t+1$ is $\tilde{y}^j/(1+r)$, where r is the safe interest rate. This is the same as in the case of no debt because the present value debt is zero.

With uncertainty, let a single firm j take on debt b^j. The value of the firm's payments to stockholders ($\tilde{y}^j - b^j$) and creditors (b^j) at $t+1$ is

$$\sum_{s=1}^{S} \tilde{p}_s(\tilde{y}_s^j - b^j) + p_b b^j.$$

Since the price of debt is

$$p_b = \sum_{s=1}^{S} \tilde{p}_s,$$

this coincides with

$$v^j = \sum_{s=1}^{S} \tilde{p}_s \tilde{y}_s^j,$$

i.e., the value of the unlevered firm .

Now the general case. Let b^j denote the debt issued by firm j. b^j is exogenous, so we can discuss the effects of exogenous changes in the firms' capital structure. Let b^i denote the debt held by individual i. Shares θ^{ij} in j entitle the holder to a fraction θ^{ij} of the proceeds of j's debt issue $p_b b^j$ at t, in addition to the same share in its date-$t+1$ profits (in a model with investment this would translate into a reduction in the contributions to the investment outlays). We rule out bankruptcy:

$$\tilde{y}^j_s \geq b^j, \quad s = 1, \ldots, S,\ j = 1, \ldots, J.$$

This means that any two firms' debt obligations are perfect substitutes for the consumers. We call this the **stock and debt economy (SDE)**.

i's budget constraints become

$$c_0^i - y_0^i \leq -\sum_{s=1}^{S} \tilde{p}_s \tilde{z}_s^i - \sum_{j=1}^{J} (\theta^{ij} - \bar{\theta}^{ij}) v^j - p_b b^i + \sum_{j=1}^{J} \theta^{ij} p_b b^j$$

$$c_s^i \leq \tilde{z}_s^i + \sum_{j=1}^{J} \theta^{ij} (\tilde{y}_s^j - b^j) + b^i,$$

where v^j is the value at which the shares of firm j trade in the stock market.

Definition: *An allocation* $(\mathbf{c}^i)_{i=1}^I$*, AS holdings* $(\tilde{\mathbf{z}}^i)_{i=1}^I$*, shareholdings* $(\theta^i)_{i=1}^I$*, debt holdings* $(b^i)_{i=1}^I$*, a vector of AS prices* $\tilde{\mathbf{p}}$*, a vector of firm values* $\mathbf{v}$*, and a price of debt* p_b *are an* ***equilibrium of the stock and debt economy (ESDE)*** *if*

- $(\mathbf{c}^i, \tilde{\mathbf{z}}^i, \theta^i, b_t^i)$ *maximizes* U^i *subject to the individual's budget constraints for all consumers* i *and*
- *markets clear:*

$$\sum_{i=1}^{I} \mathbf{c}^i = \mathbf{y}$$

$$\sum_{i=1}^{I} \tilde{\mathbf{z}}^i = \mathbf{0}$$

$$\sum_{i=1}^{I} \theta^i = \mathbf{1}$$

$$\sum_{i=1}^{I} b^i = \sum_{j=1}^{J} b^j.$$

Theorem (Modigliani-Miller Theorem): *Let* $((\mathbf{c}^{i*}, \tilde{\mathbf{z}}^{i*}, \theta^{i*})_{i=1}^{I}, (\tilde{\mathbf{p}}, \mathbf{v}))$ *be an ESME. Let*

$$p_b = \sum_{s=1}^{S} \tilde{p}_s$$

and

$$b^{i*} = \sum_{j=1}^{J} \theta^{ij*} b^j, \quad i = 1, \ldots, I.$$

Then $((\mathbf{c}^{i*}, \tilde{\mathbf{z}}^{i*}, \theta^{i*}, b^{i*})_{i=1}^{I}, \tilde{\mathbf{p}}, \mathbf{v}, p_b)$ *is an ESDE.*

Proof: Let $B^{i''''}$ denote the set of affordable $\mathbf{c}^i$'s in the SDE. As usual, we show

$$\mathbf{c}^{i*} \in B^{i'''''} \subseteq B^{i''''}.$$

Since $\mathbf{c}^{i*} \in B^{i'''}$, the budget constraints for the SME hold with equality:

$$c_0^{i*} - y_0^i = -\sum_{s=1}^{S} \tilde{p}_s \tilde{z}_s^{i*} - \sum_{j=1}^{J} (\theta^{ij*} - \bar{\theta}^{ij}) v^j$$

$$c_s^{i*} = \tilde{z}_s^{i*} + \sum_{j=1}^{J} \theta^{ij*} \tilde{y}_s^j.$$

$(\mathbf{c}^{i*}, \tilde{\mathbf{z}}^{i*}, \boldsymbol{\theta}^{i*}, b^{i*})$ satisfies the SDE budget constraints. This follows from the observation that the additional terms on the right-hand sides drop out by virtue of the equation for b^{i*} in the theorem. So $\mathbf{c}^{i*} \in B^{i'''''}$.

