Lecture course on C^*-algebras and C^*-categories

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1 Algebras over $\mathbb{C}$

In this lecture course all algebras are over the field of complex numbers $\mathbb{C}$. We let $\mathbf{Vect}_{\mathbb{C}}$ denote the category of $\mathbb{C}$-vector spaces with the usual tensor product $\otimes$. We start with the definition of not necessarily unital algebras in $\mathbf{Vect}_{\mathbb{C}}$.

Let A be in $\mathbf{Vect}_{\mathbb{C}}$.

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Definition 1.1. *An associative product on A is a map $\mu : A \otimes A \to A$ such that*

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mathtt{id}_A \otimes \mu} & A \otimes A \\ \downarrow{\scriptstyle \mu \otimes \mathtt{id}_A} & & \downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow{\mu} & A \end{array}$$

commutes.

In view of the universal property of the tensor product we can interpret the product equivalently as a bilinear map $\mu : A \times A \to A$.

Definition 1.2. *An algebra is a pair (A, μ) of A in $\mathbf{Vect}_{\mathbb{C}}$ and an associative product μ.*

Note that we do not require the existence of a unit element.

Let (A, μ) be a algebra. For a, a' in A we will use the notation aa' instead of $\mu(a \otimes a')$. Usually we will denote algebras simply by a symbol like A or similar.

Example 1.3. If V is in $\mathbf{Vect}_{\mathbb{C}}$, then its set of endmorphisms $\mathtt{End}(V)$ has again a structure of an object of $\mathbf{Vect}_{\mathbb{C}}$. The composition of endomorphisms defines an associative and bilinear product. Hence $\mathtt{End}(V)$ becomes an algebra.

For $V = \mathbb{C}^n$, using the standard basis, we identify $\mathtt{End}(V)$ with the n-by-n-matrices $\mathtt{Mat}(n)$ with the usual matrix multiplication. □

Example 1.4. If A is an algebra and X is a set, then we can form a new algebra A^X of functions from X to A with the pointwise vector space structure and product. □

Example 1.5. Let G be a magma (a set with an associative product). Then we consider the vector space $\mathbb{C}[G]$ generated by G. The element in $\mathbb{C}[G]$ corresponding to g in G will be denoted by $[g]$. The associative product $G \times G \to G$ induces an associative product

$$\mathbb{C}[G] \otimes \mathbb{C}[G] \cong \mathbb{C}[G \times G] \to \mathbb{C}[G] \ .$$

We have $[g][g'] = [gg']$. The algebra $\mathbb{C}[G]$ is called the magma ring of G.

We have $\mathbb{C}[\emptyset] = 0$.

We can identity $\mathbb{C}[\mathbb{N}]$ with the polynomial ring $\mathbb{C}[x]$ by sending $[n]$ to x^n. Similarly we have $\mathbb{C}[\mathbb{Z}] \cong \mathbb{C}[x, x^{-1}]$. □

Example 1.6. Let A be an algebra with product μ and I be a subvectorspace of A. If the composition $I \otimes I \to A \otimes A \xrightarrow{\mu} A$ takes values in I, then we get a subalgebra $(I, \mu_{|I \otimes I})$. □

Definition 1.7.

1. *A linear subspace I of A is called a left ideal if for all a in A and i in I we have $ai \in I$.*

2. *A linear subspace I of A is called a right ideal if for all a in A and i in I we have $ia \in I$.*

3. *A linear subspace I of A is called an (two-sided) ideal if it is a left- and right ideal.*

If I is a (possibly one-sided) ideal in an algebra A, then it is in particular a subalgebra.

Example 1.8. Let V be in $\mathbf{Vect}_{\mathbb{C}}$. Then the subset $\mathtt{End}^{\mathrm{fr}}(V)$ of $\mathtt{End}(V)$ of finite rank endomorphisms is an ideal. □

Example 1.9. Let $(A_i)_{i \in I}$ be a family of algebras. Then we can form the sum $\bigoplus_{i \in I} A_i$ of underlying vector spaces. It carries an algebra structure with the component wise product $\oplus_i a_i \oplus_i a_i' := \oplus_i a_i a_i'$.

If $(A)_{x \in X}$ is a constant family indexed by a set X, then $\bigoplus_{x \in X} A$ can be identified with the ideal in the algebra A^X consisting of the functions with finite support. □

Example 1.10. Let A be an algebra with product μ. If I is an ideal in A, then the quotient vector space A/I has an algebra structure which will be denoted by $\bar{\mu}$. It is given by the unique factorization

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\mu} & A \\ \downarrow & & \downarrow \\ A/I \otimes A/I & \xrightarrow{\bar{\mu}} & A/I \end{array}$$

of μ which is easily seen to exist by the condition on I being a two-sided ideal. Here we use the identification $A/I \otimes A/I \cong A \otimes A/(A \otimes I + I \otimes A)$. □

Example 1.11. Let H be a Hilbert space. The bounded operators $B(H)$ forms an algebra with respect to the composition of operators. The subspace of compact operators $K(H)$ is an ideal in $B(H)$. The quotient algebra $Q(H) := B(H)/K(H)$ is called the Calkin algebra of H. □

Example 1.12. We can consider the algebra $D(\mathbb{C})$ of differential operators with polynomial coefficients on $\mathbb{C}$. Typical elements are x (multiplication by x) and ∂ (differentiation by x). The element $e = x\partial$ is called the Euler operator. We have the relations $\partial x - x\partial = 1$ and $ex^n - x^n e = nx^n$. □

We now describe the category of algebras. We consider two algebras A, B with products μ_A and μ_B.

Definition 1.13. *A homomorphism $f : A \to B$ of algebras is morphism $f : A \to B$ in* $\mathbf{Vect}_{\mathbb{C}}$ *such that*

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\mu_A} & A \\ \downarrow{\scriptstyle f\otimes f} & & \downarrow{\scriptstyle f} \\ B \otimes B & \xrightarrow{\mu_B} & B \end{array}$$

commutes.

We get the category $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ of algebras and homomorphisms. The superscript nu stands for non-unital since we do not require the existence of units nor that maps preserve units, if they exist.

Example 1.14. Let V', V'' be in $\mathbf{Vect}_{\mathbb{C}}$ and consider their sum $V := V' \oplus V''$. Then we get a homomorphism of algebras $\mathtt{End}(V') \to \mathtt{End}(V)$ which sends ϕ to $\phi \oplus 0$. This is called the left-upper-corner inclusion.

For n, m in $\mathbb{N}$ with $n \geq m$ we have $\mathbb{C}^n \cong \mathbb{C}^m \oplus \mathbb{C}^{n-m}$. This gives the left-upper-corner inclusion $\mathtt{Mat}(m) \to \mathtt{Mat}(n)$. □

Example 1.15. Let A be an algebra and $f : X \to Y$ be a map of sets. We get a homomorphism $f^* : A^Y \to A^X$ given by restriction of functions along f. □

Example 1.16. If $G \to H$ is a morphism of magmas, then the induced map $\mathbb{C}[G] \to \mathbb{C}[H]$ is a homomorphism of algebras.

The inclusion $\mathbb{N} \to \mathbb{Z}$ induces an inclusion $\mathbb{C}[\mathbb{N}] \to \mathbb{C}[\mathbb{Z}]$ which can be identified with the inclusion $\mathbb{C}[x] \to \mathbb{C}[x, x^{-1}]$. □

Example 1.17. Let $f : A \to B$ be a homomorphism, I be an ideal in A and J be an ideal in B such that $f(I) \subseteq J$. Then the natural factorization $\bar{f}$ in

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ A/I & \xrightarrow{\bar{f}} & B/J \end{array}$$

is a homomorphism. On elements it is given by $[f]([a]) := [f(a)]$. □

Let A be in $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$

Definition 1.18. *A is called unital if it admits an element 1_A such that $1_A a = a = a 1_A$ for all a in A.*

Such an element 1_A is called a unit of A.

Lemma 1.19. *If A is unital, then the unit of A is uniquely determined.*

Proof. Let 1_A and $1'_A$ be two units. Then using the defining relation for $1'_A$ and 1_A we have $1_A = 1_A 1'_A = 1_{A'}$. $\square$

So units in algebras are a property, not data.

Example 1.20. If V is in $\mathbf{Vect}_{\mathbb{C}}$, then $\mathtt{End}(V)$ is unital with unit $1_{\mathtt{End}(V)} = \mathtt{id}_V$. If $\dim(V) = \infty$, then $\mathtt{End}^{\mathrm{fr}}(V)$ is not unital. $\square$

Example 1.21. If A is unital and X is a set, then A^X is unital, where the unit 1_{A^X} is the constant function with value 1_A.

If X is infinite, then the subalgebra $\bigoplus_{x \in X}$ of A^X is not unital. $\square$

Example 1.22. The algebra $D(\mathbb{C})$ is unital with unit 1. $\square$

Example 1.23. A monoid is a magma with an identity element. For a magma G the magma algebra $\mathbb{C}[G]$ is unital if and only of G is a monoid. In this case the unit is given by $1_{\mathbb{C}[G]} := [e]$, where e is the unit of G. $\square$

Example 1.24. For a Hilbert space H the algebra $B(H)$ is unital. The algebra $K(H)$ is unital if and only if H is finite-dimensional. $\square$

Let A and B be unital algebras and $f : A \to B$ be a homomorphism of algebras.

Definition 1.25. *f is called unital if $f(1_A) = 1_B$.*

We get the category $\mathbf{Alg}_{\mathbb{C}}$ of unital algebras and unital homomorphisms.

Example 1.26. Note that $\mathtt{Mat}(n)$ and $\mathtt{Mat}(n+1)$ are unital. But the left upper corner inclusion $\mathtt{Mat}(n) \to \mathtt{Mat}(n+1)$ is a morphism between algebras which does not preserve units. $\square$

We have an inclusion functor

$$\mathrm{incl} : \mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}} \to \mathbf{Alg}_{\mathbb{C}} \ .$$

This functor is faithful, but not full and not essentially surjective. In the following lemma we show that it has a left-adjoint. The latter will be called the unitalization functor.

Lemma 1.27. *We have an adjunction* $(-)^u : \mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}} \leftrightarrows \mathbf{Alg}_{\mathbb{C}} : \mathrm{incl}$.

Proof. We show the existence of the adjunction by providing an explicit construction of the left adjoint and of the unit and counit of the adjunction. We start with constructing the unitalization functor $(-)^u$.

1. objects: Let A be in $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$. Then the underlying vector space of A^u is $A \oplus \mathbb{C}$. The product is defined by
$$(a,\lambda)(a',\lambda') := (aa' + \lambda a' + a\lambda', \lambda\lambda') \ .$$
One checks associativity by calculation. The algebra A^u has a unit which is given by $1_{A^u} = (0,1)$.

2. morphisms: If $f : A \to B$ is a homomorphism, then we define $f^u : A^u \to B^u$ by $f^u(a,\lambda) := (f(a),\lambda)$. One checks by calculation that this is a unital homomorphism.

One checks by calculation that this construction defines a functor.

In order to define the adjunction we provide the unit and counit transformations:

1. unit: $A \to \mathrm{incl}(A^u)$ is given by $a \mapsto (a,0)$.

2. counit: $\mathrm{incl}(B)^u \to B$ is given by $(b,\lambda) \mapsto b + \lambda 1_B$.

One checks that these formulas define morphisms in the respective categories and are natural. One further checks by calculation that the following map of sets
$$\mathtt{Hom}_{\mathbf{Alg}_{\mathbb{C}}}(A^u,B) \stackrel{\mathrm{incl}}{\to} \mathtt{Hom}_{\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}}(\mathrm{incl}(A^u),\mathrm{incl}(B)) \stackrel{unit^*}{\to} \mathtt{Hom}_{\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}}(A,\mathrm{incl}(B))$$
is a bijection with inverse
$$\mathtt{Hom}_{\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}}(A,\mathrm{incl}(B)) \stackrel{(-)^u}{\to} \mathtt{Hom}_{\mathbf{Alg}_{\mathbb{C}}}(A^u,\mathrm{incl}(B)^u) \stackrel{counit_*}{\to} \mathtt{Hom}_{\mathbf{Alg}_{\mathbb{C}}}(A^u,B) \ .$$
□

For now on we will usually omit incl from the notation.

