Lecture course on coarse geometry

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Contents

1 Bornological spaces

In this section we introduce the notion of a bornological space. We show that the category of bornological spaces and proper maps is cocomplete and almost cocomplete. We explain how bornologies can be constructed and how they are applied.

Let X be a set. By $\mathcal{P}_X$ we denote the power set of X. Let $\mathcal{B}$ be a subset of $\mathcal{P}_X$.

Definition 1.1. *$\mathcal{B}$ is called a bornology if it has the following properties:*

1. *$\mathcal{B}$ is closed under taking subsets.*
2. *$\mathcal{B}$ is closed under forming finite unions.*
3. $\bigcup_{B\in\mathcal{B}} B = X$.

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The elements of the bornology are called the bounded subsets of X. 2 in Definition 1.1

Definition 1.2. *$\mathcal{B}$ is a generalized bornology if it satisfies the Conditions 1 and 2 in Definition 1.1.*

Thus we we get the notion of a generalized bornology by dropping Condition 3 in Definition 1.1.

Remark 1.3. Let x be in X. Then the singleton $\{x\}$ belongs to any bornology. Indeed, by Condition 3 there exists an element B in $\mathcal{B}$ such that $x \in B$. Then $\{x\} \subseteq B$ and hence $\{x\} \in \mathcal{B}$ by Condition 1.

If $\mathcal{B}$ is a generalized bornology, then a point x in X is called bounded if $\{x\} \in \mathcal{B}$. Otherwise it is called unbounded. A generalized bornology is a bornology if and only all points of X are bounded.

If X is a generalized bornological space, then we have a disjoint decomposition $X = X_b \sqcup X_u$ into the subsets of bounded and unbounded points. Then $\mathcal{B}$ becomes a bornology on X_b. $\square$

Remark 1.4. Let A be an abelian group and consider the abelian group A^X of functions from X to A. For f in A^X we let

$$\mathtt{supp}(f) := \{x \in X \mid f(x) \neq 0\}$$

be the support of f. We have $\mathtt{supp}(f + f') \subseteq \sup(f) \cup \mathtt{supp}(f')$.

If $\mathcal{B}$ is a generalized bornology on X, then we can consider the subset

$$C_{\mathcal{B}}(X, A) := \{f \in A^X \mid \mathtt{supp}(f) \in \mathcal{B}\} \ .$$

We shall see that this is a subgroup of A^X. Indeed, if f, f' are in $C_{\mathcal{B}}(X, A)$, then $\mathtt{supp}(f)$ and $\mathtt{supp}(f)$ belong to $\mathcal{B}$. Since $\mathtt{supp}(f + f') \subseteq \mathtt{supp}(f) \cup \mathtt{supp}(f')$ we see that also $\mathtt{supp}(f + f')$ belongs to $\mathcal{B}$ so that $f + f'$ is in $C_{\mathcal{B}}(X, A)$. Similarly, $\mathtt{supp}(-f) = \mathtt{supp}(f)$ so that with f also $-f$ belongs to $C_{\mathcal{B}}(X, A)$.

In this way $\mathcal{B}$ determines a subgroup of A^X of functions with bounded support.

If A is a ring, then A^X is also a ring and $C_{\mathcal{B}}(X, A)$ is a subring since $\mathtt{supp}(ff') \subseteq \mathtt{supp}(f) \cap \mathtt{supp}(f')$. $\square$

We now turn to examples and constructions of bornologies.

Let X be a set and $(\mathcal{B}_i)_{i \in I}$ be a family of (generalized) bornologies on X.

Lemma 1.5. *The intersection $\bigcap_{i \in I} \mathcal{B}_i$ is a (generalized) bornology.*

Proof. Let $\mathcal{B} := \bigcap_{i\in I} \mathcal{B}_i$. Consider B in $\mathcal{B}$ and let B' be a subset of B. Then $B \in \mathcal{B}_i$ for all i in I and hence $B' \in \mathcal{B}_i$ for all i in I. Hence $B' \in \mathcal{B}'$.

Assume that B, B' belong to $\mathcal{B}$. Then $B \in \mathcal{B}_i$ and $B' \in \mathcal{B}_i$ for all i in I. Hence $B \cup B' \in \mathcal{B}_i$ for all i in I and hence $B \cup B' \in \mathcal{B}$.

This finishes the case of generalized bornologies. For the case of bornologies we consider x in X. Then $\{x\} \in \mathcal{B}_i$ for every i in I and hence $\{x\} \in \mathcal{B}$. Hence $x \in \bigcup_{B\in\mathcal{B}} B$. □

Let $\mathcal{A}$ be a subset of $\mathcal{P}_X$. Then there is a smallest bornology containing $\mathcal{A}$ given by

$$\mathcal{B}\langle\mathcal{A}\rangle := \bigcap_{\mathcal{B},\, \mathcal{A}\subseteq\mathcal{B}} \mathcal{B}\ ,$$

where the intersection runs over all bornologies $\mathcal{B}$ on X. Similarly there is a smallest generalized bornology containing $\mathcal{A}$

$$\tilde{\mathcal{B}}\langle\mathcal{A}\rangle = \bigcap_{\mathcal{B},\, \mathcal{A}\subseteq\tilde{\mathcal{B}}} \tilde{\mathcal{B}}\ ,$$

where the intersection runs over all generalized bornologies $\mathcal{B}$ on X containing $\mathcal{A}$.

We can describe $\mathcal{B}\langle\mathcal{A}\rangle$ explicitly. We will assume that $\bigcup_{A\in\mathcal{A}} A = X$. Since singletons belong to every bornology (see Example 1.3) we can enlarge $\mathcal{A}$ by singletons without changing $\mathcal{B}\langle\mathcal{A}\rangle$ in order to ensure this condition.

