INTRODUCTION TO COARSE GEOMETRY
EXERCISE SHEET 2

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Exercise 1 (5)**.** Consider the set of natural numbers $\mathbb{N}$ as a coarse space with respect to the metric coarse structure induced by the Euclidean metric d. Let $X := \{n^2 \mid n \in \mathbb{N}\}$ denote the subset of square numbers and equip X with the induced coarse structure.

Show that the coarse structure on X is not finitely generated.

Exercise 2 ($1\frac{1}{2} + 1\frac{1}{2} + \frac{1}{2} + 1\frac{1}{2}$)**.** Consider the half-open interval $[0,1)$ as the subset $[0,1] \setminus \{1\}$ of the closed unit interval $[0,1]$. Show the following:

(1) Let U be a continuously controlled entourage on $[0,1)$. Define

$$b_U \colon [0,1) \to [0,1], \quad x \mapsto \sup\{|x-x'| \mid x' \in [0,1) \colon (x,x') \in U \text{ or } (x',x) \in U\}.$$

Then $\lim_{x \to 1} b_U(x) = 0$.

(2) Let $b\colon [0,1) \to [0,1]$ be a function satisfying $\lim_{x\to 1} b(x) = 0$. Then

$$U_b := \{(x,x') \mid |x-x'| \leq \min(b(x), b(x'))\}$$

is a continuously controlled entourage on $[0,1)$.

(3) For every continuously controlled entourage U on $[0,1)$, we have $U \subseteq U_{b_U}$.

(4) The continuously controlled coarse structure on $[0,1)$ is not countably generated.

Exercise 3 ($2+1+2$)**.** Let X be a set and let $\mathcal{A}$ be a set of entourages on X. Let $w\colon \mathcal{A} \to \mathbb{N}_{>0}$ be a function. For $x,y \in X$, define

$$d_w(x,y) := \min\left\{ \sum_{i=0}^{n} w(U_i) \mid \exists\, n \in \mathbb{N}\ \exists\, U_0, \ldots, U_n \in \mathcal{A} \colon (x,y) \in U_n^{\pm 1} \circ \ldots \circ U_0^{\pm 1} \right\}.$$

Show the following:

(1) d_w is a quasi-metric.

Denote by $\mathcal{C}_w$ the associated metric coarse structure.

(2) $\mathcal{C}\langle \mathcal{A} \rangle \subseteq \mathcal{C}_w$.

(3) If w is finite-to-one, then $\mathcal{C}_w \subseteq \mathcal{C}\langle \mathcal{A} \rangle$.

Exercise 4 ($2+2+1$)**.** Let G be a group. Define the *canonical* coarse structure on G as the coarse structure generated by the set $\{G(F \times F) \mid F \subseteq G \text{ finite}\}$.

(1) If G is countable, there exists a proper, left G-invariant metric on G such that the metric coarse structure coincides with the canonical coarse structure. (Recall that a metric is proper if balls of finite diameter are finite, and a left G-invariant d metric on G is required to satisfy $d(gg_1, gg_2) = d(g_1, g_2)$ for all $g, g_1, g_2 \in G$.)

(2) Any two proper, left G-invariant metrics on G induce the canonical coarse structure.

Due on 17th November 2020, 6 pm.