Suppose $\mathbf{c}^i \in B^{i''''}$, so there is $(\mathbf{c}^i, \tilde{\mathbf{z}}^i, \boldsymbol{\theta}^i, b^i)$ that satisfies the budget constraints of the SDE. Suppose, in the SME, i buys the same amounts of shares θ^{ij} and

$$\tilde{z}_s^{i\,\prime} = \tilde{z}_s^i + b^i - \sum_{j=1}^{J} \theta^{ij} b^j$$

ASs.
Using the pricing formula for debt in the theorem, it follows that $(\mathbf{c}^i, \tilde{\mathbf{z}}^{i\prime}, \boldsymbol{\theta}^i)$ satisfies the date-t budget constraint in the SME:

$$
\begin{aligned}
c_0^i - y_0^i &\leq -\sum_{s=1}^{S} \tilde{p}_s \tilde{z}_s^i - \sum_{j=1}^{J} (\theta^{ij} - \bar{\theta}^{ij}) v^j - p_b b^i + \sum_{j=1}^{J} \theta^{ij} p_b b^j \\
&= -\sum_{s=1}^{S} \tilde{p}_s \tilde{z}_s^i - \sum_{j=1}^{J} (\theta^{ij} - \bar{\theta}^{ij}) v^j - \left(b^i - \sum_{j=1}^{J} \theta^{ij} \right) \sum_{s=1}^{S} \tilde{p}_s \\
&= -\sum_{s=1}^{S} \tilde{p}_s \left(\tilde{z}_s^i + b^i - \sum_{j=1}^{J} \theta^{ij} b^j \right) - \sum_{j=1}^{J} (\theta^{ij} - \bar{\theta}^{ij}) v^j \\
&= -\sum_{s=1}^{S} \tilde{p}_s \tilde{z}_s^{i\,\prime} - \sum_{j=1}^{J} (\theta^{ij} - \bar{\theta}^{ij}) v^j.
\end{aligned}
$$

Analogously, the date-$t+1$ SME budget constraints hold:

$$\begin{aligned} c_s^i &\leq \tilde{z}_s^i + \sum_{j=1}^{J} \theta^{ij}(\tilde{y}_s^j - b^j) + b^i \\ &= \tilde{z}_s^i + b^i - \sum_{j=1}^{J} \theta^{ij} b^j + \sum_{j=1}^{J} \theta^{ij} \tilde{y}_s^j \\ &= \tilde{z}_s^{i\,\prime} + \sum_{j=1}^{J} \theta^{ij} \tilde{y}_s^j. \end{aligned}$$

This proves that $\mathbf{c}^i \in B^{i''''}$ implies $\mathbf{c}^i \in B^{i'''}$, i.e., $B^{i''''} \subseteq B^{i'''}$.

Goods market clearing in the SDE follows from goods market clearing in the ESME.
Since the demands for ASs and stocks are unchanged, the same holds true for the markets for ASs and stocks.
Finally, the debt market also clears:

$$\sum_{i=1}^{I} b^{i*} = \sum_{i=1}^{I} \sum_{j=1}^{J} \theta^{ij*} b^{j} = \sum_{j=1}^{J} b^{j} \underbrace{\sum_{i=1}^{I} \theta^{ij*}}_{=1} = \sum_{j=1}^{J} b^{j}.$$

Q.E.D.

The firm values v^j are unaffected by changes in capital structure. So the "corporate finance MM theorem" is a direct corollary of our "general equilibrium MM theorem".

The intuition behind the theorem is this:

- From (the b^j terms in) the SDE budget constraints, households' consumption vectors would change if they did not react to the change in capital structure (i.e., if they chose $b^i = 0$), viz., by $\sum_{j=1}^{J} \theta^{ij} p_b b^j = p_b b^{i^*}$ in period t and by $-b^{i^*}$ in period $t+1$.