Remark 1.28. For A in $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ we have a split exact sequence
$$0 \to A \stackrel{unit}{\to} A^u \stackrel{(a,\lambda)\mapsto\lambda}{\to} \mathbb{C} \to 0$$
with split $\mathbb{C} \to A^u$ given by $\lambda \mapsto (0,\lambda)$. If A is unital, then we have an isomorphism of algebras
$$A^u \cong A \oplus \mathbb{C} \ , \quad (a,\lambda) \mapsto (a + \lambda 1_A, \lambda) \ .$$
□

We have a functor $U : \mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}} \to \mathbf{Vect}_{\mathbb{C}}$ which forgets the product and takes the underlying vector space. It restricts to a functor $U : \mathbf{Alg}_{\mathbb{C}} \to \mathbf{Vect}$.

Lemma 1.29. *We have adjunctions*

$$T : \mathbf{Vect}_{\mathbb{C}} \leftrightarrows \mathbf{Alg}_{\mathbb{C}} : U$$

and

$$T^{\geq 1} : \mathbf{Vect}_{\mathbb{C}} \leftrightarrows \mathbf{Alg}_{\mathbb{C}} : U \ .$$

Proof. (sketch) We again provide explicit constructions of the left-adjoints and the units and counits. The functor T associates to a vector space V the tensor algebra

$$T(V) := \bigoplus_{n \geq 0} V^{n\otimes}$$

with the concatenation product. The unit $V \to U(T(V))$ of the adjunction is the inclusion $V \to T(V)$ into the summand for $n = 1$, and the counit $T(U(A)) \to A$ sends $a_1 \otimes \cdots \otimes a_n$ in the n'th summand to $a_1 \dots a_n$. It further sends λ in the 0'th summand $\mathbb{C} = V^{0\otimes}$ to $\lambda 1_A$.

The functor $T^{\geq 1}$ is the subfunctor of T which sends V to

$$T^{\geq 1}(V) := \bigoplus_{n \geq 1} V^{n\otimes} \ .$$

The unit and counit are given by the same description. One checks that for A in $\mathbf{Alg}_{\mathbb{C}}$

$$\mathtt{Hom}_{\mathbf{Vect}_{\mathbb{C}}}(V, U(A)) \stackrel{T}{\to} \mathtt{Hom}_{\mathbf{Alg}_{\mathbb{C}}}(T(V), T(U(A)) \stackrel{counit_*}{\to} \mathtt{Hom}_{\mathbf{Alg}_{\mathbb{C}}}(T(V), A)$$

and for A in $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$

$$\mathtt{Hom}_{\mathbf{Vect}_{\mathbb{C}}}(V, U(A)) \stackrel{T^{\geq 1}}{\to} \mathtt{Hom}_{\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}}(T^{\geq 1}(V), T^{\geq 1}(U(A)) \stackrel{counit_*}{\to} \mathtt{Hom}_{\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}}(T^{\geq 1}(V), A)$$

are bijections. □

Let $S : \mathbf{Vect}_{\mathbb{C}} \to \mathbf{Set}$ be the functor which takes the underlying set. We have an adjunction

$$\mathbb{C}[-] : \mathbf{Set} \leftrightarrows \mathbf{Vect}_{\mathbb{C}} : S \ ,$$

where the right-adjoint takes the underlying set of a vector space. The vector space $\mathbb{C}[X]$ is the vector space generated by the set X. Composing this adjunction with the adjunctions in Lemma 1.29 we get:

Corollary 1.30. *We have adjunctions*

$$\mathrm{Free} : \mathbf{Set} \leftrightarrows \mathbf{Alg}_{\mathbb{C}} : S \ , \quad \mathrm{Free}^{\mathrm{nu}} : \mathbf{Set} \leftrightarrows \mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}} : S \ , \tag{1.1}$$

where

$$\mathrm{Free}(X) := T(\mathbb{C}[X]) \quad \textit{and} \quad \mathrm{Free}^{\mathrm{nu}}(X) := T^{\geq 1}(\mathbb{C}[X])$$

The right-adjoints take the underlying sets of an algebra.

The algebras Free(X) and $\mathrm{Free}^{\mathrm{nu}}$ are called the free unital and non-unital algebras generated by the set X.

Recall that a category is complete if it admits limits for all small diagrams. Similarly it is called cocomplete if it admits colimits for all small diagrams.

Proposition 1.31. *The categories* $\mathbf{Alg}_{\mathbb{C}}$ *and* $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ *are complete and cocomplete.*

Proof. (sketch) In view of the adjunctions (1.1) the forgetful functor from algebras to sets preserves limits. Thus limits in $\mathbf{Alg}_{\mathbb{C}}$ and $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ are obtained by forming the limits in **Set** and equipping the results with the induced vector space and algebra structures. In order to show completeness it suffices to show the existences of products of small families and equalizers. We discuss products and equalizers in the Examples 1.32 and 1.33 below.

The explicit description of colimits is more complicated. But to show cocompleteness it suffices to show the existence of coproducts of small families and coequalizers which are discussed in Examples 1.34 and 1.35 below. □

Example 1.32. Let $(A_i)_{i\in I}$ be a family of algebras. Then the product of the family is given by the product of the underlying sets $\prod_{i\in I} A_i$ with the factorwise operations. The structure maps are the projections to the factors.

If the family consists of unital algebras, then $(1_{A_i})_{i\in I}$ is the unit of the product.

If $(A_i)_{i\in I}$ is a finite family, then $\prod_{i\in I} A_i \cong \bigoplus_{i\in I} A_i$.

If $(A)_{x\in X}$ is a constant family, then $\prod_{x\in X} A \cong A^X$. □

Example 1.33. Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

be a diagram in $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$. The limit of this diagram is called the equalizer of f and g and given by the subalgebra

$$\mathrm{Eq}(f,g) := \{a \in A \mid f(a) = g(a)\}$$

of A. The structure map is the canonical inclusion $\mathrm{Eq}(f,g) \to A$.

If A, B are unital and f and g preserve units, then $1_A \in \mathrm{Eq}(f,g)$ so that the equalizer is unital. □

Example 1.34. In this example we describe coproducts in $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$. For every A in $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ we have an exact sequence

$$0 \to I_A \to \mathrm{Free}^{\mathrm{nu}}(S(A)) \stackrel{counit}{\to} A \to 0 \ ,$$

where the ideal I_A is defined as the kernel of the counit.

Let A, B be in $\mathbf{Alg}_{\mathbb{C}}^{\mathrm{nu}}$. Then we form $\mathrm{Free}^{\mathrm{nu}}(S(A) \sqcup S(B))$. The map $S(A) \to A \sqcup B$ induces a map

$$I_A \to \mathrm{Free}^{\mathrm{nu}}(S(A)) \to \mathrm{Free}^{\mathrm{nu}}(S(A) \sqcup S(B))$$

and similarly for B. We let I be the two-sided ideal in $\mathrm{Free}^{\mathrm{nu}}(S(A) \sqcup S(B))$ generated by the images of I_A and I_B. Then we have factorizations

$$\begin{array}{ccc} \mathrm{Free}^{\mathrm{nu}}(S(A)) & \longrightarrow & \mathrm{Free}^{\mathrm{nu}}(S(A) \sqcup S(B)) \\ \Big\downarrow \scriptstyle{counit} & & \Big\downarrow \\ A & \dashrightarrow & \mathrm{Free}^{\mathrm{nu}}(S(A) \sqcup S(B))/I \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{Free}^{\mathrm{nu}}(S(B)) & \longrightarrow & \mathrm{Free}^{\mathrm{nu}}(S(A) \sqcup S(B)) \\ \Big\downarrow \scriptstyle{counit} & & \Big\downarrow \\ B & \dashrightarrow & \mathrm{Free}^{\mathrm{nu}}(S(A) \sqcup S(B))/I \end{array}$$

These maps present $\mathrm{Free}^{\mathrm{nu}}(S(A) \sqcup S(B))/I$ as the coproduct $A \sqcup B$ in $\mathbf{Alg}_{\mathbb{C}}^{\mathrm{nu}}$.

It is often denoted by $A * B$ and called the free product of A and B.

Note that for commutative algebras A, B the coproduct $A * B$ in $\mathbf{Alg}_{\mathbb{C}}^{\mathrm{nu}}$ is non-commutative and differs from the coproduct in commutative algebras which is given by the vector space $A \otimes B$ with the induced algebra structure.

E.g. in $\mathbb{C}[x] * \mathbb{C}[y]$ we have $xy \neq yx$.

A similar construction works for $\mathbf{Alg}_{\mathbb{C}}$ and for coproducts of arbitrary families of objects. □

Example 1.35. In this example we describe coequalizers. Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

be a coequalizer diagram in $\mathbf{Alg}_{\mathbb{C}}^{\mathrm{nu}}$. Then the elements $f(a) - g(a)$ for all a in A generate a two-sided ideal I in B. The projection $B \to B/I$ presents B/I as the coequalizer $\mathrm{Coeq}(f, g)$.

The same construction works in the unital case. □

Example 1.36. The initial algebra in $\mathbf{Alg}_{\mathbb{C}}^{\mathrm{nu}}$ is the coproduct of the empty family. It is the zero algebra 0. Note that the zero algebra happens to be unital with unit $1_0 = 0$. But it is not the initial object in $\mathbf{Alg}_{\mathbb{C}}$. E.g. there is no unital morphism $0 \to \mathbb{C}$. In fact, $\mathbb{C}$ is an initial algebra in $\mathbf{Alg}_{\mathbb{C}}$.

The zero algebra is the final algebra in $\mathbf{Alg}_{\mathbb{C}}^{\mathrm{nu}}$ and $\mathbf{Alg}_{\mathbb{C}}$. □

Example 1.37. An exact sequence $0 \to I \to A \to A/I \to 0$ of algebras in $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ can be interpreted as a push-out diagram

$$\begin{array}{ccc} I & \longrightarrow & A \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & A/I \end{array} \quad .$$

□

We finally introduce the concept of the spectrum of an element in an algebra. Let A be in $\mathbf{Alg}_{\mathbb{C}}$ and a be in A.

Definition 1.38. *a is invertible if there exists and element b in A such that $ba = 1_A$ and $ab = 1_A$.*

The element b is called and inverse of a.

Lemma 1.39. *An inverse is uniquely determined.*

Proof. Let b, b' be two inverses of a. Then we have $b = b1_A = bab' = 1_A b' = b'$. □

We usually use the notation a^{-1} for the inverse of a.

Let $f : A \to B$ be a morphism in $\mathbf{Alg}_{\mathbb{C}}$ and a be in A.

Lemma 1.40. *If a is invertible, then $f(a)$ is invertible and $f(a)^{-1} = f(a^{-1})$.*

Proof. We calculate $f(a)f(a^{-1}) = f(aa^{-1}) = f(1_A) = 1_B$ and similarly $f(a^{-1})f(a) = 1_B$. □

Note that we use in this proof that f preserves units.

In the following for A in $\mathbf{Alg}_{\mathbb{C}}$ we use the notation

$$\lambda := \lambda 1_A$$

Let a be in A.

Definition 1.41. *The spectrum of a is the set*

$$\sigma(a) := \{\lambda \in \mathbb{C} \mid (\lambda - a) \text{ is not invertible in } A\} .$$

The complement $\rho(a) := \mathbb{C} \setminus \sigma(a)$ is called the resolvent set.

Example 1.42. Let a be in $\mathtt{Mat}(n)$. Then the spectrum $\sigma(a)$ is set of eigenvalues of a. $\square$

Example 1.43. For a in $\mathbb{C}^X$ we have $\sigma(a) = a(X)$ $\square$

Let A be in $\mathbf{Alg}_\mathbb{C}$ and a be in A. Let p be in $\mathbb{C}[x]$. Then we can form $p(a)$ in A in the obvious way.

Lemma 1.44. *We have* $p(\sigma(a)) \subseteq \sigma(p(a))$.