Lemma 1.6. *Assume that $\bigcup_{A\in\mathcal{A}} A = X$. Then a subset B in $\mathcal{P}_X$ belongs to $\mathcal{B}\langle\mathcal{A}\rangle$ if and only if of there exists a finite family $(A_i)_{i\in I}$ in $\mathcal{A}$ such that $B \subseteq \bigcup_{i\in I} A_i$.*

Proof. We consider the subset $\mathcal{B}'$ of $\mathcal{P}$ of subsets B of X such that there exists a finite family $(A_i)_{i\in I}$ in $\mathcal{A}$ with $B \subseteq \bigcup_{i\in I} A_i$.

One checks that $\mathcal{B}'$ is a bornology which contains $\mathcal{A}$. Therefore $\mathcal{B}\langle\mathcal{A}\rangle \subseteq \mathcal{B}'$. On the other hand, $\mathcal{B}'$ is contained in any other bornology which contains $\mathcal{A}$. This implies that $\mathcal{B}' \subseteq \mathcal{B}\langle\mathcal{A}\rangle$. □

Remark 1.7. The generalized bornology $\tilde{\mathcal{B}}\langle\mathcal{A}\rangle$ has a similar description. We let $X_b := \bigcup_{A\in\mathcal{A}} A$. Then $\mathcal{A}$ generates a bornology $\mathcal{B}'$ on X_b whose elements are described as in Lemma 1.6. We have $\tilde{\mathcal{B}}\langle\mathcal{A}\rangle = \mathcal{B}'$.

Example 1.8. If X is a set, then it has the minimal bornology $\mathcal{B}_{min}$ of finite subsets and the maximal bornology $\mathcal{B}_{max}$ of all subsets.

It has the empty generalized bornology. □

Example 1.9. Let X be a topological space. The set $\mathcal{B}_{qc}$ of subsets of quasi-compact subsets of X is a bornology.

The set $\mathcal{B}_{rc}$ of relatively quasi compact subsets (subsets whose closures are quasi compact) is a generalized bornology. It might happen that the closure of a point is not quasi compact. This can happen of X is not Hausdorff.

If we replace the condition “quasi compact” by “compact”, then in general we do not get a bornology since the union of two compact subsets is not necessarily compact. Again this can happen since compactness includes the condition of being Hausdorff and a union of two Hausdorff subsets need not be Hausdorff. □

Example 1.10. Let d be a quasi-metric (infinite distances are allowed) on X. Then the metrically bounded subsets generate a bornology $\mathcal{B}_d$. Note that we must say “generate” since the union of two bounded sets might be unbounded.

But if d is a metric, then the bounded sets form a bornology since Condition 2 follows from the triangle inequality for the metric.

The bornology $\mathcal{B}_d$ is generated by the set of balls $\{B(x,r) \mid x \in X \ r \in [0,\infty)\}$. □

Example 1.11. Let Y be a topological space, A be a subset, and $X := Y \setminus A$.

Let $\mathcal{B} := \{Z \subseteq X \mid \bar{Z} \cap A = \emptyset\}$. This is a generalized bornology. If Y is Hausdorff then it is a bornology. □

Example 1.12. Recall that a filter on a set X is a subset $\mathcal{F}$ of $\mathcal{P}_X$ with the following properties:

1. $\emptyset \notin \mathcal{F}$ and $X \in \mathcal{F}$.
2. $\mathcal{F}$ is closed under forming finite intersections.
3. $\mathcal{F}$ is closed under taking supersets.

If $\mathcal{F}$ is a filter on X, then the set of complements

$$\mathcal{F}^c := \{X \setminus F \mid F \in \mathcal{F}\}$$

is a generalized bornology on X. Vice versa the complements of a generalized bornology $\mathcal{B}$ is a filter if and only if $X \notin \mathcal{B}$, i.e., $\mathcal{B}$ is not the maximal bornology.

The generalized bornology $\mathcal{F}^c$ is a bornology if and only if the filter is free, i.e., if and only if $\bigcap_{F \in \mathcal{F}} F = \emptyset$. Hence, upon taking complements, non-maximal bornologies correspond to free filters on X. □

Example 1.13. Let $\mathcal{F}$ be a subset of $\mathcal{P}_X$. Then

$$\mathcal{F}^\perp := \{Y \subseteq X \mid (\forall L \in \mathcal{F} \mid |L \cap Y| < \infty\}$$

is a bornology. It is called the dual bornology to $\mathcal{F}$. We have an obvious inclusion $\mathcal{F} \subseteq (\mathcal{F}^{\perp})^{\perp}$. □

Definition 1.14. *A (generalized) bornological space is a pair $(X, \mathcal{B})$ of a set with a (generalized) bornology.*

Usually we write X for generalized bornological spaces and let $\mathcal{B}_X$ denote its generalized bornology.

Example 1.15. For a set X we write X_{min} and X_{max} for X equipped with the minimal and maximal bornology. We write $X_{\emptyset}$ for the generalized bornological space with the empty bornology. □

Example 1.16. We consider a bornological space X and a subset L:

Definition 1.17. *L is called locally finite if $|L \cap B| < \infty$ for all B in $\mathcal{B}_X$.*

We let $\mathcal{LF}(X)$ denote the set of locally finite subsets of X. By Example 1.13, the subset $\mathcal{LF}(X)$ of $\mathcal{P}_X$ is a bornology on X.

We write $(X, \mathcal{LF}(X)) =: X^{\perp}$. With this notation we have $(X_{min})^{\perp} = X_{max}$ and $(X_{max})^{\perp} = X_{min}$. □

Let $f : X \to Y$ be a map between the underlying sets of generalized bornological spaces.

Definition 1.18. *f is called:*

1. *proper if $f^{-1}(\mathcal{B}_Y) \subseteq \mathcal{B}_X$.*
2. *bornological, if $f(\mathcal{B}_X) \subseteq \mathcal{B}_Y$.*

Let $\mathcal{A}$ be a subset of $\mathcal{P}_Y$. By the following lemma we can check properness of a map on generators.

Lemma 1.19.