- So the debt holdings in the theorem imply that the effects of the change in capital structure are neutralized: they cost $p_b b^{i^*}$ in period t and yield b^{i^*} in period $t+1$. One can (but need not, due to the perfect substitutes property) assume that the households buy debt issued by the firms of which they hold stocks in proportion to their shareholdings. The first part of the proof of the theorem ($\mathbf{c}^{i*} \in B^{i''''}$) shows that this neutralization strategy is feasible.
- The second part of the proof of the theorem ($B^{i''''} \subseteq B^{i'''}$) shows that this neutralization strategy maximizes utility: any consumption vector that is affordable after the debt issue was also affordable before. The fact that i chose $\mathbf{c}^{i*}$ means that this is still the best choice.

According to the MM theorem, for given investment decisions, firms' financial choices are irrelevant. Put differently, what matters for a firm's value is how it operates and not how it slices the claims to the proceeds. The MM theorem has been criticized on several grounds (Gifford, 1998). Maybe the most important caveat is that when the firm still has to finance the expansion of its operations, asymmetric information between the firm and suppliers of capital makes the choice of the financial instruments used to raise capital relevant. Intermediation, in particular by commercial banks, plays a vital role then, and disintermediation is crucial for the severity of financial crises.

http://pages.stern.nyu.edu/~adamodar/New_Home_Page/articles/MM40yearslater.htm

Literature:

- Gifford, Dun Jr., (1998), "After the Revolution: Forty years ago, the Modigliani-Miller propositions started a new era in corporate finance. How does M&M hold up today?", *CFO Magazine*.
- Modigliani, Franco, and Merton H. Miller (1958), "The Cost of Capital, Corporation Finance, and the Theory of Investment", *American Economic Review* 48, 261-297.
- Stiglitz, Joseph E. (1969), "A Re-Examination of the Modigliani-Miller Theorem", *American Economic Review* 59, 784-793.

11 The Capital Asset Pricing Model

The Capital Asset Pricing Model (CAPM) is usually derived from Markowitz portfolio theory. It can also be derived from our equilibrium model. To do so, assume the consumers' utility functions are additively separable and quadratic (so that they could be represented by a (μ, σ) utility function, as employed in portfolio theory), and each consumer has the same discount factor β:

$$U^i = c_0^i - \frac{b^i}{2}\left(c_0^i\right)^2 + \beta \sum_{s=1}^{S} \pi_s \left[c_s^i - \frac{b^i}{2}\left(c_s^i\right)^2\right].$$

Assume that the b^i's are sufficiently small so that the marginal utilities $1 - b^i c_0^i$ and $1 - b^i c_s^i$ are positive.

Define the rate of return on j's stocks as

$$r_s^j = \frac{\tilde{y}_s^j}{v^j} - 1.$$

Let $v^M = \sum_{j=1}^{J} v^j$ the total market capitalization. Define the return on the market as

$$r_s^M = \frac{y_s}{v^M} - 1.$$

Denote the covariance between the returns of asset j and the market as σ^{jM} and the variance of the market rate of return as σ^{M2}. The safe rate of return is denoted r. Er^j and Er^M are the expected returns of firm j and the market, respectively.

For the sake of simplicity, we return to the assumption of no debt and consider the SME with a complete set of ASs.

Theorem (CAPM): *In an ESME, given quadratic utility and uniform discount factors, the risk premium of firm j is*

$$Er^j - r = \beta^j \left(Er^M - r \right),$$

where

$$\beta^j = \frac{\sigma^{jM}}{(\sigma^M)^2}.$$

Proof: Quadratic utility implies that the AS prices are linear functions of aggregate output $y_s = \sum_{j=1}^{J} \tilde{y}_s^j$ alone. To see this, consider i's MRS between consumption in state s in $t+1$ and consumption in t:

$$-\frac{dc_0^i}{dc_s^i} = \pi_s \beta \frac{1 - b^i c_s^i}{1 - b^i c_0^i}.$$

Each consumer adjusts his MRS to the corresponding AS price $\tilde{p}_s$, so the MRSs of any two consumers i and i' are the same:

$$\pi_s \beta \frac{1 - b^i c_s^i}{1 - b^i c_0^i} = \pi_s \beta \frac{1 - b^{i'} c_s^{i'}}{1 - b^{i'} c_0^{i'}}.$$

Rearranging terms yields

$$c_s^{i'} = \frac{1}{b^{i'}} \left[1 - \frac{1 - b^i c_s^i}{1 - b^i c_0^i} \left(1 - b^{i'} c_0^{i'} \right) \right].$$