Proof. Since $p(\lambda) - p(x)$ vanishes at $x = \lambda$ we can write $p(\lambda) - p(x) = (\lambda - x)q(x)$ for some q in $\mathbb{C}[x]$. The equality

$$p(\lambda) - p(a) = (\lambda - a)q(a)$$

implies that if the left-hand side is invertible, so the two factors on the right-hand side. $\square$

Let $f : A \to B$ be morphism in $\mathbf{Alg}_\mathbb{C}$ and a in A.

Lemma 1.45. *We have* $\sigma(f(a)) \subseteq \sigma(a)$.

Proof. Consider λ in $\mathbb{C}$. If $\lambda \notin \sigma(a)$, then $(\lambda - a)^{-1}$ exists in A and hence $f((\lambda - a)^{-1}) = (\lambda - f(a))^{-1}$ exists in B. Hence $\lambda \notin \sigma(f(a))$ $\square$

Example 1.47 shows that $\sigma(f(a))$ can be strictly smaller than $\sigma(a)$.

We consider A in $\mathbf{Alg}_\mathbb{C}$ and a in A.

Definition 1.46. *We define the spectral radius* $r(a) := \sup |\sigma(a)|$.

It is an element of $[-\infty, \infty]$. We have $-\infty$ iff $\sigma(a) = \emptyset$ and ∞ if $\sigma(a)$ is unbounded

Example 1.47. We consider x in $\mathbb{C}[x]$. Then $\sigma(x) = \mathbb{C}$. We have $r(x) = \infty$.

If we consider x in $\mathbb{C}(x)$ (the quotient field of $\mathbb{C}[x]$), then $\sigma(x) = \emptyset$. In this case $r(x) = -\infty$. $\square$

We consider a morphism $f : A \to B$ in $\mathbf{Alg}_\mathbb{C}$ and a in A. The following is immediate from Lemma 1.45.

Corollary 1.48. $r(f(a)) \leq r(a)$

By Example 1.47 the inequality in Corollary 1.48 can be strict.

The notion of the spectrum is extended to the non-unital case as follows. Let A be in $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ and a be in A. Then we consider $(a,0) \in A^u$

Definition 1.49. *We define* $\sigma^u(a) := \sigma((a,0))$.

We always have $0 \in \sigma^u(a)$.

Lemma 1.50. *If A is unital, then* $\sigma^u(a) = \sigma(a) \cup \{0\}$.

Proof. We use that $A^u \cong A \oplus \mathbb{C}$ is given by $(a,\lambda) \mapsto (a + \lambda 1_A, \lambda)$. Under this identification $\lambda 1_{A^u} - (a,0) \mapsto (\lambda 1_A - a, \lambda)$. We read off that λ in $\sigma^u(a)$ iff $\lambda \in \sigma(a)$ or $\lambda = 0$ □

2 Banach algebras

We consider a norm $\| - \|$ on a vector space V. The pair $(V, \| - \|)$ is called a normed vector space. The norm induces a metric $d(v,v') := \|v - v'\|$ on V. A normed vector space is called complete if it is complete (in the sense of metric spaces) with respect to this metric.

A Banach space is a topological vector space whose topology is induced from a norm and which is complete with respect to the induced metric. This condition does not depend on the choice of the metric. In other words, when we talk about Banach spaces, then we only care about the topology, but not about the specific norm generating the topology.

For a Banach space we let B^* denote the space of continuous linear maps $B \to \mathbb{C}$. It will be equipped with the topology of uniform convergence on bounded subsets of B (defined using some norm). It is again a Banach space. By the Hahn-Banach theorem we know that the canonical map $B \to (B^*)^*$ is injective.

Let B be a Banach space, U be an open subset of $\mathbb{C}$, and $f : U \to B$ be a continuous function. We consider a curve $\gamma : [0,1] \to U$. Then we can consider the Riemann integral

$$\int_\gamma f(z)dz \ .$$

It is an element of B defined as the limit over Riemann sums

$$\sum_{i\in 1}^{n} f(\gamma(t_i))(t_i - t_{i-1})$$

over the filtered poset of finite partitions of the interval $[0,1]$ here given as sequences

$$0 = t_0 \le t_1 \le \cdots \le t_{n-1} \le t_n = 1 \ .$$

Thereby the partitions are are partially ordered by refinement. One shows convergence using uniform continuity of $f \circ \gamma$ in the same way as in scalar case. Note that here we use the completeness of B in order to ensure existence of the limit.

If $A : B \to B'$ is a linear continuous map between Banach spaces, then it preserves integrals:

$$A(\int_\gamma f(z)dz) = \int_\gamma A(f(z))dz \ .$$

One can talk about holomorphic functions $f : U \to B$. But for simplicity in the course we will only use the notion of weak holomorphy which reduces everything to the scalar case.

Definition 2.1. *f is called weakly holomorphic if $\phi(f) : U \to \mathbb{C}$ is holomorphic for every continuous functional ϕ in B^**

Lemma 2.2. *If f is weakly holomorphic and γ is closed and contractible in U, then we have $\int_\gamma f(z)dz = 0$.*

Proof. Since $B \to (B^*)^*$ is injective it suffices to show that

$$\phi(\int_\gamma f(z)dz) = 0$$

for all ϕ in B^*. Since ϕ is continuous we have

$$\phi(\int_\gamma f(z)dz) = \int_\gamma \phi(f(z))dz \ .$$

Finally by the Cauchy integral theorem we have

$$\int_\gamma \phi(f(z))dz = 0 \ .$$

□

We now consider the interplay between norms and the product on an algebra. Let A be in $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ and $\|-\|$ be a norm on the underlying vector space.

Definition 2.3. *We say that $\|-\|$ is sub-multiplicative if $\|aa'\| \le \|a\|\|a'\|$ for all a, a' in A.*

Definition 2.4. *A normed algebra $(A, \|-\|)$ is pair of an algebra and a sub-multiplicative norm.*

We add the adjective unital in order to express the compatibility of the norm with the unit of the algebra.

Definition 2.5. *A unital normed algebra* $(A, \|-\|)$ *is a normed algebra such that* $A \in \mathbf{Alg}_{\mathbb{C}}$ *and* $\|1_A\| = 1$*.*

Example 2.6. Let $(V, \|-\|_V)$ be a normed vector space. Then $\mathtt{End}(V)$ has the operator norm given by

$$\|A\| := \sup_{v \in V, \|v\|_V = 1} \|Av\|_V \ .$$

Then $(\mathtt{End}(V), \|-\|)$ is a unital normed algebra. □

Example 2.7. We consider a normed algebra $(A, \|-\|)$ and assume that A is unital. We explicitly do not require that A is unitally normed. In general we then have

$$\|1_A\| \geq 1 \ .$$

In order to see this we start with

$$1_A^n = 1_A \ .$$

Using the sub-multiplicativity of the norm we get

$$\|1_A\| = \|1_A^n\| \leq \|1_A\|^n \ .$$

We insert $n = 2$ and conclude the desired inequality. In Example 2.8 we show that this inequality may be strict. □

Example 2.8. We consider a unital normed algebra $(A, \|-\|)$. Then we set $\|-\|' := 2\|-\|$. Then $(A, \|-\|')$ is normed algebra, but $\|1_A\| = 2 \neq 1$ □

Recall that two norms $\|-\|$ and $\|-\|'$ on a vector space V are called equivalent if there exists C in $(0, \infty)$ such that

$$C^{-1}\|v\| \leq \|v\|' \leq C\|v\|$$

for all v in V. The equivalence class of norms defining the topology of a Banach space is uniquely determined.

Let $(A, \|-\|)$ be a normed algebra and A be unital.

Lemma 2.9. *There is an equivalent sub-multiplicative norm* $\|-\|'$ *on* A *with* $\|1_A\|' = 1$*.*

Proof. We have a map

$$A \to \mathtt{End}_{\mathbf{Vect}_{\mathbb{C}}}(A) \ , \quad b \mapsto (a \mapsto ba) \ .$$

The operator norm on $\mathtt{End}_{\mathbf{Vect}_{\mathbb{C}}}(A)$ induces a sub-multiplicative seminorm (i.e. we allow that $\|a\|' = 0$ for non-zero a) $\|-\|'$ on A such that $\|1_A\|' = 1$.

We must show that $\|-\|'$ and $\|-\|$ are equivalent. This also implies that $\|-\|'$ is a norm.

For all b in A we have

$$\|b\|' = \sup_{a,\|a\|=1} \|ba\| \leq \|b\|$$

and

$$\|b\| = \|b1_A\| \leq \|b\|'\|1_A\| \ .$$

□

Definition 2.10. *A (unital) Banach algebra is a (unital) algebra A with admits a norm $\|-\|$ such that the underlying normed vector space is a Banach space.*

Example 2.11. Let H be a Hilbert space. Then $\mathbb{B}(H)$ (with the operator norm) is a unital Banach algebra.

If we specialize H to $\mathbb{C}^n$ with the standard scalar product and associated norm, then we see that $\mathtt{Mat}(n, \mathbb{C}) \cong \mathbb{B}(\mathbb{C}^n)$ is a Banach algebra. □

Example 2.12. We consider a topological space X and a Banach algebra A with norm $\|-\|$. The subalgebra $C_b(X, A)$ of the space $C(X, A)$ of bounded continuous functions from X to A is a Banach algebra with respect to sup-norm

$$\|f\|_\infty := \sup_{x \in X} \|f(x)\| \ .$$

If A is unital, then so is $C_b(X, A)$.

We can also consider the closure $C_0(X, A)$ of the compactly supported functions with respect to $\|-\|_\infty$. If X is not compact, then this a is non-unital Banach algebra. Note that if X is not locally compact, then $C_0(X, A)$ might be very small. □

Next we show that unitalization preserves Banach algebras. Let A be in $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$.

Lemma 2.13. *If A is Banach, then A^u is unital Banach.*

Proof. We define a norm $\|(a, \lambda)\| := \|a\| + |\lambda|$ and check the submultiplicativity

$$\|(a, \lambda)(a', \lambda')\| = \|aa' + \lambda a' + \lambda' a\| + |\lambda\lambda'| \leq (\|a\| + |\lambda|)(\|a'\| + |\lambda'|) \ .$$

It is clear that A^u with this norm is Banach.

Furthermore we have $\|(0, 1)\| = 1$ □

We now study the spectrum of elements in a Banach algebra using analytic arguments. Let A be a unital Banach algebra.

Lemma 2.14. *The set $GL_1(A)$ of invertible elements in A is open. Furthermore the map $a \mapsto a^{-1}$ is continuous.*

Proof. Let a be in $GL_1(A)$ and set $r := \|a^{-1}\|$. If b is in A such that $\|a - b\| < r$, then b is in $GL_1(A)$. In order to see this we write

$$b = a + (b - a) = a(1_A + a^{-1}(b - a)) \ .$$

Then

$$b^{-1} = (1_A + a^{-1}(b - a))^{-1}a^{-1} \ ,$$

where

$$(1_A + a^{-1}(b - a))^{-1} = \sum_{n=0}^{\infty}(-1)^n[a^{-1}(b - a)]^n \ .$$

The sum (Neumann series) converges absolutely since $\|a^{-1}(b - a)\| \leq r\|b - a\| < 1$. The identity

$$(1_A + a^{-1}(b - a))\sum_{n=0}^{n}(-1)^n[a^{-1}(b - a)]^n = 1_A + (-1)^n[a^{-1}(b - a)]^{n+1}$$

shows by considering the limit as $n \to \infty$ that the infinite sum represents the inverse of $1_A + a^{-1}(b - a)$.

We furthermore get

$$b^{-1} - a^{-1} = \sum_{n=1}^{\infty}(-1)^n[a^{-1}(b - a)]^n a^{-1} \ .$$

As long as $\|b - a\| \leq r/2$ this gives an estimate

$$b^{-1} - a^{-1} \leq \|b - a\|C$$

for some constant which does not depend on b. This shows the continuity of the inverse map. □

We consider a unital Banach algebra A with norm $\| - \|$ and a in A. Recall from Definition 1.46 that $r(a)$ denotes the spectral radius of a.