1. *We assume that $Y = \bigcup_{A \in \mathcal{A}} A$ and $\mathcal{B}_Y = \mathcal{B}\langle \mathcal{A} \rangle$. Then the map f is proper if and only if $f^{-1}(A) \in \mathcal{B}_X$ for all A in $\mathcal{A}$.*
2. *We assume that $\mathcal{B}_Y := \tilde{\mathcal{B}}\langle \mathcal{A} \rangle$. Then the map f is proper if and only if $f^{-1}(A) \in \mathcal{B}_X$ for all A in $\mathcal{A}$.*

Proof. We show Assertion 1. If f is proper, then $f^{-1}(A) \in \mathcal{B}_X$ for all A in $\mathcal{A}$ since $\mathcal{A} \subseteq \mathcal{B}_Y$.

For the converse we assume that B is in $\mathcal{B}_Y$. Then by Lemma 1.6 there exists a finite family $(A_i)_{i\in I}$ in $\mathcal{A}$ such that $B \subseteq \bigcup_{i\in I} A_i$. Then $f^{-1}(B) \subseteq f^{-1}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f^{-1}(A_i) \in \mathcal{B}_X$ since $f^{-1}(A_i) \in \mathcal{B}_X$ for every i in I and $\mathcal{B}_X$ is closed under forming finite unions.

For Assertion 2 we argue similarly using Remark 1.7. We do not have to assume that $\mathcal{A}$ covers Y. □

Example 1.20. Let $f : X \to Y$ be a proper map between bornological spaces. Let A be a group. Then the pull-back $f^* : A^Y \to A^X$ restricts to a homomorphism $f^* : C_{\mathcal{B}_Y}(Y, A) \to C_{\mathcal{B}_X}(X, A)$. This follows from the relation $\mathtt{supp}(f^*\phi) \subseteq f^{-1}(\mathtt{supp}(\phi))$ for all ϕ in A^Y. □

Example 1.21. Let $f : X \to Y$ be a map between the underlying sets of bornological spaces. If $f : X \to Y$ is proper, then $f : X^\perp \to Y^\perp$ is bornological. Indeed, let L be in $\mathcal{LF}(X)$ and B be in $\mathcal{B}_Y$. Then $f(L) \cap B = f(L \cap f^{-1}(B))$ is the image under f of a finite subset of X and hence finite. We see hat $f(L) \in \mathcal{LF}(B)$. □

Example 1.22. Let X be a bornological space, Y be a set, and $f : Y \to X$ be a map. We set

$$f^{-1}(\mathcal{B}_X) := \{f^{-1}(B) \mid B \in \mathcal{B}_X\} .$$

Then $\mathcal{B}\langle f^{-1}(\mathcal{B}_X)\rangle$ is called the induced bornology. It is the minimal bornology on Y such that $f : X \to Y$ becomes a proper map of bornological spaces.

Similarly one can define the induced generalized bornology $\tilde{\mathcal{B}}\langle f^{-1}(\mathcal{B}_X)\rangle$. It is the minimal generalized bornology such that f becomes a proper map of generalied bornological spaces. If Y contains unbounded points, then the inclusion $\tilde{\mathcal{B}}\langle f^{-1}(\mathcal{B}_X)\rangle \subseteq \mathcal{B}\langle f^{-1}(\mathcal{B}_X)\rangle$ might be proper. □

Example 1.23. Let X be a (generalized) bornological space and $f : X \to Y$ be a map of sets. Then we can consider the maximal (generalized) bornology on Y such that f becomes a proper map. This bornology is given by

$$f_*\mathcal{B}_X := \{B \subseteq Y \mid f^{-1}(B) \in \mathcal{B}_X\} .$$

We call this bornology the coinduced bornology. □

We let **Born** denote the category of bornological spaces and proper maps. Furthermore, we write $\widetilde{\mathbf{Born}}$ for the category of generalized bornological spaces and proper maps. Our next goal is to study these categories and various functors relating them with the category of sets.

We have a forgetful functors $S : \mathbf{Born} \to \mathbf{Set}$ and $S : \widetilde{\mathbf{Born}} \to \mathbf{Set}$.

Proposition 1.24. *We have an adjunctions*

$$(X \mapsto X_{max}) : \mathbf{Set} \leftrightarrows \mathbf{Born} : S \ , \quad (X \mapsto X_{max}) : \mathbf{Set} \leftrightarrows \widetilde{\mathbf{Born}} : S \ .$$

Proof. For the case of **Born** on checks the equality

$$\mathtt{Hom}_{\mathbf{Born}}(X_{max}, Y) = \mathtt{Hom}_{\mathbf{Set}}(X, S(Y))$$

for all sets X and bornological spaces Y.

The argument for $\widetilde{\mathbf{Born}}$ is similar. □

Proposition 1.25. *We have an adjunction*

$$S : \widetilde{\mathbf{Born}} \leftrightarrows \mathbf{Set} : (X \mapsto X_{\emptyset}) \ .$$

Proof. One checks the equality

$$\mathtt{Hom}_{\widetilde{\mathbf{Born}}}(S(Y), X) = \mathtt{Hom}_{\mathbf{Set}}(Y, X_{\emptyset})$$

for all sets X and generalized bornological spaces Y. □

Note that there is no such adjunction in the case of **Born**.

Proposition 1.26. *The categories* **Born** *and* $\widetilde{\mathbf{Born}}$ *are cocomplete.*

Proof. Let $X : \mathbf{I} \to \mathbf{Born}$ be a small diagram. We then define the set $Y := \mathtt{colim}_{\mathbf{I}}\, S(X)$. It comes with a family $(e_i : S(X) \to Y)_{i \in \mathbf{I}}$ of maps of sets exhibiting Y as a colimit of $S(X)$ in **Set**. We equip Y with the intersection of the coinduced bornologies

$$\mathcal{B}_Y := \bigcap_{i \in \mathbf{I}} e_{i,*}\mathcal{B}_{X_i} \ .$$

In other words a subset A of Y belongs to $\mathcal{B}_Y$ if and only $e_i^{-1}(A)$ is bounded for all i in $\mathbf{I}$.