Inserting this into the market clearing condition

$$\sum_{i'=1}^{I} c_s^{i'} = y_s,$$

rearranging terms, and using $\sum_{i'=1}^{I} c_0^{i'} = y_0$ yields

$$\frac{1 - b^i c_s^i}{1 - b^i c_0^i} = \frac{\sum_{i'=1}^{I} \frac{1}{b^{i'}} - y_s}{\sum_{i'=1}^{I} \frac{1}{b^{i'}} - y_0}.$$

Equality of the MRS and the AS price for state s implies

$$\tilde{p}_s = \pi_s(a - by_s),$$

where

$$a = \beta \frac{\sum_{i'=1}^{I} \frac{1}{b^{i'}}}{\sum_{i'=1}^{I} \frac{1}{b^{i'}} - y_0}$$

and

$$b = \beta \frac{1}{\sum_{i'=1}^{I} \frac{1}{b^{i'}} - y_0}.$$

An AS is expensive if the underlying state occurs with high probability and aggregate production is low. We now use this formula for the AS prices in order to price stock j, the market, and a riskless bond. This will yield the CAPM formula.

Consider any risky income α_s. The value as of date *t* is

$$v^\alpha = \sum_{s=1}^{S} \tilde{p}_s \alpha_s.$$

The rate of return is defined as

$$r_s^\alpha = \frac{\alpha_s}{v^\alpha} - 1,$$

so that

$$Er^\alpha = \frac{E\alpha}{v^\alpha} - 1.$$

The covariance $\sigma^{\alpha M}$ between r^α and r^M satisfies

$$E[(1 + r^\alpha)(1 + r^M)] = \sigma^{\alpha M} + (1 + Er^\alpha)(1 + Er^M).$$

Hence,

$$
\begin{aligned}
v^{\alpha} &= \sum_{s=1}^{S} \tilde{p}_s \alpha_s \\
&= \sum_{s=1}^{S} \pi_s (a - b y_s) \alpha_s \\
&= aE\alpha - bE(\alpha y) \\
1 &= aE\left(\frac{\alpha}{v^{\alpha}}\right) - b v^M E\left(\frac{\alpha}{v^{\alpha}} \frac{y}{v^M}\right) \\
&= a(1 + Er^{\alpha}) - b v^M \left[\sigma^{\alpha M} + (1 + Er^{\alpha})\left(1 + Er^M\right)\right] \\
&= (1 + Er^{\alpha})\left[a - b v^M \left(1 + Er^M\right)\right] - b v^M \sigma^{\alpha M}.
\end{aligned}
$$

Letting $\alpha_s = 1$, we get

$$\frac{1}{1+r} = a - bv^M \left(1 + Er^M\right).$$

This can be used to write the corresponding expressions for $\alpha_s = \tilde{y}_s^j$ and $\alpha_s = y_s$ as

$$(1+r)bv^M \sigma^{jM} = Er^j - r$$

and

$$(1+r)bv^M (\sigma^M)^2 = Er^M - r.$$

Solving for $Er^j - r$ yields the CAPM formula. Q.E.D.

Consumption-based asset pricing and the standard CAPM are not competing theories of asset pricing. Both are ways to express asset prices in equilibrium.

From the pricing formula $v^j = \sum_{s=1}^{S} \tilde{p}_s \tilde{y}_s^j$, asset j is expensive if it pays high dividends $\tilde{y}_s^j$ in states with a high AS price $\tilde{p}_s$. From the equation for the AS prices $\tilde{p}_s = \pi_s(a - by_s)$ and the expression for the market return $r_s^M = y_s/v^M - 1$, these are states with low aggregate output and low market returns. Hence, j is expensive if the correlation of its output and return with the market are low. The similarity to consumption-based asset pricing is evident.

The CAPM does not perform very well empirically. Fama and French (1992) find a positive but insignificant value for beta using U.S. data for 1963-1990. beta even turns negative (but remains insignificant) if one controls for size. Fama and French propose a three-factor model that relates returns to beta, size, and book-to-market.

https://www.ivey.uwo.ca/cmsmedia/3775518/the_cross-section_of_expected_stock_returns.pdf

Literature:

- Allen, Franklin, and Douglas Gale, *Comparing Financial Systems*, MIT Press (2000), Section 7.2.
- Fama, Eugene F.. and Kenneth R. French (1992), “The Cross-Section of Expected Stock Returns”, *Journal of Finance* 47, 427-465.
- Sharpe, William F. (1964), “Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk”, *Journal of Finance* 19, 425-452.

MULTI-PERIOD MODEL

12 Time and uncertainty

All we had to say about uncertainty so far is that the probabilities π_s of states $s = 1, \ldots, S$ satisfy $\pi_s > 0$ and $\sum_{s=1}^{S} \pi_s = 1$. In the multi-period setup uncertainty unfolds as time proceeds. This raises some subtle questions.