Corollary 2.15. $r(a) \leq \|a\|$.

Proof. Assume that λ is in $\mathbb{C}$ and $|\lambda| > \|a\|$. Then $\lambda - a = \lambda(1 - \lambda^{-1}a)$ is invertible since $\|\lambda^{-1}a\| < 1$. □

We consider a unital Banach algebra A and a in A.

Lemma 2.16. *$\rho(a)$ is open and*

$$\rho(a) \ni \lambda \mapsto (\lambda - a)^{-1} \in A$$

is continuous and weakly holomorphic.

Proof. The map

$$\mathbb{C} \ni \lambda \mapsto \lambda - a \in A$$

is continuous. Hence the set $\rho(a)$ is open since it is the preimage of the open subset $GL_1(A)$ of A under this map. Furthermore, $\lambda \mapsto (\lambda - a)^{-1}$ is continuous on $\rho(a)$. It remains to show weak holomorphy. Let ϕ be in A^*. Then for μ in $\rho(a)$ and λ close to μ the Neumann series implies the formula

$$(\lambda - a)^{-1} = \sum_{k=0}^{\infty} (\mu - a)^{-k-1}(\mu - \lambda)^k$$

and hence

$$\phi((\lambda - a)^{-1}) = \sum_{k=0}^{\infty} \phi((\mu - a)^{-k-1})(\mu - \lambda)^k$$

where we have used the continuity of ϕ in order to bring it inside of the sum. We already know from the proof of Lemma 2.14 that this sum converges absolutely for λ near μ and therefore defines a holomorphic function in λ. □

We consider a unital Banach algebra A and a in A.

Lemma 2.17 (Formula for spectral radius). *We have the equality*

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} \ .$$

Proof. Note that this includes the assertion that the limit exists.

If λ is in $\sigma(a)$, then $\lambda^n \in \sigma(a^n)$ by Lemma 1.44 applied to $p(x) = x^n$. This implies

$$|\lambda|^n \le \|a^n\|$$

for all n in $\mathbb{N}$ and hence

$$|\lambda| \le \liminf_{n\in\mathbb{N}} \|a^n\|^{1/n} \ .$$

We conclude that

$$r(a) \le \liminf_{n\in\mathbb{N}} \|a^n\|^{1/n} \ . \tag{2.1}$$

We now take s in $(0, \infty)$ and use the notation $B_s := \{|z| < s\}$ for the s-ball and $S_s := \partial B_s$ for its boundary. We assume that $r(a) < s$. Then $\lambda \mapsto (\lambda - a)^{-1}$ exists and is continuous and weakly holomorphic on an open neighbourhood of $\mathbb{C} \setminus B_s$. We claim that

$$a^n = \frac{1}{2\pi i} \int_{S_{s'}} \frac{\lambda^n d\lambda}{\lambda - a} \tag{2.2}$$

for all s' in $(0, \infty)$ with $s' \ge s$. Indeed, the right-hand side is independent of s' (apply arbitrary ϕ in A^* and the Cachy integral theorem) and for $s' > \limsup_{n\to\infty} \|a^n\|^{1/n}$ it is equal to

$$\frac{1}{2\pi i} \int_{S_{s'}} \sum_{k=0}^{\infty} \lambda^{n-1-k} a^k d\lambda = a^n \ .$$

We estimate the norm of the integral by the integral of the norm of the integrand. Then we get the estimate

$$\|a^n\| \le \frac{\sup_{z\in S_s} \|\frac{1}{z-a}\|}{2\pi} s^{\prime,n}$$

We now take the n'th root and consider the limit of the right-hand side as $n \to \infty$. Since we can choose s' arbitrary close to s we get

$$\limsup_{n\to\infty} \|a^n\|^{1/n} \le s \ .$$

We now vary s and get

$$\limsup_{n\to\infty} \|a^n\|^{1/n} \le \max\{0, r(a)\} \ . \tag{2.3}$$

We exclude the case $r(a) = -\infty$ as follows. In this case $(\lambda - a)^{-1}$ is holomorphic on all of $\mathbb{C}$. By (2.2) for $n = 1$ get get $a = 0$. But in this case $r(0) = 0$, a contradiction.

Combining (2.1) and (2.3) we get the desired assertion. □

Corollary 2.18. *If A is a unital Banach algebra and a is in A, then $\sigma(a) \neq \emptyset$*

Proof. This follows from $0 \le r(a)$. □

Corollary 2.19 (Gelfand-Mazur)**.** *Assume that A is unital Banach algebra such that every non-zero element is invertible. Then $A \cong \mathbb{C}$.*

Proof. We argue by contradiction. Let a be in A and assume that $a \neq \lambda$ for all λ in $\mathbb{C}$. Then $\lambda - a$ is invertible for all λ in $\mathbb{C}$ This implies that $\sigma(a) = \emptyset$ and hence $r(a) = -\infty$. This is a contradiction.

□

We consider a unital Banach algebra A and an ideal I in A. Then also $\bar{I}$ is an ideal.

Lemma 2.20. *If I is proper, then also $\bar{I}$.*

Proof. We show that for every b in I we have $\|1_A - b\| \ge 1$. This then implies that $1_A \notin \bar{I}$.

Assume by contradiction that $\|1 - b\| < 1$. Then b is invertible and $I = A$, a contradiction. □

3 $*$-algebras and C^*-algebras

In this section in consider algebras with an involution called $*$-algebras. We investigate the categories unital and non-unital of $*$-algebras. We then introduce the condition of compatibility of a norm with the involution called the C^*-equality and study its consequences.

Let A be in $\mathbf{Alg}_{\mathbb{C}}$.

Definition 3.1. *An involution on A is a complex antilinear map $* : A \to A$ (written on elements as $a \mapsto a^*$) such that:*

1. $* \circ * = \mathtt{id}$
2. $(aa')^* = a'^* a^*$

Definition 3.2. *A $*$-algebra is an algebra over $\mathbb{C}$ with an involution. A $*$-homomorphism between $*$-algebras is a homomorphism between algebras which preserves the involution.*

A $*$-homomorphism f between $*$-algebras is thus a homomorphism between algebras which in addition satisfies the identity $f(a^*) = f(a)^*$. By ${}^*\mathbf{Alg}_{\mathbb{C}}$ and ${}^*\mathbf{Alg}_{\mathbb{C}}^{\mathrm{nu}}$ we denote the categories of unital and not-necessarily unital $*$-algebras and corresponding $*$-homomorphisms.

Example 3.3. Complex conjugation is an involution on the algebra $\mathbb{C}$ turning it into a unital $*$-algebra.

If A is a $*$-algebra and X is a set, then A^X is a $*$-algebra with the involution defined pointwise by $(a^*)(x) := a(x)^*$.

If B is a subalgebra of a $*$-algebra which is invariant (as a set) by the involution, then B with the restriction of the involution to B is a $*$-algebra. □

Example 3.4. If H is a Hilbert space, then $B(H)$ is a $*$-algebra with the involution which sends an operator to its adjoint. The algebra of compact operators $K(H)$ is a $*$-subalgebra.

For $H = \mathbb{C}^n$ we get an involution on $\mathtt{Mat}(n)$. In this case $a^* = \bar{a}^t$, the complex conguate of the transposed matrix. □

Example 3.5. If G is a group, then on the group ring $\mathbb{C}[G]$ we have the involution given by

$$(\sum_{g \in G} \lambda_g [g])^* := \sum_{g \in G} \bar{\lambda}_g [g^{-1}] \ .$$

Note that here we need inverses in G and this definition does not work for general monoids. □

Example 3.6. The polynomial ring $\mathbb{C}[z]$ has an involution extending $z^* = z$. This extends further to the quotient field $\mathbb{C}(z)$

The differential operators $D(\mathbb{C})$ have the involution determined by $\partial^* := -\partial$ and $z^* := z$. Indeed the defining relation $\partial z - z\partial = 1$ is preserved:

$$1^* = 1 \ , \quad (\partial z - z\partial)^* = z^*\partial^* - \partial^* z* = -z\partial - (-\partial z) = \partial z - z\partial \ .$$

□

Let I be a two-sided ideal in a $*$-algebra A. For a subset S of A we write $S^* = \{s^* | s \in S\}$.

Definition 3.7. *I is a $*$-ideal if $I = I^*$.*

Example 3.8. If I is a $*$-subalgebra and a left or right-ideal, then it is automatically a $*$-ideal. Assume that I is right ideal and a $*$-subalgebra. Then for every a in A we have $ai = (i^* a^*)^* \in I^* = I$, hence I is also a left ideal. □

Example 3.9. Let A be in ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ and I be a $*$-ideal in A. Then A/I is $*$-algebra with involution given $[a]^* := [a^*]$. The map $A \to A/I$ is initial for $*$-homomorphisms from A to $*$-algebras which send the elements of I to zero.

Example 3.10. If S is a a subset of A, then we can form the smallest $*$-ideal

$${}^*(S) := \bigcap_{S \subseteq I} I$$

containing S, where the intersection runs over all $*$-ideals I of A containing S. In generaal ${}^*(S)$ can be larger than the two-sided ideal (S) generated by S. It is easy to see that ${}^*(S) = (S \cup S^*)$. The quotient map

$$A \to A/{}^*(S)$$

is initial for $*$-homomorphisms from A to $*$-algebras which send the elements of S to zero. □

The unitalization adjunction extends to $*$-algebras.

Lemma 3.11. *We have an adjunction*

$$(-)^u : {}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}} \leftrightarrows {}^*\mathbf{Alg}_{\mathbb{C}} : \mathrm{incl}$$

Proof. Let A in ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$. Then A^u has an involution given by $(a, \lambda)^* := (a^*, \bar{\lambda})$. One checks that the unit and counit of the adjunction from Lemma 1.27 are morphisms of $*$-algebras. This implies the assertion. □

The underlying set $S(A)$ of a $*$-algebra carries an action of the group C_2 such that the non-trivial element σ in C_2 acts by $\sigma a := a^*$. We therefore have a forgetful functor $S : {}^*\mathbf{Alg}_{\mathbb{C}} \to C_2\mathbf{Set}$.

Lemma 3.12. *We have adjunctions*

$$\mathrm{Free}^{*,\mathrm{nu}} : C_2\mathbf{Set} \leftrightarrows {}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}} : S \ , \quad \mathrm{Free}^{*} : C_2\mathbf{Set} \leftrightarrows {}^*\mathbf{Alg}_{\mathbb{C}} : S \ .$$

Proof. (sketch) We give an explicit construction of the left-adjoint. For X in $C_2\mathbf{Set}$ we first consider the vector space $\mathbb{C}[X]$ in with the anti-linear action of C_2 which extends the action on X. So

$$\sigma(\lambda_1[x_1] + \cdots + \lambda_n[x_n]) = \bar{\lambda}_1[\sigma x_1] + \cdots + \bar{\lambda}_n[\sigma x_n] \ .$$

Then we equip the tensor algebra $T(\mathbb{C}[X])$ with the anti-linear involution characterized by

$$(v_1 \otimes \cdots \otimes v_n)^* := \sigma(v_n) \otimes \cdots \otimes \sigma(v_1)$$

The resulting $*$-algebra in ${}^*\mathbf{Alg}_{\mathbb{C}}$ will be denoted by $\mathrm{Free}^*(X)$. It is the free unital $*$-algebra generated by the set X. It is straightforward to extend this construction to morphisms of sets so that we obtain the functor $\mathrm{Free}^* : \mathbf{Set} \to {}^*\mathbf{Alg}_{\mathbb{C}}$.

The unit of the adjunction is given by the canonical inclusion of C_2-sets $X \mapsto S(\mathrm{Free}^*(X))$ sending x in X to the basis vector $[x]$ of $\mathbb{C}[X]$ considered as a summand of $\mathrm{Free}^*(X)$. One then checks that for every A in ${}^*\mathbf{Alg}_{\mathbb{C}}$ the composition

$$\mathtt{Hom}_{{}^*\mathbf{Alg}_{\mathbb{C}}}(\mathrm{Free}^*(X), A) \stackrel{S}{\to} \mathtt{Hom}_{C_2\mathbf{Set}}(S(\mathrm{Free}^*(X)), S(A)) \stackrel{unit^*}{\to} \mathtt{Hom}_{C_2\mathbf{Set}}(X, S(A))$$

is a bijection.