From now on Y denotes the bornological space $(Y, \mathcal{B}_Y)$. Then $e_i : X \to Y$ are morphisms in **Born**. We now check that $(Y, (e_i)_{i \in \mathbf{I}})$ is a colimit of X in **Born**. Let T be in **Born** arbitrary. By construction the family of maps $(e_i)_{i \in \mathbf{I}}$ induces a bijection

$$\mathtt{Hom}_{\mathbf{Set}}(S(Y), S(T)) \stackrel{\cong}{\to} \lim_{\mathbf{I}^{\mathrm{op}}} \mathtt{Hom}_{\mathbf{Set}}(S(X), S(T)) \ , \quad g \mapsto (g \circ e_i)_{i \in \mathbf{I}}$$

Since e_i is a morphisms **Born** for every i in $\mathbf{I}$ one first observes that this bijection restricts to a (necessarily injective) map

$$\mathtt{Hom}_{\mathbf{Born}}(Y, S) \to \lim_{\mathbf{I}^{\mathrm{op}}} \mathtt{Hom}_{\mathbf{Born}}(X, T) \ .$$

In order to show surjectivity assume that $(f_i : X \to T)_{i \in \mathbf{I}}$ represents an element in $\mathtt{lim}_{\mathbf{I}}\, \mathtt{Hom}_{\mathbf{Born}}(X, T)$ and let $g : S(Y) \to S(T)$ be the corresponding map of underlying sets. We must show that $g \in \mathtt{Hom}_{\mathbf{Born}}(Y, S)$, i.e., that g is proper. If B is in $\mathcal{B}_T$, then $f_i^{-1}(B) = e_i^{-1}(g^{-1}(B))$ is in $\mathcal{B}_{X_i}$ for all i in I. By the definition of $\mathcal{B}_Y$ we conclude that $g^{-1}(B) \in \mathcal{B}_Y$. Hence g is proper.

The same argument works for generalized bornological spaces. □

Example 1.27. Let $(X_i)_{i\in I}$ be a family of bornological spaces. Then we can describe its coproduct $X := \coprod_{i\in I}$ in **Born** explicitly. The underlying set is the disjoint union $S(X) := \bigsqcup_{i\in I} S(X_i)$ of the underlying sets of the X_i. We consider the sets X_i as subsets of X. A subset B of X is bounded if and only if $B \cap X_i$ is bounded in X_i for all i in I.

For example, for a set Y we have $Y_{max} \cong \bigsqcup_{y\in Y}\{y\}$.

Proposition 1.28. *The category* $\widetilde{\mathbf{Born}}$ *is complete.*

Proof. Let $X : \mathbf{I} \to \widetilde{\mathbf{Born}}$ be a diagram. We define the set $Y := \mathtt{lim}_{\mathbf{I}}\, S(X)$. It comes with a family of maps $(p_i : Y \to S(X))_{i\in\mathbf{I}}$ exhibiting Y as a limit of $S(X)$. We equip Y with the generalized bornology

$$\mathcal{B}_Y := \tilde{\mathcal{B}}\langle \bigcup_{i\in\mathbf{I}} p_i^{-1}(\mathcal{B}_{X_i})\rangle \ .$$

Then $p_i : Y \to X$ becomes a morphism in $\widetilde{\mathbf{Born}}$ for every i in $\mathbf{I}$. One now checks that $(Y,(p_i)_{i\in\mathbf{I}})$ is a limit of X. Let T be in $\widetilde{\mathbf{Born}}$. By construction the family $(p_i)_{i\in\mathbf{I}}$ induces a bijection

$$\mathtt{Hom}_{\mathbf{Set}}(S(T), S(Y)) \stackrel{\cong}{\to} \mathtt{lim}_{\mathbf{I}}\, \mathtt{Hom}_{\mathbf{Set}}(S(T), S(X)) \ , \quad g \mapsto (p_i \circ g)_{i\in\mathbf{I}}$$

Since p_i is a morphism in $\widetilde{\mathbf{Born}}$ for every i in $\mathbf{I}$ it restricts to a (necessarily injective) map

$$\mathtt{Hom}_{\widetilde{\mathbf{Born}}}(T, Y) \to \mathtt{lim}_{\mathbf{I}}\, \mathtt{Hom}_{\widetilde{\mathbf{Born}}}(T, X) \ .$$

In order to show surjectivity we consider a family $(f_i : T \to X_i)_{i\in\mathbf{I}}$ in $\mathtt{lim}_{\mathbf{I}}\, \mathtt{Hom}_{\widetilde{\mathbf{Born}}}(T, X)$ and let $g : S(T) \to S(Y)$ be the corresponding map of underlying sets.

We must show that g is proper. We use Lemma 1.19 in order to check properness on generators. Fix i in $\mathbf{I}$ and assume that B is bounded in X_i. Then $p_i^{-1}(B)$ is a typical enerator of the generalized bornology of Y. Then $g^{-1}(p_i^{-1}(B)) = f_i^{-1}(B)$ is bounded in T since f_i is proper.

We conclude that g is proper. □

The empty limit in $\widetilde{\mathbf{Born}}$ is a final object $*_\emptyset$. It does not belong to **Born**.