- What does it mean formally that uncertainty unfolds?
- What are agents' expectations of future payoffs, prices, etc.?
- What are agents' expectations of their future expectations?

Let $\mathbb{S} = \{1, 2, \ldots, S\}$ denote the set of states. The basic idea is that at some final date decision makers know the true state $s \in \mathbb{S}$, but earlier on, they have imperfect information.
More concretely, at each date there is a partitioning $\{\sigma_1, \ldots, \sigma_J\}$ of $\mathbb{S}$ (i.e., the σ_j's are disjoint and $\dot{\cup}_j \sigma_j = \mathbb{S}$). Agents know σ_j. This means that they know that at the final date some $s \in \sigma_j$ will be realized. The unfolding of information means that the sets which comprise the partitioning of $\mathbb{S}$ become smaller, i.e., the partitioning becomes finer. Formally, if agents observe σ, then at each later date there is a partitioning $\{\sigma'_1, \ldots \sigma'_J\}$ of σ and agents know σ'_j.
Information is symmetric: at each date, the partitioning $\{\sigma_1, \ldots, \sigma_J\}$ of $\mathbb{S}$ is the same for each agent. (Asymmetric information raises subtle questions about knowing what others know.)

Let x be a random variable, i.e., x takes on the value x_s ($s \in \mathbb{S}$) at the final date. Suppose agents observe σ at some date. The conditional expectation of x is then

$$E(x|\,\sigma) = \sum_{s' \in \sigma} \frac{\pi_{s'}}{\pi_\sigma} x_{s'},$$

where $\pi_\sigma = \sum_{s' \in \sigma} \pi_{s'}$.

What are agents' expectations of their future expectations? Suppose they know σ. At each later date, there is a partitioning $\{\sigma'_1, \dots \sigma'_J\}$ of σ and they will know $\sigma' \in \{\sigma'_1, \dots \sigma'_J\}$. Their expectation at the earlier date of their expectation at the later date is

$$E\left(E(x|\,\sigma')\,|\,\sigma\right) = \sum_{j=1}^{J} \sum_{s \in \sigma'_j} \frac{\pi_s}{\pi_\sigma} E(x|\,\sigma'_j).$$

We are now in a position to state the fundamental result of this section: the law of iterated expectations. The law says that the expectation of future expectations is simply today's expectations. This is not surprising: the detour must not improve the accuracy of forecasts. But it's crucial to any multi-period equilibrium model, in which today's prices depend on expected future prices (resale values), which depend on the future expectation of prices then in the future, and so on.

Theorem (The law of iterated expectations):

$$E(x|\,\sigma) = E\left(E(x|\,\sigma')\,|\,\sigma\right).$$

Let $\pi_{\sigma'_j} = \sum_{s \in \sigma'_j} \pi_s$. Then:

Proof:

$$
\begin{aligned}
E(x|\,\sigma) &= \sum_{s'\in\sigma} \frac{\pi_{s'}}{\pi_\sigma} x_{s'} \\
&= \sum_{j=1}^{J} \sum_{s'\in\sigma'_j} \frac{\pi_{s'}}{\pi_\sigma} x_{s'} \\
&= \sum_{j=1}^{J} \sum_{s'\in\sigma'_j} \frac{\pi_{s'}}{\pi_\sigma} x_{s'} \sum_{s\in\sigma'_j} \frac{\pi_s}{\pi_{\sigma'_j}} \\
&= \sum_{j=1}^{J} \sum_{s\in\sigma'_j} \frac{\pi_s}{\pi_\sigma} \sum_{s'\in\sigma'_j} \frac{\pi_{s'}}{\pi_{\sigma'_j}} x_{s'} \\
&= \sum_{j=1}^{J} \sum_{s\in\sigma'_j} \frac{\pi_s}{\pi_\sigma} E(x|\,\sigma'_j) \\
&= E\left(E(x|\,\sigma')\,|\,\sigma\right).
\end{aligned}
$$

We usually encounter iterated expectations when individuals form expectations at t about their expectations at $t+1$ about the realization x_T of a random variable x at $T > t+1$. They know σ_t and that σ'_{t+1} is in a partitioning $\{\sigma'_{t+1,1}, \ldots, \sigma'_{t+1,J}\}$ of σ_t. We then write

$$E(x_T|\,\sigma_t) = E_t(x_T)$$

and

$$E\left(E(x|\,\sigma'_{t+1}) \mid \sigma_t\right) = E_t[E_{t+1}(x_T)],$$

so that $E_t(x_T) = E_t[E_{t+1}(x_T)]$.