The non-unital case is similar with $\mathrm{Free}^{*,\mathrm{nu}} := T^{\geq 1}(\mathbb{C}[X])$. □

Let $\mathcal{F} : {}^*\mathbf{Alg}_{\mathbb{C}} \to \mathbf{Alg}_{\mathbb{C}}$ or $\mathcal{F}^{\mathrm{nu}} : {}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}} \to \mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ denote the functors which forget the $*$-operation.

Lemma 3.13. *We have adjunctions*

$$L : \mathbf{Alg}_{\mathbb{C}} \leftrightarrows {}^*\mathbf{Alg}_{\mathbb{C}} : \mathcal{F} \ , \quad L^{\mathrm{nu}} : \mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}} \leftrightarrows {}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}} : \mathcal{F}^{\mathrm{nu}} \ .$$

Proof. (sketch) Let A be in $\mathbf{Alg}_{\mathbb{C}}$ and consider the exact sequence

$$0 \to I \to \mathrm{Free}(A) \to A \to 0 \ .$$

We form the C_2-set $A \sqcup A$ with the action of C_2 flipping the components. Let $i_0, i_1 : A \to A \sqcup A$ denote the two inclusions. We have a homomorphisms

$$\mathrm{Free}(i_0) : \mathrm{Free}(A) \to \mathrm{Free}(A \sqcup A)$$

induced by the inclusion of the first copy of A.

We now note that $\mathrm{Free}(A \sqcup A)$ has the structure of a $*$-algebra $\mathrm{Free}^*(A \sqcup A)$ as in Lemma 3.12. We let J be the $*$-ideal generated by $\mathrm{Free}(i_0)(I)$ and form $L(A) := \mathrm{Free}(A \sqcup A)/J$. We then have the diagram in $C^*\mathbf{Alg}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & \mathrm{Free}(A) & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \mathrm{Free}(i_0)} & & \downarrow{\scriptstyle unit} & & \\ 0 & \longrightarrow & \mathcal{F}(J) & \longrightarrow & \mathcal{F}(\mathrm{Free}^*(A \sqcup A)) & \longrightarrow & \mathcal{F}(L(A)) & \longrightarrow & 0 \end{array}$$

defining the unit map.

It is straightforward to define L on morphisms so that one gets a functor $L : \mathbf{Alg}_{\mathbb{C}} \to {}^*\mathbf{Alg}_{\mathbb{C}}$. Finally one checks that for every B in ${}^*\mathbf{Alg}_{\mathbb{C}}$ the composition

$$\mathtt{Hom}_{{}^*\mathbf{Alg}_{\mathbb{C}}}(L(A), B) \stackrel{\mathcal{F}}{\to} \mathtt{Hom}_{\mathbf{Alg}_{\mathbb{C}}}(\mathcal{F}(L(A)), \mathcal{F}(B)) \stackrel{unit^*}{\to} \mathtt{Hom}_{\mathbf{Alg}_{\mathbb{C}}}(A, \mathcal{F}(B))$$

is a bijection.

The non-unital case is analoguous using $\mathrm{Free}^{\mathrm{nu}}$. □

Proposition 3.14. *The categories* ${}^*\mathbf{Alg}_{\mathbb{C}}$ *and* ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ *are complete and cocomplete.*

Proof. By Lemma 3.13 and Corollary 1.30 the forgetful functors ${}^*\mathbf{Alg}_{\mathbb{C}} \to \mathbf{Set}$ and ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}} \to \mathbf{Set}$ are right-adjoints. Therefore limits in ${}^*\mathbf{Alg}_{\mathbb{C}}$ and ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ are calculated on the level of underlying sets, respectively. The results then equipped with an involution induced by functoriality.

The argument for colimits and is similar as in Prop. 1.31. It is enough to show the existence of coequalizers and coproducts. This is done in Examples 3.15 and 3.16 below. □

Example 3.15. Let $(A_i)_{i\in I}$ be a family in ${}^*\mathbf{Alg}_{\mathbb{C}}$. Then we consider for every i the exact sequence

$$0 \to I_i \to \mathrm{Free}^*(A_i) \to A_i \to 0 \ .$$

Then we form the $*$-algebra $\mathrm{Free}^*(\bigsqcup_{i\in I} A_i)$ and the $*$-ideal J generated by the images of the ideals I_i. The quotient $\mathrm{Free}^*(\bigsqcup_{i\in I} A_i)/J$ together with the family $(e_i)_{i\in I}$ of $*$-homomorphisms $e_i : A_i \to \mathrm{Free}^*(\bigsqcup_{i\in I} A_i)/J$ induced by

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_i & \longrightarrow & \mathrm{Free}^*(A_i) & \longrightarrow & A_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle e_i} & & \\ 0 & \longrightarrow & J & \longrightarrow & \mathrm{Free}^*(\bigsqcup_{i\in I} A_i) & \longrightarrow & \mathrm{Free}^*(\bigsqcup_{i\in I} A_i)/J & \longrightarrow & 0 \end{array}$$

represent the coproduct of the family.

The same construction works in ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ using $\mathrm{Free}^{*,\mathrm{nu}}$ in place of Free^*.

Example 3.16. Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

be a coequalizer diagram in ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ or ${}^*\mathbf{Alg}_{\mathbb{C}}$. Then the elements $f(a) - g(a)$ for all a in A generate a $*$-ideal I in B. The projection $B \to B/I$ presents B/I as the coequalizer $\mathrm{Coeq}(f, g)$ in ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ or ${}^*\mathbf{Alg}_{\mathbb{C}}$, respectively.

We consider an algebra A with involution $*$ and a norm $\|-\|$.

Definition 3.17. $(A, *, \|-\|)$ *is a normed* $*$*-algebra if* $\|a^*\| = \|a\|$ *for all* a *in* A.

In other words, for a normed $*$-algebra we require that $*$ acts isometrically.

Example 3.18. If $\|-\|$ is any norm on a $*$-algebra, then we can form a new norm $\|-\|'$ by

$$\|-\|' = \max\{\|a\|, \|a^*\|\} .$$

Then $(A, *, \|-\|')$ is a normed $*$-algebra.

In general it is not clear that $\|-\|'$ is equivalent to $\|-\|$. But if A is a Banachalgebra with norm $\|-\|$ and $*$ is continuous, then $\|-\|'$ is equivalent to $\|-\|$. □

Usually we use the symbol A in order to denote a $*$-algebra or a normed $*$-algebra.

Let A be a normed $*$-algebra

Definition 3.19. $\|-\|$ *has the* C^**-property if* $\|a^*a\| = \|a\|^2$ *for all* a *in* A.

We call such a norm a C^*-norm

Definition 3.20. *A* C^**-*$*$*-algebra is a* $*$*-algebra which admits a norm turning it into a normed Banach algebra and which satisfies the* C^**-property.*

Remark 3.21. Being C^*-algebra is property of a $*$-algebra. Note that the norm is not part of the data for a C^*-algebra. We just require existence. But it will turn out later that it is actually uniquely determined.

In order to show that a given $*$-algebra is a C^*-algebra one usually produces a norm and shows that it has the required properties.

Our definition is not the standard definition of a C^*-algebra, but equivalent to it, as we shall see below. □

We let $C^*\mathbf{Alg}^{\mathrm{nu}}$ be the full subcategory of ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ of C^*-algebras and $C^*\mathbf{Alg}$ be the full subcategory of ${}^*\mathbf{Alg}_{\mathbb{C}}$ of unital C^*-algebras.

Example 3.22. If A is a C^*-algebra and B is $*$-subalgebra of A which is closed w.r.t a norm exhibiting A as a C^*-algebra, then B is a C^*-algebra, too.

Any closed $*$-subalgebra of $\mathbb{B}(H)$ is a C^*-algebra.

Actually, one classical definition of the notion of a C^*-algebra is as a closed $*$-subalgebra of $B(H)$ form some Hilbert space H. One can show that every C^*-algebra is isomorphic to such a subalgebra. □

Example 3.23. Let X be a topological space and A be a C^*-algebra. Then $C_b(X, A)$ is again a C^*-algebra exhibited by the norm $\|a\|_\infty := \sup_{x\in X} \|a(x)\|_A$. Furthermore, $C_0(X, A)$ is a closed subalgebra of $C_b(X, A)$ and hence a C^*-algebra. □

We consider a C^*-algebra with norm $\|-\|$. Let a be in A. Note that $r(a^*a)$ (the spectral radius of a^*a) only depends on the algebra A and not on the norm.

Lemma 3.24 (C^*-norm completely determined by $*$-algebra structure)**.** *We have* $\|a\|^2 = r(a^*a)$ *for all* a *in* A*.*

Proof. We use the formula for the spectral radius of a^*a given in Lemma 2.17. For k in $\mathbb{N}$, using the C^*-property repeatedly, we have

$$\|a\|^{2k+1} = \|a^*a\|^{2^k} = \|(a^*a)^2\|^{2^{k-1}} = \cdots = \|(a^*a)^{2^k}\| \ .$$

This gives $\|a\|^2 = \|(a^*a)^{2^k}\|^{2^{-k}}$ for all k in $\mathbb{N}$. We take the limit as $k \to \infty$ and get $\|a\|^2 = r(a^*a)$. □

Corollary 3.25. *A C^*-algebra has a unique C^*-norm.*

We now consider a morphism $f : A \to B$ in $C^*\mathbf{Alg}^{\mathrm{nu}}$. Note that by definition f is just a homomorphism of algebras which is compatible with the $*$-operation. The following Lemma shows that it is a contraction (in particular continuous) provided we equip the algebras with their unique C^*-norms $\|-\|_A$ and $\|-\|_B$.

Corollary 3.26 (automatic continuity of morphisms)**.** *For all* a *in* A *we have* $\|f(a)\|_B \leq \|a\|_A$*.*

Proof. We use Corollary 1.48 and Lemma 3.24 in order to calculate

$$\|f(a)\|_B^2 = r(f(a)^*f(a)) = r(f(a^*a)) \leq r(a^*a) = \|a\|_A^2 \ .$$

□

Next we show that unitalization is compatible with C^*-algebras.

Lemma 3.27. *We have an adjunction*

$$(-)^u : C^*\mathbf{Alg}^{\mathrm{nu}} \leftrightarrows C^*\mathbf{Alg} : \mathrm{incl} \ .$$

Proof. We restrict the adjunction from Lemma 3.11 to C^*-algebras. It suffices to show that $(-)^u$ preserves C^*-algebras.

First assume that A is unital. Then we have an isomorphism of $*$-algebras $A^u \overset{\cong}{\to} A \oplus \mathbb{C}$ given by $(a, \lambda) \mapsto (a + \lambda 1_A, \lambda)$. Therefore the norm $\|(a, \lambda)\| := \max\{|\lambda|, \|a + \lambda 1_A\|_A\}$ is a C^*-norm on A^u.

We now assume that A is non-unital. Note that the obvious norm on A^u is not a C^*-norm: We have

$$\|(a, \lambda)^*(a, \lambda)\| = \|(a^*a + \bar{\lambda}a + \lambda a^*, |\lambda|^2)\| = \|a^*a + \bar{\lambda}a + \lambda a^*\| + |\lambda|^2$$

which is in general not equal to

$$(\|a\| + |\lambda|)^2 = \|a^*a\| + 2|\lambda|\|a\| + |\lambda|^2 \ .$$

Note that A^u acts on A be left multiplication $(b, \lambda)a = (ba + \lambda a)$. We let $\| - \|'$ be the operator norm, i.e.

$$\|(b, \lambda)\|' := \sup_{a \in A \ , \|a\|=1} \|ba + \lambda a\| \ .$$

It is a $*$-norm.