Proposition 1.29. *The category* **Born** *admits all non-empty limits.*

Proof. The same argument as for Proposition 1.28 works. In this case $\mathcal{B}_{X_i}$ is a bornology for every i in $\mathbf{I}$. The condition $\mathbf{I} \neq \emptyset$ ensures that every point in Y belongs to $\tilde{\mathcal{B}}\langle \bigcup_{i\in\mathbf{I}} p_i^{-1}(\mathcal{B}_{X_i})\rangle$ so that this generalized bornology is a bornology. □

Example 1.30. Let $(X_i)_{i\in I}$ be a family of bornological spaces. Then we can describe its cartesian product $X := \prod_{i\in I} X_i$ explicitly. The underlying set of the product is the cartesian product $S(X) := \prod_{i\in I} S(X_i)$ of underlying sets. The bornology on X is generated by the cylinder sets $p_i^{-1}(B)$ for all i in I and B in $\mathcal{B}_{X_i}$. $\square$

The categories **Born** and $\widetilde{\mathbf{Born}}$ have a symmetric monoidal structure which will be denoted by $\otimes$. It will be obtained from the cartesian product of the underlying sets by equipping the products with bornology specified as follows:

Definition 1.31. *We define the functor*

$$-\otimes - : \widetilde{\mathbf{Born}} \times \widetilde{\mathbf{Born}} \to \widetilde{\mathbf{Born}}$$

such that it sends X, X' in $\widetilde{\mathbf{Born}}$ to the set $X \times X'$ with the bornology generated by $B \times B'$ for all B in $\mathcal{B}_X$ and B' in $\mathcal{B}_{X'}$.

If $f : X \to Y$ and $f' : X' \to Y'$ are proper, then $f \otimes f' : X \otimes X' \to Y \otimes Y'$ is again proper since it is obvious that preimages of generators are again generators. The space $\{*\}_{\max}$ is the tensor unit of this structure.

The tensor structure on $\widetilde{\mathbf{Born}}$ restricts to a structure on **Born**.

We have a morphism $X \times X' \to X \otimes X'$ given by the identity of the underlying sets, but this map is general not a morphism except if both X' and X' are bounded.

Example 1.32. For sets X, Y we have $X_{min} \otimes Y_{min} \cong (X \times Y)_{min}$.

We have $X \otimes \{*\}_\emptyset \cong S(X)_\emptyset$ which shows that $\{*\}_\emptyset$ does not act as a tensor unit. $\square$

2 Coarse spaces

In this section we introduce the category of coarse spaces and proper map. We show that it is complete and cocomplete. We explain various ways how coarse structures can appear, and how they are used to define subalgebras of matrix algebras.

Let X be a set. We call subsets U of $X \times X$ entourages. The diagonal $\mathtt{diag}(X)$ is an example of an entourage. For an entourage U of X we define its inverse by

$$U^{-1} := \{(y,x) \in X \times X \mid (x,y) \in U\} \ .$$

If V is a second entourage, then we define the composition of V and U by

$$V \circ U := \{(x,z) \in X \times X \mid (\exists y \in X \mid (x,y) \in V \text{ and } (y,z) \in U\} \ .$$

We have the relations

$$\mathtt{diag}(X) \circ U = U \circ \mathtt{diag}(X) = U \ , \quad (W \circ V) \circ U = W \circ (V \circ U) \ .$$

Let U be an entourage of X and B be a subset of X. Then we call the subset

$$U[B] := \{x \in X \mid (\exists b \in B \mid (x, b) \in U)\}$$

of X the U-thickening of B.

Write $x \sim_U y$ if $(x, y) \in U$. So $U[Y]$ consists of all points x in X such that $x \sim_U y$.

Let Y, Z be subsets of X and U be an entourage on X.

Definition 2.1. *We say that Y is U-separated from Z if $Y \cap U[Z] = \emptyset$.*

This means that there is no pair of points y in Y and z in Z such that $y \sim_U z$.

Example 2.2. Let X be a set with subsets Y and Z. Let U and V be entourages of X. Then we have the following assertions.

If Y is U-separated from Z, then Z is U^{-1}-separated from Y. Indeed $y \sim_U z$ is equivalent to $z \sim_{U^{-1}} y$.

If Y is $V \circ U$-separated from Z, then Y is V-separated from $U[Z]$. If Y were not V-separated from $U[Z]$, then there exists y in Y, x in X and z in Z such that $y \sim_V x$ and $x \sim_U z$. But then $y \sim_{V \circ U} z$. □

Example 2.3. Let U be an entourage on X. Then U is an equivalence relation if and only if

1. $\mathtt{diag}(X) \subseteq U$
2. $U = U^{-1}$
3. $U \circ U = U$.

In this case, for x in X the set $U[\{x\}]$ is the equivalence class of x. □

Example 2.4. Let $f : X \to X$ be a map. Then the graph of f

$$\mathrm{graph}(f) := \{(f(x), x) \mid x \in X\}$$

is an entourage. We have the relations

$$\mathrm{graph}(f' \circ f) = \mathrm{graph}(f') \circ \mathrm{graph}(f)$$

and if f^{-1} exists, also

$$\mathrm{graph}(f^{-1}) = \mathrm{graph}(f)^{-1} \ .$$

Note that

$$\mathrm{graph}(\mathtt{id}_X) = \mathtt{diag}(X) \ .$$

Entourages can be considered as generalized maps, which may be multivalued and not everwhere defined. □

Let $\mathcal{C}$ be a subset of $\mathcal{P}_{X\times X}$.

Definition 2.5. *$\mathcal{C}$ is called a coarse structure if it has the following properties:*

1. $\mathtt{diag}(X) \in \mathcal{C}$.
2. *$\mathcal{C}$ is closed under finite unions and taking subsets.*
3. *If U, V are in $\mathcal{C}$, then $V \circ U \in \mathcal{C}$.*
4. *If U is in $\mathcal{C}$, then $U^{-1} \in \mathcal{C}$.*

The elements of $\mathcal{C}$ are called the coarse entourages of X.

Remark 2.6. The notion of a coarse structure and the main ideas of coarse geometry as presented here have been invented by John Roe [**?**].

Example 2.7. Let X be a set. Let R be a ring, and consider the R-module $R[X]$. We let $[x]$ denote the basis element corresponding to x in X. For every subset Y of X we can consider a projection $\mu(Y)$ in $\mathtt{End}(R[X])$ determined by the condition that

$$\mu(y)[x] := \begin{cases} [x] & x \in Y \\ 0 & x \notin Y \end{cases} \ .$$

Let A be in $\mathtt{End}(R[X])$ and U be an entourage of X.