Literature:

- Magill, Michael, and Martine Quinzii, *Theory of Incomplete Markets*, MIT (2002), Section 26.

13 Equilibrium and the fundamental equations of asset pricing

In this section, we show that the fundamental equations of asset pricing derived in the two-period economy hold true without any modification in a multi-period exchange economy. We return to an exchange economy with one consumption good. The time horizon T (≥ 1) is finite. (With an infinite horizon the question of whether asset prices may contain bubble components becomes harder to answer.)

Let $\mathbf{c}_t^i = (c_{1,t}^i, \dots, c_{S,t}^i)$ denote the vector of realizations of his random date-t consumption c_t^i. Only one of the components of $\mathbf{c}_t^i$ is realized at t, but i has to contemplate different possibilities when he makes decisions prior to date t). The endowment vector $\mathbf{y}_t^i$ is defined analogously.
In any state s, aggregate utility from date t onwards is

$$U_{s,t}^i(\mathbf{c}_t^i, \dots, \mathbf{c}_T^i) = \sum_{\tau=t}^{T} (\beta^i)^{\tau-t} E_t \left[u^i(c_\tau^i) \right].$$

There are K assets, all in zero net supply. Asset k pays a dividend $a_{sk,t}$ and is traded at price $p_{sk,t}$ in state s at date t. The vector of date-t asset prices is denoted $\mathbf{p}_t = (p_{11,t}, \ldots, p_{sk,t}, \ldots, p_{SK,t})$.
i's date-t state-s position in asset k is denoted $z^i_{sk,t}$, his date-t portfolio is $\mathbf{z}^i_t = (z^i_{11,t}, \ldots, z^i_{sk,t}, \ldots, z^i_{SK,t})$. i's state-s date-t budget constraints are:

$$c^i_{s,t} + \sum_{k=1}^{K} p_{sk,t}\left(z^i_{sk,t} - z^i_{sk,t-1}\right) \leq y^i_{s,t} + \sum_{k=1}^{K} a_{sk,t} z^i_{sk,t-1},$$

with $p_{sk,T} = 0$ for all s and all k.

In the two-period model, the reason why the necessary optimality conditions for utility maximization give rise to a single SDF is that the MRSs are uniform across individuals because consumers adjust their MRS to asset prices or because of Pareto optimality. To simplify things, we now assume that all individuals are alike. They have

- the same utility function u,
- the same subjective discount factor β,
- at each date t, the same endowment y_t (y_t may vary across time), and
- the same information.

In this setup, the MRSs are uniform across individuals simply because all individuals are alike and choose the same consumption levels. We drop the superscript i.

Consider an equilibrium in which individuals consume their endowments and do not invest in any asset:

Definition: $\mathbf{c}_t = \mathbf{y}_t$, $\mathbf{z}_t = \mathbf{0}$, *and* $\mathbf{p}_t$, $t = 0, \ldots, T$, *are a* **no-trade equilibrium (NTE)** *if* $c_{s,t} = y_{s,t}$ *and* $z_{sk,t} = 0$ *solve the utility maximization problem for all* $s = 1, \ldots, S$, $k = 1, \ldots, K$, *and* $t = 0, 1, \ldots, T$.

An NTE looks trivial, since it leads to the same allocation as no markets at all. However, the question is: how do asset prices have to adjust in order to induce the individuals to optimally choose not to trade?

Theorem (NTE): If $(\mathbf{c}_t, \mathbf{z}_t, \mathbf{p}_t)_{t=0}^{T}$ is an NTE if, then

$$p_{sk,t} = E_t\left[\beta \frac{u'(y_{t+1})}{u'(y_{s,t})}(p_{k,t+1} + a_{k,t+1})\right].$$

Proof: Suppose not. Let a consumer change his holdings of asset k in s at t by $dz_{sk,t} \neq 0$, holding all other current and future asset demands constant. Date-t consumption changes by $dc_{s,t} = -p_{sk,t}dz_{sk,t}$ and date-$t+1$ consumption by $dc_{t+1,s} = (p_{sk,t+1} + a_{sk,t+1})dz_{sk,t}$.

The induced change in utility is

$$\begin{aligned} dU^i_{s,t} &= -u'(y_{s,t})p_{sk,t}dz_{sk,t} \\ &\quad +E_t\left[\beta u'(y_{t+1})(p_{k,t+1}+a_{k,t+1})\right]dz_{sk,t} \\ &= -u'(y_{s,t})dz_{sk,t}\left\{p_{sk,t}-E_t\left[\beta\frac{u'(y_{t+1})}{u'(y_{s,t})}(p_{k,t+1}+a_{k,t+1})\right]\right\} \\ &\neq 0. \end{aligned}$$

This contradicts utility maximization. Q.E.D.