We first verify the C^*-property for $\| - \|'$. We have

$$\begin{aligned}
\|(b, \lambda)^*(b, \lambda)\|' &= \sup_{\|a\|=1} \|b^*ba + \bar{\lambda}ba + \lambda b^*a + |\lambda|^2 a\| \\
&= \sup_{\|a\|=1} \|a^*\|\|b^*ba + \bar{\lambda}ba + \lambda b^*a + |\lambda|^2 a\| \\
&\geq \sup_{\|a\|=1} \|a^*(b^*ba + \bar{\lambda}ba + \lambda b^*a + |\lambda|^2 a)\| \\
&= \sup_{\|a\|=1} \|(a^*b^* + a^*\bar{\lambda})(ba + \lambda a)\| \\
&= \sup_{\|a\|=1} \|ba + \lambda a\|^2 = \|(b, \lambda)\|'^{,2}.
\end{aligned}$$

We get the reverse inequality

$$\|(b, \lambda)^*(b, \lambda)\|' \leq \|(b, \lambda)\|'^{,2}$$

using the sub-multiplicativity of the norm $\|-\|'$ (a general property of the operator norm) and the fact that it is a $*$-norm.

It remains to show that $\|-\|$ is equivalent to the obvious norm. This implies that A^u is complete w.r.t to $\|-\|'$. The inequality

$$\|(a,\lambda)\|' \leq \|a\| + |\lambda|$$

is clear. We claim that $\|(a,\lambda)\|' = 0$ implies $(a,\lambda) = 0$. This claim implies that $\|-\|'$ is equivalent to $\|-\|$. Indeed, if A' denotes the Banach-closure of A with respect to $\|-\|'$, then the map $A \to A'$ is a continuous surjective map of Banach spaces. The condition implies injectivity. Hence $A \to A'$ is an isomorphism by the bounded inverse theorem.

We now show the claim. Assume that $\lambda \neq 0$. Then $ab + \lambda b = 0$ for all b in A implies $\lambda^{-1}a$ is a unit of A. Since A was non-unital this is impossible.

If $\lambda = 0$, then in particular $aa^* = 0$ and hence $\|a\|^2 = \|aa^*\| = 0$ which implies that $a = 0$. □

Let A be in ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ and a be in A.

Definition 3.28. *a is called:*

1. *selfadjoint if $a^* = a$*
2. *normal if $[a^*, a] = 0$*
3. *a projection if $a^2 = a$*
4. *an orthogonal projection if $a^2 = a$ and $a^* = a$*
5. *a partial isometry if a^*a and aa^* are (necessarily orthogonal) projections*
6. *an isometry if $a^*a = 1_A$ (for A in ${}^*\mathbf{Alg}_{\mathbb{C}}$)*
7. *a unitary if $a^*a = 1_A$ and $aa^* = 1_A$ (for A in ${}^*\mathbf{Alg}_{\mathbb{C}}$)*

Let A be a C^*-algebra and a be in A.

Lemma 3.29. *If a is a partial isometry or an orthogonal projection, then $\|a\| \in \{0,1\}$.*

Proof. Let a be an orthogonal projection. Then we have

$$\|a\|^2 = r(a^*a) = r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \lim_{n\to\infty} \|a\|^{1/n} \in \{0,1\} \ .$$

Assume now that a is a partial isometry. Then

$$\|a\|^2 = \|a^*a\| \in \{0, 1\}$$

by the first case since a^*a is aorthogonal projection. □

Let A be in $C^*\mathbf{Alg}$ and u be in A.

Lemma 3.30. *If u is unitary, then $\sigma(u) \subseteq U(1)$.*

Proof. We consider λ in $\mathbb{C}$. We first assume that $|\lambda| > 1$. Then we have

$$(\lambda - u) = \lambda(1 - \lambda^{-1}u)$$

and the right-hand side is invertible since $\|\lambda^{-1}u\| < 1$ We now assume that $|\lambda| < 1$. Then

$$(\lambda - u) = -u(1 - \lambda u^*)$$

and the right-hand side is invertible since $\|\lambda u^*\| < 1$.

In both cases we see that $\lambda \in \rho(u)$. □

Let A be in $C^*\mathbf{Alg}^{\mathrm{nu}}$ and a be in A.

Lemma 3.31. *If a is self-adjoint, then $\sigma(a) \subseteq \mathbb{R}$.*

Proof. By considering the image $(a, 0)$ of a in A^u we can assume that A is unital. We assume that λ in $\sigma(a)$. Then we can define e^{ia} by a convergent power series. We claim that $e^{i\lambda} \in \sigma(e^{ia})$ and that $e^{i\lambda} \in U(1)$. These two assertions imply that $\lambda \in \mathbb{R}$.

We now show the claim. We consider

$$b := \sum_{n=1}^{\infty} \frac{i^n(a-\lambda)^n}{n!} \ .$$

Then

$$e^{ia} - e^{-\lambda} = e^{i\lambda}(e^{i(a-\lambda)} - 1) = (a - \lambda)e^{i\lambda}b$$

Since $(a - \lambda)$ not invertible also $e^{ia} - e^{-\lambda}$ is not invertible. Hence $e^{i\lambda} \in \sigma(a)$.

In general, if c, d are in A and $[c, d] = 0$, then we have $e^c e^d = e^{c+d}$ by a calculation with the power series. We furthermore have the relation$(e^c)^* = e^{c^*}$.

We have $(e^{ia})^* = e^{-ia}$. This implies that $e^{ia}(e^{ia})^* = 1_A = (e^{ia})^* e^{ia}$. Thus e^{ia} is unitary and we have $\sigma(e^{ia}) \subseteq U(1)$ by Lemma 3.30. Hence $e^{i\lambda} \in U(1)$ as claimed. □

4 Gelfand duality

In this section we consider the structure of the subcategory of $C^*\mathbf{Alg}$ of commutative C^*-algebras.

Let $L : \mathcal{C} \to \mathcal{D}$ be a functor.

Definition 4.1. *We say that L is a reflective localization if it fits into an adjunction*

$$L : \mathcal{C} \leftrightarrows \mathcal{D} : R$$

where R is fully faithful.

Example 4.2. Here we present some examples of reflective localizations for the purpose of illustration.

Let $\mathbf{Metr}$ be the category of metric spaces and isometries, and $\mathbf{Metr}_{\mathrm{compl}}$ be the full subcategory of complete metric spaces. Then

$$\mathrm{compl} : \mathbf{Metr} \leftrightarrows \mathbf{Metr}_{\mathrm{compl}} : \mathrm{incl}$$

is a reflective localization, where compl sends a metric space to its completion.

Let $\mathbf{Hausd}$ be the full subcategory of $\mathbf{Top}$ of Hausdorff spaces. Then we have a reflective localization

$$(-)_{\mathrm{Hausd}} : \mathbf{Top} \leftrightarrows \mathbf{Hausd} : \mathrm{incl}\ ,$$

where for X in $\mathbf{Top}$ we denote by X_{Hausd} is the maximal Hausdorf quotient of X.

We have a reflective localization

$$K_0 : \mathbf{Monoids} \leftrightarrows \mathbf{Groups} : \mathrm{incl}\ ,$$

where K_0 sends a monoid to its group completion (Grothendieck construction). □

For every topological space X we have a commutative C^*-algebra $C_b(X)$ of continuous bounded $\mathbb{C}$-valued functions on X. If $f : X \to X'$ is a continuous map, then we have a homomorphism

$$f^* : C_b(X') \to C_b(X)\ , \quad a \mapsto a \circ f\ .$$

Let $C^*\mathbf{Alg}^{\mathrm{comm}}$ and $C^*\mathbf{Alg}^{\mathrm{nu,comm}}$ be the full subcategories of $C^*\mathbf{Alg}$ and $C^*\mathbf{Alg}^{\mathrm{nu}}$ of unital and not necessarily unital commutative C^*-algebras. Then this construction determines a functor

$$C_b : \mathbf{Top}^{\mathrm{op}} \to C^*\mathbf{Alg}^{\mathrm{comm}}\ .$$

The main theorem of this section is:

Theorem 4.3. *There is a reflective localization*

$$C_b : \mathbf{Top} \leftrightarrows (C^*\mathbf{Alg}^{\mathrm{comm}})^{\mathrm{op}} : G$$

where G identifies $(C^*\mathbf{Alg}^{\mathrm{comm}})^{\mathrm{op}}$ with the full subcategory of $\mathbf{Hausd}^{\mathrm{comp}}$ of $\mathbf{Top}$ of compact Hausdorff spaces.

The functor G is called the Gelfand transformation. It sends a commutative C^*-algebra A to its space $G(A)$ of characters, i.e., homomorphisms $A \to \mathbb{C}$. The proof of the theorem will be given after some preparations about characters.

Let A be in $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$.

Definition 4.4. *A character of A is a homomorphism $A \to \mathbb{C}$.*

The set of characters of A is thus the morphism set $\mathtt{Hom}_{\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}}(A, \mathbb{C})$. We actually have a functor

$$\mathtt{Hom}_{\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}}(-, \mathbb{C}) : (\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}})^{\mathrm{op}} \to \mathbf{Set}$$

represented by $\mathbb{C}$.

Non-zero characters on a unital algebra are automatically unital. Assume that A is in $\mathbf{Alg}_{\mathbb{C}}$ and that $\phi : A \to \mathbb{C}$ is a character.

Lemma 4.5. *If $\phi \neq 0$, then $\phi(1_A) = 1$.*

Proof. Indeed, let a in A be such that $\phi(a) \neq 0$. Then we have $\phi(a) = \phi(1_A a) = \phi(1_A)\phi(a)$. This implies that $\phi(1_A) = 1$. □

Characters of C^*-algebras are automatically morphisms of $*$-algebras. Indeed, let A be in $C^*\mathbf{Alg}^{\mathrm{nu}}$ and ψ be a character on A.

Lemma 4.6. *ψ is a $*$-homomorphism.*

Proof. We can extend ψ to a homomorphism $\psi^u : A^u \to \mathbb{C}$ such that $\psi^u(a, 0) = \psi(a)$. We have $\psi^u(a, 0) \in \sigma^u(a, 0)$ by Lemma 1.45. This implies $\psi(a) \in \sigma(a) \cup 0$ by Lemma 1.50.

First assume that a is selfadjoint. Then using Lemma 3.31 (this lemma applies since we assume that A is C^*-algebra and not only a $*$-algebra)

$$\psi(a) \in \sigma(a) \cup \{0\} \subseteq \mathbb{R} .$$

If a is general, then we write a as a sum of selfadjoints

$$a = \frac{a + a^*}{2} + i\frac{a - a^*}{2i} .$$

We then have

$$\psi(a) = \psi(\frac{a+a^*}{2}) + i\psi(\frac{a-a^*}{2i}) \ , \quad \psi(a^*) = \psi(\frac{a+a^*}{2}) + i\psi(\frac{a^*-a}{2i}) = \overline{\psi(a)}$$

□

Recall that $*$-homomorphisms between C^*-algebras are automatically continuous. So characters on a C^*-algebra are automatically continuous.

Let A be in $C^*\mathbf{Alg}^{\mathrm{nu}}$. By A^* we denote the dual of A in the sense of Banach spaces. The C^*-norm on A induces a norm in A^*. On A^* we also have the weak topology generated by the maps $(\phi \mapsto \phi(a)) : A^* \to \mathbb{C}$ for all a in A. The unit ball $B(A^*)$ is compact w.r.t. to the weak topology by the theorem of Banach-Alaoglou.

We let $\hat{A}$ denote the set of non-zero characters of A. If ϕ in $\hat{A}$, then $\phi \in B(A^*)$. We equip $\hat{A}$ with the weak topology. The set $\hat{A} \cup \{0\}$ is a closed subset of $B(A^*)$. In fact, if $(\phi_i)_{i \in I}$ is a converging net in $\hat{A}$, then $\mathtt{lim}_{i\in I}\, \phi_i$ is again a character. Indeed, for a, b in A we have

$$\mathtt{lim}_{i\in I} \phi_i(ab) = \mathtt{lim}_{i\in I} \phi_i(a)\phi_i(b) = \mathtt{lim}_{i\in I} \phi_i(a) \, \mathtt{lim}_{i\in I} \phi_i(b) \ .$$

Note that it may happen that $\mathtt{lim}_{i\in I}\, \phi_i = 0$.