Definition 2.8. *We say that A is U-controlled if for all pairs of subsets Y, Z of X such that Y is U-separated from Z we have $\mu(Y)A\mu(Z) = 0$.*

Let now $\mathcal{C}$ be a coarse structure on X.

We consider the subset

$$\mathtt{End}^{\mathcal{C}}(R[X]) := \{A \in \mathtt{End}(R[X]) \mid (\exists U \in \mathcal{C} \mid A \text{ is } U\text{-controlled})\} \ .$$

The first three axioms of a coarse structure for $\mathcal{C}$ imply that $\mathtt{End}^{\mathcal{C}}(R[X])$ is a subalgebra of $\mathtt{End}(R[X])$.

1. The identity $1_{R[X]}$ is $\mathtt{diag}(X)$-controlled and belongs to $\mathtt{End}^{\mathcal{C}}(R[X])$.

2. If A, B are in $\mathtt{End}^{\mathcal{C}}(R[X])$ and A is U-controlled and B is V-controlled, then $A + B$ is $U \cup V$ controlled. Indeed, if Y is $U \cup V$-separated from Z, then it is U- and V-separated from Z. Hence
$$\mu(Y)(A + B)\mu(Z) = \mu(Y)A\mu(Z) + \mu(Y)B\mu(Z) = 0 + 0 = 0 \ .$$

3. If A, B are in $\mathtt{End}^{\mathcal{C}}(R[X])$ and A is U-controlled and B is V-controlled, then $A \circ B$ is $U \circ V$-controlled. Assume that Y is $U \circ V$-separated from Z. Let $W := V[Z]$. Then Y is still U-separated from W, and $X \setminus W$ is V-separated from Z. Hence using $1_{R[X]} = \mu(W) + \mu(X \setminus W)$ we get
$$\mu(Y)(A \circ B)\mu(Z) = \mu(Y)A\mu(W)B\mu(Z) + \mu(Y)A\mu(X \setminus W)B\mu(Z) = 0 + 0 = 0 \ .$$

□

Example 2.9. Note that the Example 2.7 does not yet motivate the fourth condition that a coarse structure is stable under taking inverses. This is achieved with the following related example.

Let X be a set. We equip X with the counting measure. Then we consider the Hilbert space $L^2(X)$. We have an orthonormal basis $([x])_{x \in X}$. For a subset Y of X we can define the orthogonal projections $\mu(Y)$ in $B(L^2(X))$ as before such that
$$\mu(y)[x] := \begin{cases} [x] & x \in Y \\ 0 & x \notin Y \end{cases} \ .$$

We define the notion of U-control as in Definition 2.8. Let now $\mathcal{C}$ be a coarse structure. We let
$$B^{\mathcal{C}}(L^2(X)) := \{A \in B(L^2(X)) \mid (\exists U \in \mathcal{C} \mid A \text{ is } U\text{-controlled})\} \ .$$
We claim that $B^{\mathcal{C}}(L^2(X))$ is a $*$-subalgebra. It is a subalgebra by the same argument as in Example 2.7. Furthermore, it is closed under taking adjoints since if A is U-controlled, then A^* is U^{-1}-controlled. This follows from the fact that for subsets Y, Z of X we have: Y is U-separated from Z if and only if Z is U^{-1}-separated from Y.

Note that in general $B^{\mathcal{C}}(L^2(X))$ is not (topologically) closed, i.e., a C^*-algebra. But it is so if $\mathcal{C}$ has a maximal entourage, i.e., if it is generated by an equivalence relation.

Definition 2.10. *The C^*-algebra $C_u^*(X, \mathcal{C}) := \overline{B^{\mathcal{C}}(L^2(X))}$ obtained by forming the closure of $B^{\mathcal{C}}(L^2(X))$ in $B(L^2(X))$ is called the uniform Roe algebra of the coarse space $(X, \mathcal{C})$.*

□

We now turn to examples and constructions of coarse structures. Let X be a set and $(\mathcal{C}_i)_{i\in I}$ be a family of coarse structures on X.

Lemma 2.11. *The intersection $\bigcap_{i\in I}\mathcal{C}_i$ is a coarse structure.*

Proof. We set $\mathcal{C} := \bigcap_{i\in I}\mathcal{C}_i$.

Since $\mathtt{diag}(X)\in\mathcal{C}_i$ for every i in I we conclude that $\mathtt{diag}(X)\in\mathcal{C}$.

Assume that U and V are in $\mathcal{C}$, and that W is a subset of U. Then for every i in I we have $U\in\mathcal{C}_i$ and $V\in\mathcal{C}_i$. This implies W in $\mathcal{C}_i$, $U\cup V\in\mathcal{C}_i$, $V\circ U\in\mathcal{C}_i$ and $U^{-1}\in\mathcal{C}_i$ for every i in I. Hence W in $\mathcal{C}$, $U\cup V\in\mathcal{C}$, $V\circ U\in\mathcal{C}$ and $U^{-1}\in\mathcal{C}$. □

Let $\mathcal{A}$ be a subset of $\mathcal{P}_{X\times X}$. Then there is a smallest coarse structure containing $\mathcal{A}$ is given by

$$\mathcal{C}\langle\mathcal{A}\rangle = \bigcap_{\mathcal{C},\,\mathcal{A}\subseteq\mathcal{C}}\mathcal{C}\ ,$$

where the intersections runs over the coarse structures on X containing $\mathcal{A}$.

Let X be a set an $\mathcal{A}$ be a subset of $\mathcal{P}_{X\times X}$. We can describe the elements of $\mathcal{C}\langle\mathcal{A}\rangle$ explicitly. Since any coarse structure contains $\mathtt{diag}(X)$ we can add $\mathtt{diag}(X)$ to $\mathcal{A}$ without changing the coarse structure generated by $\mathcal{A}$. Furthermore with U in $\mathcal{A}$ we have $U\cup U^{-1}\in\mathcal{C}\langle\mathcal{A}\rangle$. So by enlarging the generators without changing $\mathcal{C}\langle\mathcal{A}\rangle$ we can assume that all elements in $\mathcal{A}$ are symmetric.