Letting

$$M_{t+1,s} = \beta \frac{u'(y_{t+1,s})}{u'(y_t)},$$

one gets the fundamental asset pricing equations:

$$p_{k,t} = E_t \left[M_{t+1}(p_{k,t+1} + a_{k,t+1}) \right].$$

The pricing formula for a riskless one-period bond is obtained by setting $p_{k,t} = 1/(1 + r_{t+1})$, $p_{k,t+1} = 0$, and $a_{k,t+1} = 1$. The applications of the fundamental asset pricing equations in Section 6 go through without any modification.

With risk neutrality (i.e., $u'(c) = 1$) the SDF ($M_{t+1} = \beta$) is not stochastic), and

$$p_{k,t} = E_t \left[\beta(p_{k,t+1} + a_{k,t+1})\right].$$

This equation gives a precise statement of the random walk result mentioned in Application 5 in Section 6. If $\beta = 1$, then

$$p_{k,t} = E_t(p_{k,t+1})$$

for each trading date before a date without a dividend payment (i.e., for all t such that $a_{sk,t+1} = 0$ for all s). This is the “no free lunch” hypothesis. It also sometimes called an efficient markets hypothesis.

The evidence on the validity of the no free lunch hypothesis is mixed. Short-run momentum and long-run mean reversion in asset prices contradict the non-predictability of future price movements. Jegadeesh and Titman (1993) show that in the U.S. stock market between 1965 and 1989 buying each month the top-10% performers of the previous six months yielded a 0.8% monthly return. Financing this portfolio by shorting the bottom-10% performers increased the return to 1.0% per month. On the other hand, the fact that even for professional investors it is hard to consistently beat the market over an extended period of time is evidence for no free lunch (Malkiel, 2003).

https://onlinelibrary.wiley.com/doi/epdf/10.1111/1468-036X.00205

Literature:

- Jegadeesh, Narasimhan, and Sheridan Titman (1993), "Returns to Buying Winners and Selling Losers: Implications for Stock Market Efficiency", *Journal of Finance* 48, 65-91.
- Malkiel, Burton G. (2003), "Passive Investment Strategies and Efficient Markets", *European Financial Management* 9, 1-10.

14 Fundamental value

The fundamental equations of asset pricing are expectational difference equations in the multi-period model. Which price sequences satisfy these equations?

In this section, we show that the only price sequence consistent with the fundamental equations is fundamental value, i.e., the present expected value of the asset payoffs. This is a strong version of the efficient markets hypothesis.

We start with the case of risk neutrality (i.e. $u'(c) = 1$). As shown above, $M_{t+1} = \beta$ and

$$p_{k,t} = E_t \left[\beta (p_{k,t+1} + a_{k,t+1}) \right] .$$

Theorem (Fundamental value with risk neutrality): If consumers are risk-neutral, asset prices are uniquely determined by

$$p_{k,t} = \sum_{\tau=t+1}^{T} \beta^{\tau-t} E_t(a_{k,\tau}).$$

Proof: Uniqueness of the sequence of equilibrium prices follows from the fundamental equation of asset pricing via backward induction: $p_{k,T-1} = \beta E_{T-1}(a_{k,T})$ is unique, so $p_{k,T-2} = \beta E_{T-2}[\beta(a_{k,T-1} + p_{k,T-1})$ is unique, and so on. The formula in the theorem follows via backward induction. The above formula for $p_{k,T-1}$ satisfies the formula in the theorem for $t = T - 1$. Using the law of iterated expectations, we have

$$
\begin{aligned}
p_{k,t} &= E_t\left[\beta(p_{k,t+1} + a_{k,t+1})\right] \\
&= E_t\left[\beta \sum_{\tau=t+2}^{T} \beta^{\tau-(t+1)} E_{t+1}(a_{k,\tau}) + \beta a_{k,t+1}\right] \\
&= \sum_{\tau=t+2}^{T} \beta^{\tau-t} E_t[E_{t+1}(a_{k,\tau})] + \beta E_t(a_{k,t+1}) \\
&= \sum_{\tau=t+2}^{T} \beta^{\tau-t} E_t(a_{k,\tau}) + \beta E_t(a_{k,t+1}) \\
&= \sum_{\tau=t+1}^{T} \beta^{\tau-t} E_t(a_{k,\tau}).
\end{aligned}
$$

Q.E.D.