It follows that $\hat{A} \cup \{0\}$ is compact. Consequently, $\hat{A}$ is locally compact.

Lemma 4.7. *If A is unital, then $\hat{A}$ is compact.*

Proof. For every ϕ in $\hat{A}$ we have $\phi(1_A) = 1$. Hence any limit point of $\hat{A}$ in $B(A^*)$ satisfies this condition, too. It follows that 0 is isolated in $\hat{A} \cup \{0\}$ and hence $\hat{A}$ itself is closed and hence compact. □

Let A be in $C^*\mathbf{Alg}^{\mathrm{nu}}$ and consider a in A. Then we define a function

$$g_A(a) : \hat{A} \to \mathbb{C} \ , \quad \phi \mapsto g_A(a)(\phi) := \phi(a) \ .$$

Definition 4.8. *The function $g_A : \hat{A} \to \mathbb{C}$ is called the Gelfand transform of a.*

Lemma 4.9. *We have $g_A(a) \in C_b(\hat{A})$ and $\|g_A(a)\|_\infty \le \|a\|$.*

Proof. The function $g_A(a)$ is continuous by the very definition of the weak topology. Furthermore, since a $*$-homomorphism between C^*-algebras is contractive we have

$$\|g_A(a)\|_\infty = \sup_{\phi \in \hat{A}} |\phi(a)| \le \sup_{\phi \in \hat{A}} \|\phi\| \|a\| \le \|a\| \ .$$

□

Let A be in $C^*\mathbf{Alg}^{\mathrm{nu}}$ and consider a in A. Then we consider the set of values of the Gelfand transformation $g_A(a)$.

Lemma 4.10. *We have $g_A(a)(\hat{A}) \cup \{0\} \subseteq \sigma^u(a)$. If A is commutative, then this is an equality.*

Proof. We have already seen (in the proof of Lemma 4.6) that

$$g_A(a)(\hat{A}) \subseteq \sigma^u(a) \ .$$

Indeed, for ϕ in $\hat{A}$ we have $g_A(a)(\phi) = \phi(a) \in \sigma^u(a)$.

We now assume that A is commutative and show that

$$\sigma^u(a) \subseteq g_A(a)(\hat{A}) \cup \{0\} \ .$$

We assume that λ is in $\mathbb{C} \setminus \{0\}$ and $\lambda \in \sigma^u(a)$. Then $(\lambda - a)A^u$ is proper ideal in A^u since otherwise $(\lambda - a)$ would be invertible. Here we use commutativity in order to conclude that the ideal is two-sided. Using Zorn's Lemma we find a maximal proper ideal I with $(\lambda - a)A^u \subseteq I$

We claim that I is closed. If it is not closed, $\bar{I}$ would larger and also proper by Lemma 2.20. Then A^u/I is a field and a Banach algebra. It follows from Corollary 2.19 (Gelfand-Mazur) that $A^u/I \cong \mathbb{C}$. We define the character

$$\psi : A \to A^u \to A^u/I = \mathbb{C} \ .$$

By construction $\psi(a) = \lambda \neq 0$ so that $\psi \in \hat{A}$. Hence $\lambda \in g_a(\hat{A})$. □

We consider A in $C^*\mathbf{Alg}^{\mathrm{comm}}$. The following is the key result leading to the main theorem of this section.

Theorem 4.11 (Gelfand)**.** *The Gelfand transform $g_A : A \to C_b(\hat{A})$ is an isomorphism of C^*-algebras .*

Proof. We first show that g_A is a homomorphism of $*$-algebras. It is linear since

$$g_A(a + \lambda b)(\phi) = \phi(a + \lambda\phi) = \phi(a) + \lambda\phi(b) = (g_A(a) + \lambda g_A(b))(\phi)$$

and multiplicative since

$$g_A(ab)(\phi) = \phi(ab) = \phi(a)\phi(b) = g_A(a)(\phi)g_A(b)(\phi) = (g_A(a)g_A(b))(\phi) \ .$$

Finally we show that g_A is compatible with the involution:

$$g_A(a^*)(\phi) = \phi(a^*) = \overline{\phi(a)} = g_A(a)^*(\phi) \ .$$

Next we show that g_A is isometric. We indeed have

$$\|g_A(a)\|_\infty^2 = \sup_{\phi\in\hat{A}} |\phi(a)|^2 = \sup_{\phi\in\hat{A}} |\phi(a)^*\phi(a)| = \sup_{\phi\in\hat{A}} |\phi(a^*a)| \stackrel{!}{=} r(a^*a) \stackrel{!!}{=} \|a\|^2 \ .$$

Here we use Lemma 4.10 at the equality marked by ! and Lemma 3.24 at the equality marked by !!.

This implies that g_A is injective.

We now show that g_A is surjective. We are going to apply the Stone-Weierstrass theorem to the compact space $\hat{A}$. We observe that $g_A(A)$ is C^*-subalgebra of $C_b(\hat{A})$. It clearly separates points. Finally for every ϕ in $\hat{A}$ there exists a in A such that $g_A(a)(\phi) = \phi(a) \neq 0$ (since $\hat{A}$ consists of non-zero characters). By the Stone-Weierstrass we conclude that $g_A(A)$ is dense in $C_b(\hat{A})$. Since g_A is an isometry we have $g_A(A) = C_b(\hat{A})$. □

Proof of Theorem 4.3. We define the functor

$$G : (C^*\mathbf{Alg}^{\mathrm{comm}})^{\mathrm{op}} \to \mathbf{Top} \ , \quad G(A) := \hat{A} \ .$$

If $f : A \to B$ is a morphism in $C^*\mathbf{Alg}^{\mathrm{comm}}$, then $f^* : B^* \to A^*$ is continuous for the weak topology. Indeed, for every a in A the function $f^*(-)(a) = (-)(f(a)) : B^* \to \mathbb{C}$ is continuous.

Since $f(1_A) = 1_B$ the pull-back preserves non-zero characters. This implies that the restriction of f^* to non-zero characters is a continuous map $G(f) : G(B) \to G(A)$.

In order to construct the adjunction claimed in Theorem 4.3 we construct the unit and counit.

The counit $u : C_b \circ G \to \mathtt{id}$ will be given by the family $(g_A^{\mathrm{op}})_{A\in(C^*\mathbf{Alg}^{\mathrm{comm}})^{\mathrm{op}}}$. Note that without opping we have $g_A : A \to C_b(G(A))$.

We check naturality of u. Let $f : A \to B$ be a morphism in $C^*\mathbf{Alg}^{\mathrm{comm}}$. We must check that

$$\begin{CD} A @>{f}>> B \\ @V{g_A}VV @VV{g_B}V \\ C_b(G(A)) @>{C_b(G(f))}>> C_b(G(B)) \end{CD}$$

commutes. For a in A and ϕ in $G(B)$ we indeed have

$$g_B(f(a))(\phi) = \phi(f(a)) = G(f)(\phi)(a) = g_A(a)(G(f)(\phi)) = (C_b(G(f))(g_A)(a))(\phi) \ .$$

The unit will $h : \mathtt{id} \to G \circ C_b$ is given by the family $(h_X)_{X\in\mathbf{Top}}$ is defined by

$$h_X : X \to G(C_b(X)) \ , \quad h_X(x) := (a \mapsto a(x)) \ .$$

We must check that h_X is continuous. To this end we observe that for every a in $C_b(X)$ the function $x \mapsto h_X(x)(a) = a(x)$ is continuous. In view of the definition of the weak topology on $G(C_b(X))$ this implies that $X \to G(C_b(X))$ is continuous.

We check naturality of h. Let $f : X \to Y$ be a morphism in **Top**. Then we must check that

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle h_X} & & \downarrow{\scriptstyle h_Y} \\ G(C_b(X)) & \xrightarrow{G(C_b(f))} & G(C_b(Y)) \end{array}$$

commutes. For x in X and b in $C_b(Y)$ we have

$$h_Y(f(x))(b) = b(f(x))$$

and

$$G(C_b(f))(h_X(x))(b) = h_X(x)(C_b(f)(b)) = b(f(x)) \ .$$

We next show that u, h define an adjunction. To this end we consider the maps

$$\alpha : \mathtt{Hom}_{(C^*\mathbf{Alg}^{\mathrm{comm}})^{\mathrm{op}}}(C_b(X), B) \stackrel{G}{\to} \mathtt{Hom}_{\mathbf{Top}}(G(C_b(X)), G(B)) \stackrel{h_X^*}{\to} \mathtt{Hom}_{\mathbf{Top}}(X, G(B)) \ .$$

$$\beta : \mathtt{Hom}_{\mathbf{Top}}(X, G(B)) \stackrel{C_b}{\to} \mathtt{Hom}_{(C^*\mathbf{Alg}^{\mathrm{comm}})^{\mathrm{op}}}(C_b(X), C_b(G(B))) \stackrel{u_{B,*}}{\to} \mathtt{Hom}_{(C^*\mathbf{Alg}^{\mathrm{comm}})^{\mathrm{op}}}(C_b(X), B) \ .$$

We show that these maps are inverse to each other. We have for $r : B \to C_b(X)$, b in B, and x in X that

$$\begin{aligned} (\beta(\alpha(r))(b))(x) &= (C_b(\alpha(r)) \circ u_B)(b)(x) \\ &= C_b(\alpha(r))(u_B(b))(x) \\ &= u_B(b)(\alpha(r)(x)) \\ &= \alpha(r)(x)(b) \\ &= (G(r) \circ h_X)(x)(b) \\ &= r(b)(x) \ , \end{aligned}$$

hence $\beta \circ \alpha = \mathtt{id}$. We furthermore calculate for $s : X \to G(B)$, x in X and b in B

$$\begin{aligned} \alpha(\beta(s))(x)(b) &= (G(\beta(s)) \circ h_X)(x)(b) \\ &= G(\beta(s))(\mathtt{ev}_x)(b) \\ &= \beta(s)(b)(x) \\ &= (C_b(s) \circ u_B)(b)(x) \\ &= C_b(s)(u_B(b))(x) \\ &= u_B(b)(s(x)) = s(x)(b), \end{aligned}$$

hence $\alpha \circ \beta = \mathtt{id}$. This finishes the construction of the adjunction. By Theorem 4.11 the counit of the adjunction is an isomorphism. Consequently, the right-adjoint is fully faithful.

Its image consists of compact Hausdorff spaces. We must show that all compact Hausdorff spaces belong to the essential image.

Let X be a compact Hausdorff space. Then we have a map $h_X : X \to G(C_b(X))$. In order to show that it is an isomorphism it suffices to show that it is a bijection.

We first show that h_X is injective. Let x, x' be X be distinct points. Since $C_b(X)$ separates points (by the Urysohn Lemma since compact Hausdorff spaces are normal) exists f in $C_b(X)$ such that $f(x) \neq f(x')$. Then $h_X(x)(f) = f(x) \neq f(x') = h_X(x')(f)$. Hence $h_X(x) \neq h_X(x')$.

We now show surjectivity of h_X. We consider a non-zero character $\psi : C_b(X) \to \mathbb{C}$. Then $\mathtt{ker}(\psi)$ is a proper closed $*$-subalgebra of $C_b(X)$. This subalgebra also separates points as we will show now. Let x, x' in X be distinct and choose a, a' in $C_b(X)$ such that $a(x) = 1$ and $a(x') = 0$ and $a'(x) = 0$ and $a'(x') = 1$. Then $\psi(a')a - \psi(a)a' \in \mathtt{ker}(\psi)$. Then either $\psi(a) = 0$ and a does the job, or otherwise $(\psi(a')a - \psi(a)a')(x) = \psi(a') - \psi(a)$ and $(\psi(a')a - \psi(a)a')(x') = \psi(a') + \psi(a)$ are distinct and $\psi(a')a - \psi(a)a'$ separates the two points.