Lemma 2.12. *For simplicity we assume that $\mathtt{diag}(X)\in\mathcal{A}$ and that $\mathcal{A}$ consists of symmetric entourages. An entourage V of X belongs to $\mathcal{C}\langle\mathcal{A}\rangle$ if and only if there exists a finite family of families $((U_{j,i})_{i\in 1,\dots,n_j})_{j\in J}$ of elements of $\mathcal{A}$ such that*

$$V\subseteq\bigcup_{j\in J}U_{j,1}\circ\dots\circ U_{j,n_j}\ .$$

Proof. Let $\mathcal{C}'$ be the subset of $\mathcal{P}_{X\times X}$ of entourages V such that there exists a finite family of families $((U_{j,i})_{i\in 1,\dots,n_j})_{j\in J}$ of elements of $\mathcal{A}$ such that

$$V\subseteq\bigcup_{j\in J}U_{j,1}\circ\dots\circ U_{j,n_j}\ .$$

Then $\mathcal{C}'$ is a coarse structure. Indeed, $\mathtt{diag}(X)\in\mathcal{C}'$ since $\mathtt{diag}(X)\in\mathcal{A}$. Furthermore, by construction $\mathcal{C}'$ is closed under taking subsets and finite unions.

If $V\subseteq\bigcup_{j\in J}U_{j,1}\circ\dots\circ U_{j,n_j}$ and $V'\subseteq\bigcup_{j\in J}U_{j',1}\circ\dots\circ U_{j',n_j}$ for families $((U_{j,i})_{i\in 1,\dots,n_j})_{j\in J}$ and $((U'_{j',i})_{i\in 1,\dots,n'_{j'}})_{j'\in J'}$, then

$$V\circ V'\subseteq\bigcup_{j\in J,j'\in J'}U_{j,1}\circ\dots\circ U_{j,n_j}\circ U'_{j',1}\circ\dots\circ U'_{j',n'_{j'}}\ .$$

It is clear that $\mathcal{C}'$ contains $\mathcal{A}$ and therefore $\mathcal{C}\langle\mathcal{A}\rangle \subseteq \mathcal{C}'$. On the other hand $\mathcal{C}'$ is contained in every coarse structure containing $\mathcal{A}$, hence $\mathcal{C}' \subseteq \mathcal{C}\langle\mathcal{A}\rangle$. $\square$

Example 2.13. Coarse structure generated by a single entourage are particularly easy to describe. Let U be an entourage. We set $U^0 := \mathtt{diag}(X)$ and

$$U^n := \underbrace{U \circ \cdots \circ U}_{n\times}$$

for all positive integers. Then we consider the coarse structure $\mathcal{C}_U := \mathcal{C}\langle\{U\}\rangle$.

Assume that $U = U^{-1}$. Then an entourage V belongs to $\mathcal{C}\langle\{U\}\rangle$ if and only if there exists n in $\mathbb{N}$ such that $V \subseteq U^n$.

If U is an equivalence relation, then $\mathcal{C}_U = \{V \in \mathcal{P}_{X\times X} \mid V \subseteq U\}$. $\square$

Example 2.14. Let X be a set. It has a minimal coarse structure $\mathcal{C}_{min}$ consisting of all subsets of $\mathtt{diag}(X)$. We have $\mathcal{C}_{min} = \mathcal{C}\langle\emptyset\rangle$.

The set $\mathcal{C}_{max} := \mathcal{P}_{X\times X}$ is the maximal coarse structure. $\square$

Example 2.15. Let (X, d) be a quasi-metric space. Then for r in $[0,\infty)$ we define the metric entourage

$$U_r := \{(x,y) \in X \times X \mid d(x,y) \le r\}$$

of width r.

The coarse structure $\mathcal{C}_d := \mathcal{C}\langle(U_r)_{r\in[0,\infty)}\rangle$ is called the metric coarse structure on X.

The triangle inequality for the quasi-metric implies that $U_r \circ U_s \subseteq U_{r+s}$. In view of Lemma 2.12 an entourage V of X belongs to $\mathcal{C}_d$ if and only if there exists an r in $[0,\infty)$ such that $V \subseteq U_r$.

The U_r-thickening of a point is given by the r-ball centered at this point: $U_r[\{x\}] = B(x,r)$. $\square$

Example 2.16. Let U be an entourage of X with $U = U^{-1}$. Then we can define a quasi-metric on X as follows:

$$d_U(x,y) := \inf\{n \in \mathbb{N} \mid (x,y) \in U^n\} .$$

It can happen that $d_U(x,y) = \infty$, namly if the argument of inf above is empty. We verify the axioms of a distance.

1. $d_U(x,x) = 0$ since $(x,x) \in U^0 = \mathtt{diag}(X)$.

2. $d_U(x,y) = d_U(y,x)$ since $U^n = (U^n)^{-1}$ for every n in $\mathbb{N}$.

3. If $d_U(x,y) = m$ and $d_U(y,z) = n$, then $(x,y) \in U^m$ and $(y,z) \in U^n$. Hence $(x,z) \in U^m \circ U^n = U^{m+n}$. Hence $d_U(x,y) \le m+n = d_U(x,y) + d_U(y,z)$.

We then have the relation

$$\mathcal{C}_U = \mathcal{C}_{d_U} \ .$$

□

Example 2.17. The space $\mathbb{R}^n$ is a coarse space with the coarse structure induced by the usual metric. If not said differently we will consider subsets of $\mathbb{R}^n$ like $\mathbb{Z}^n$, $\mathbb{R}^n_+$ or $\mathbb{Q}^n$ as coarse spaces with respect to the induced metric. □

Example 2.18. Let $\bar{X}$ be a topological space and Y be a subset of $\bar{X}$. We consider $X := \bar{X} \setminus Y$.

A subset U of $X \times X$ is called continuously controlled (w.r.t $(\bar{X}, Y)$) if for every net $((x_i, x_i'))_{i \in I}$ in U the condition $x_i \to y \in Y$ implies that $x_i' \to y$ and U^{-1} satisfies the same condition.