Next, we allow for risk aversion (so that the SDF is stochastic). Define the cumulated discount factor from τ to t by:

$$\bar{M}_{t,\tau} = \prod_{\theta=t+1}^{\tau} M_\theta.$$

Risk neutrality is the special case with $\bar{M}_{t,\tau} = \prod_{\theta=t+1}^{\tau} \beta = \beta^{\tau-t}$. Notice also that $\bar{M}_{t,t+1} = M_{t+1}$ and

$$\bar{M}_{t,\tau} = M_{t+1} \prod_{\theta=t+2}^{\tau} M_\theta = M_{t+1} \bar{M}_{t+1,\tau}.$$

Theorem (Fundamental value): Asset prices are uniquely determined by

$$p_{k,t} = \sum_{\tau=t+1}^{T} E_t(\bar{M}_{t,\tau} a_{k,\tau}).$$

Proof: The same arguments as in the risk-neutral case imply uniqueness. Analogously as before,

$$
\begin{aligned}
p_{k,t} &= E_t[M_{t+1}(p_{k,t+1} + a_{k,t+1})] \\
&= E_t\left[M_{t+1} \sum_{\tau=t+2}^{T} E_{t+1}(\bar{M}_{t+1,\tau} a_{k,\tau}) + M_{t+1} a_{k,t+1}\right] \\
&= \sum_{\tau=t+2}^{T} E_t[E_{t+1}(\bar{M}_{t,\tau} a_{k,\tau})] + E_t(M_{t+1} a_{k,t+1}) \\
&= \sum_{\tau=t+2}^{T} E_t(\bar{M}_{t,\tau} a_{k,\tau}) + E_t(M_{t+1} a_{k,t+1}) \\
&= \sum_{\tau=t+1}^{T} E_t(\bar{M}_{t,\tau} a_{k,\tau}).
\end{aligned}
$$

Q.E.D.

There are two striking pieces of evidence against fundamental value: excess volatility and bubbles. Shiller (1981) computed that that the standard deviation of the S&P 500 was 5.5 times as high as the standard deviation of the stocks' fundamental value over the period 1870-1979. The factor for the DJIA 1928-79 is 13. The volatility of actual stock prices is incompatible with fundamental value. Generally, asset prices (and prices of other storable goods) appear to be sporadically plagued by bubble-like behavior, i.e., cumulative movements away from average, followed by steep drops when the bubble bursts.

https://www.businesspundit.com/10-most-bizarre-economic-bubbles-in-history/

Literature:

- Shiller, Robert J. (1981), "Do Stock Prices Move Too Much to be Justified by Subsequent Changes in Dividends?", *American Economic Review* 71, 421-436.

Incomplete markets

Market incompleteness means that financial markets do not allow the transfer of purchasing power to each single state (the seminal model is Diamond, 1967). The implications of general equilibrium with incomplete markets for welfare, asset prices, etc. are much less clear-cut than in the case of complete markets (see Geanakoplos et al., 1990, Magill and Quinzii, 2002).

- The equilibrium allocation is not generally (unconstrained) Pareto optimal. This is obvious. In the extreme case with no financial markets at all, each individual consumes his endowments, there is no risk sharing whatsoever.

- The equilibrium allocation is not even *constrained* Pareto optimal, i.e., optimal relative to the trading opportunities generated by the existing financial markets. A special case with Pareto optimality was identified by Diamond (1967): there is a single final good, physical capital is the only input in production, stocks are the only financial assets, and uncertainty arises in the form of multiplicative shocks to firms' production functions.
- Even though there is not a common SDF, there exist state prices which can be used to price all financial assets (the fundamental theorem of asset pricing).
- Shareholders are not unanimous except in very restrictive special cases, such as the one proposed by Diamond (1967).

- The MM theorem holds.
- The analysis of the multi-period model did not make use of market completeness. So the pricing implications, including the random walk behavior of asset prices for risk neutrality, go through.

Literature:

- Diamond, Peter (1967). "The Role of a Stock Market in a General Equilibrium Model with Technological Uncertainty", *American Economic Review* 57, 759-776.
- Geanakoplos, John, Michael Magill, Martine Quinzii, and Jacques Drèze (1990), "Generic Inefficiency of Stock Market Equilibrium When Markets Are Incomplete", *Journal of Mathematical Economics* 19, 113-151.
- Magill, Michael, and Martine Quinzii, *Theory of Incomplete Markets*, MIT (2002), Section 6.