By the Stone-Weierstraß Theorem and since $\mathtt{ker}(\psi)$ is a closed proper ideal of $C_b(X)$ there exists a point x in X such that $a(x) = 0$ for all a in $\mathtt{ker}(\psi)$ (because otherwise $\mathtt{ker}(\psi) = C_b(X)$ by SWT). We now show that $\psi = h_X(x)$. Let f be in $C_b(X)$. Then we have $f = \psi(f) + (f - \psi(f))$, where $f - \psi(f) \in \mathtt{ker}(\psi)$ and hence $(f - \psi(f))(x) = 0$. We get

$$f(x) = \psi(f) + (f - \psi(f))(x) = \psi(f) \ .$$

This finishes the verification that h_X is an isomorphism for compact Hausdorff spaces X. This finishes the proof of Theorem 4.3. □

Corollary 4.12 (Gelfand duality)**.** *The functors C_b restricts to an equivalence of categories*

$$C_b : \mathbf{Hausd}^{\mathrm{comp}} \stackrel{\simeq}{\to} (C^*\mathbf{Alg}^{\mathrm{comm}})^{\mathrm{op}}$$

with inverse G.

5 The non-unital case

We extend Gelfand duality to the non-unital case.

For a category $\mathcal{C}$ and object c in $\mathcal{C}$ we can consider the slice category $\mathcal{C}_{/c}$. An object in $\mathcal{C}_{/c}$ is a morphism $c' \to c$ in $\mathcal{C}$. A morphism $(c' \to c) \to (c'' \to c)$ in $\mathcal{C}_{/c}$ is a commutative triangle

$$\begin{array}{ccc} c' & \longrightarrow & c'' \\ & \searrow \quad \swarrow & \\ & c & \end{array} \ .$$

Analoguously we define $\mathcal{C}_{c/}$ such that $\mathcal{C}_{c/} \cong (\mathcal{C}^{\mathrm{op}}_{/c})^{\mathrm{op}}$. We extend the unitalization functor to a functor

$$U : C^*\mathbf{Alg}^{\mathrm{nu}} \to C^*\mathbf{Alg}_{/\mathbb{C}} \ , \quad A \mapsto (A^u \to \mathbb{C}) \ .$$

Lemma 5.1. *The functor U is an equivalence of categories.*

Proof. The inverse $K : C^*\mathbf{Alg}_{/\mathbb{C}} \to C^*\mathbf{Alg}^{\mathrm{nu}}$ sends $(\phi : B \to \mathbb{C})$ to $\mathtt{ker}(\phi)$. A morphism in $C^*\mathbf{Alg}_{/\mathbb{C}}$ is a commuting triangle

$$\begin{array}{ccc} B & \xrightarrow{h} & B' \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\phi'} & \\ & \mathbb{C} & \end{array} \ .$$

We get an induced morphism $K(h) : K(\phi) \to K(\phi')$, where $K(h)$ is the restriction of h to $\mathtt{ker}(\phi)$ considered as a homomorphism with values in $\mathtt{ker}(\phi')$.

We now provide the natural isomorphisms $\alpha : \mathtt{id} \to K \circ U$ and $\beta : U \circ K \to \mathtt{id}$ exhibiting U and K as inverses to each other. The canonical inclusion $A \to A^u$ identifies A with $K(U(A))$. We let $\alpha_A : A \to K(U(A))$ be this canonical inclusion.

For $\phi : B \to \mathbb{C}$ in $C^*\mathbf{Alg}_{/\mathbb{C}}$ we have the canonical inclusion $K(\phi) \to B$. Since B is unital we can extend it uniquely to $U(K(\phi)) \to B$. Then we let $\beta_B : (\mathtt{ker}(\phi)^u \to \mathbb{C}) \to (\phi : B \to \mathbb{C})$ be the resulting morphism in the slice category. □

Note that $C_b(*) \cong \mathbb{C}$.

Corollary 5.2. *We have an equivalence of categories*

$$C_b : (\mathbf{Hausd}^{\mathrm{comp}})_{*/} \xrightarrow{\cong} (C^*\mathbf{Alg}^{\mathrm{comm}}_{/\mathbb{C}})^{\mathrm{op}} \overset{K}{\simeq} (C^*\mathbf{Alg}^{\mathrm{nu,comm}})^{\mathrm{op}} \ .$$

Example 5.3. An object in $(\mathbf{Hausd}^{\mathrm{comp}})_{*/}$ is a pointed compact Hausdorff space (X, x). We let $\mathbf{Hausd}^{\mathrm{lcomp}}$ be the a category whose objects are the locally compact topological spaces, and whose morphisms are partially defined maps $X \supseteq U \xrightarrow{f} X'$. The composition of such a map with $X' \supseteq U' \xrightarrow{f} X''$ is given by $X \supseteq (U \cap f^{-1}(U')) \xrightarrow{f' \circ f} X''$.

We can define a functor

$$L : (\mathbf{Hausd}^{\mathrm{comp}})_{*/} \to \mathbf{Hausd}^{\mathrm{lcomp}} \ , \quad (X, x) \mapsto X \setminus \{x\} \ .$$

A morphism $f : (X, x) \to (X', x')$ is send by L to a partially defined map

$$L(f) : X \supseteq (X \setminus f^{-1}(x')) \to X' \setminus \{x\} \ .$$

The functor L fits into an equivalence of categories

$$((-)^+, +) : \mathbf{Hausd}^{\mathrm{lcomp}} \leftrightarrows (\mathbf{Hausd}^{\mathrm{comp}})_{*/} : L \ ,$$

where $((-)^+,+)$ sends a locally compact space X to its one-point compactification $(X^+,+)$ pointed by the additional point. If $U\to X'$ is a map defined on an open subset U of X, then it extends to a map $X^+\to U^+\to X'^{,+}$, where the first map is the collapse map sending every point outside U to $+$.

We have an natural isomorphism $L(X^+,+)\cong X$. The canonical map $(X\setminus\{x\})\to X$ extends to a map $(X^+,+)\to(X,x)$. This map is a bijection between compact Hausdorff spaces and hence an isomorphism. □

We further consider the functor

$$C_0:=K(C_b((-)^+,+)):\mathbf{Hausd}^{\mathrm{lcomp}}\to(C^*\mathbf{Alg}^{\mathrm{nu,comm}})^{\mathrm{op}}\ .$$

Thus $C_0(X)$ are the continuous functions on X^+ which vanish at $+$.

Remark 5.4. In general $C_0(X)$ is bigger than the closure of the subsspace of $C_b(X)$ of functions of compact support. □

We let

$$G_0:(C^*\mathbf{Alg}^{\mathrm{nu,comm}})^{\mathrm{op}}\stackrel{U}{\to}(C^*\mathbf{Alg}^{\mathrm{comm}}_{/\mathbb{C}})^{\mathrm{op}}\stackrel{G}{\to}(\mathbf{Hausd}^{\mathrm{comp}})_{*/}\stackrel{(X,x)\mapsto X\setminus\{*\})}{\to}\mathbf{Hausd}^{\mathrm{lcomp}}$$

be the functor which sends A to $G(A^u)\setminus\{\epsilon_A\}$, where $\epsilon_A:A^u\to\mathbb{C}$ is the canonical character.

Corollary 5.5. *We have an equivalence of categories*

$$C_0:\mathbf{Hausd}^{\mathrm{lcomp}}\leftrightarrows C^*\mathbf{Alg}^{\mathrm{nu,comm}}:G_0$$

Example 5.6. We consider the functor

$$U_{\mathrm{alg}}:C^*\mathbf{Alg}^{\mathrm{nu,comm}}\stackrel{U}{\to}C^*\mathbf{Alg}^{\mathrm{comm}}_{/\mathbb{C}}\stackrel{(B\to\mathbb{C})\mapsto B}{\to}C^*\mathbf{Alg}^{\mathrm{comm}}\ .$$

This functor sends a commutative C^*-algebra to its unitaliztion A^u considered as an object in $C^*\mathbf{Alg}^{\mathrm{comm}}$. Under Gelfand duality this functor of $C^*\mathbf{Alg}^{\mathrm{nu,comm}}\to C^*\mathbf{Alg}^{\mathrm{comm}}$ corresponds to the functor

$$(-)^+:\mathbf{Hausd}^{\mathrm{lcomp}}\stackrel{((-)^+,+)}{\to}\mathbf{Hausd}_{*/}\stackrel{(X,x)\mapsto X}{\to}\mathbf{Hausd}^{\mathrm{comp}}\ .$$

This functor sends a locally compact topological space X to its one-point compactification X^+. □

We consider the functor

$$\beta:\mathbf{Top}\stackrel{C_b}{\to}C^*\mathbf{Alg}^{\mathrm{comm}}\stackrel{G}{\to}\mathbf{Hausd}^{\mathrm{comp}}\ .$$

The unit of the adjunction from Theorem 4.3 is a natural transformation

$$\iota : \mathtt{id} \to \beta \ .$$

Its evaluation at X is given by the map

$$X \to \beta(X) = G(C_b(X)) \ , \quad x \mapsto (a \mapsto a(x)) \ .$$

Definition 5.7. *The map $X \to \beta(X)$ is called the Stone-Čech compactification of X.*

The following is an immediate consequence of Theorem 4.3. We just compose the functor C_b with the equivalence $G_{|\mathbf{Hausd}^{\mathrm{comp}}}$.

Corollary 5.8. *We have an reflective localization*

$$\beta : \mathbf{Top} \to \mathbf{Hausd}^{\mathrm{comp}} : \mathrm{incl}$$

whose unit is given by β.

Example 5.9. Assume that X is locally compact. Then the natural map $X \to X^+$ extends uniquely to a map $\beta(X) \to X^+$ such that the composition $X \to \beta(X) \to X^+$ is the canonical inclusion. This shows that $X \to \beta(X)$ is injective. Thus X can be considered as a subspace of $\beta(X)$. Since the restriction of functions along $X \to \beta(X)$ is the identity $C_b(\beta(X)) = C_b(X))$ it is in particular injective. This shows that the subspace X is dense in $\beta(X)$.

A C^*-algebra is separable A if it has a countable dense subset A_0. We assume that A is separable. We call a topological space second countable if its topology has a countable base.

Note that the weak topology on A^* is generated by the functions $\mathtt{ev}_a : B(A^*) \to \mathbb{C}$ for all a in A. Since a uniform limit of continuous functions is continuous and the functions $\mathtt{ev}_a$ for a_0 in A_0 are dense in all such evaluation functions the topology is also generated by the countable family functions $(\mathtt{ev}_a)_{a \in A_0}$. Since the topology of $\mathbb{C}$ is second countable it follows that the weak topology on $B(A^*)$ is second countable. On the other hand if X is in $\mathbf{Hausd}^{\mathrm{comp}}$ and has a countable base, then $C_b(X)$ is separable.

We let $\mathbf{Hausd}^{\mathrm{comp}}_{\mathrm{sep}}$ denote the full subcategory of $\mathbf{Hausd}^{\mathrm{comp}}$ of compact Hausdorff spaces with countable base and $C^*\mathbf{Alg}^{\mathrm{comm}}_{\mathrm{sep}}$ be the full subcategory of $C^*\mathbf{Alg}^{\mathrm{comm}}$ of separable algebras.

Corollary 5.10. *The Gelfand duality restricts to an equivalence*

$$C_b : \mathbf{Hausd}^{\mathrm{comp}}_{\mathrm{sep}} \leftrightarrows (C^*\mathbf{Alg}^{\mathrm{comm}}_{\mathrm{sep}})^{\mathrm{op}} : G \ .$$

A locally compact Hausdorff space X is second countable if and only if its one-point compactification X^+ is second countable.

Similarly A in $C^*\mathbf{Alg}^{\mathrm{nu,comm}}$ is separable if and only if A^+ is separable.

Corollary 5.11. *Gelfand duality restricts to an equivalence of categories*

$$C_0 : \mathbf{Hausd}_{\mathrm{sep}}^{\mathrm{lcomp}} \leftrightarrows (C^*\mathbf{Alg}_{\mathrm{sep}}^{\mathrm{nu,comm}})^{\mathrm{op}} : G_0 \ .$$