The set of continuously controlled entourages forms a coarse structure called the continuously controlled coarse structure. We verify the axioms:

1. It is clear that the diagonal is continuously controlled.

2. It is also clear that if U is continuously controlled and V is a subset of U, then V is continuously controlled. Assume that U and V are continuously controlled. We must show that $U \cup V$ is continuously controlled. Let $((x_i, x_i'))_{i \in I}$ be a net in $U \cup V$ such that $x_i \to y \in Y$. We can find subsets I_U and I_V of I such that $I = I_U \cup I_V$ and $(x_i, x_i') \in V$ for i in I_V and $(x_i, x_i') \in U$ for i in I_U. If I_U is cofinal in I, then we conclude that $\mathtt{lim}_{i \in I_U} x_i' = y$, and similarly, if U_V is cofinal in I, we have $\mathtt{lim}_{i \in I_V} x_i' = y$. This implies $\mathtt{lim}_{i \in I} x_i' = y$.

3. Assume that U and V are continuously controlled. We show that $U \circ V$ is continuously controlled. The main argument is as follows. Let $((x_i, x_i'))_{i \in I}$ be a net in $U \circ V$ such that $\mathtt{lim}_I x_i = y \in Y$. For every i in I we find x_i'' such that $(x_i, x_i'') \in U$ and $(x_i'', x_i') \in V$. We first conclude that $\mathtt{lim}_{i \in I} x_i'' = y$ since U is continuously controlled, and then $\mathtt{lim}_{i \in I} x_i' = y$ since V is continuously controlled.

4. If U is continuously controlled, then U^{-1} is continuously controlled by definition.

□

Example 2.19. Let $\mathcal{C}$ be a coarse structure on X. Then

$$R := \bigcup_{U \in \mathcal{C}} U$$

is an equivalence relation on X. For x, y in X we have $x \sim y$ if and only of $(x,y) \in \mathcal{C}$.

Definition 2.20. *The equivalences classes for R are called the coarse components of X.*

We let $\pi_0^{\mathcal{C}}(X)$ denote the set of coarse components. □

Definition 2.21. *A coarse space is a pair $(X, \mathcal{C})$ of a set with a coarse structure.*

We usually write X for a coarse space and $\mathcal{C}_X$ for the corresponding coarse structrure.

Example 2.22. For a set X we write $X_{min} := (X, \mathcal{C}_{min})$ and $X_{max} := (X, \mathcal{P}_X)$.

Let X, Y be coarse spaces and $f : X \to Y$ be a map between the underlying sets. We write $f(U) := (f \times f)(U)$.

Definition 2.23. *The map f is called controlled if $f(\mathcal{C}_X) \subseteq \mathcal{C}_Y$.*

In details this means that for every coarse entourage U of X the set $f(U)$ is a coarse entourage of Y. We obtain the category **Coarse** of coarse spaces and controlled maps.

Let $\mathcal{A}$ be a family in $\mathcal{P}_{X\times X}$ and assume that $\mathcal{C}_X = \mathcal{C}\langle\mathcal{A}\rangle$.

Lemma 2.24. *Then the map $f : X \to Y$ is controlled if and only if $f(A) \in \mathcal{C}_Y$ for all A in $\mathcal{A}$.*

Proof. If f is controlled, then $f(A) \in \mathcal{C}_Y$ for all A in $\mathcal{A}$ since $\mathcal{A} \subseteq \mathcal{C}_X$.

We now consider the converse. Since $f(\mathtt{diag}(X)) \subseteq \mathtt{diag}(Y)$ we know that $f(\mathtt{diag}(X)) \in \mathcal{C}_Y$. Furthermore $f(U \cup U^{-1}) \subseteq f(U) \cup f(U^{-1})$. Hence we can assume that $\mathcal{A}$ contains $\mathtt{diag}(X)$ and consists of symmetric entourages.

We consider V in $\mathcal{C}_X$. By Lemma 2.12 there exists a finite family of families $((U_{j,i})_{i\in 1,\dots,n_j})_{j\in J}$ of elements of $\mathcal{A}$ such that
$$V \subseteq \bigcup_{j\in J} U_{j,1} \circ \cdots \circ U_{j,n_j} \ .$$
Then
$$f(V) \subseteq f(\bigcup_{j\in J} U_{j,1} \circ \cdots \circ U_{j,n_j}) \subseteq \bigcup_{j\in J} f(U_{j,1}) \circ \cdots \circ f(U_{j,n_j})$$
belongs to $\mathcal{C}_Y$.

Here we used the relations $f(U \circ U') \subseteq f(U) \circ f(U')$ and $f(U \cup U') = f(U) \cup f(U')$. □

Example 2.25. Assume that X and Y are metric spaces and have the metric coarse structures.

Lemma 2.26. *A map $f : X \to Y$ is controlled if and only if for all S in $[0, \infty)$ there exist R in $[0, \infty)$ such that $d_X(x, x') \leq S$ implies $d_Y(f(x), f(x')) \leq R$.*

Proof. Assume that f is controlled. If S is in $[0, \infty)$, then $U_{X,S} \in \mathcal{C}_X$ and therefore $f(U_{X,S}) \in \mathcal{C}_Y$. As explained in Example 2.15 there exists R in $[0, \infty)$ such that $f(U_S) \subseteq U_{Y,R}$. This inclusion is equivalent to the assertion that $d_X(x, x') \leq S$ implies $d_Y(f(x), f(x')) \leq R$.

We now consider the converse. Let V be in $\mathcal{C}_X$. Again by Example 2.15 there exists S in $[0, \infty)$ such that $V \subseteq U_{X,S}$. The condition on f says that there exist R in $[0, \infty)$ with $f(U_{X,S}) \subseteq U_{Y,R}$. Since $U_{Y,R} \in \mathcal{C}_Y$ and $f(V) \subseteq f(U_{X,S}) \subseteq U_{Y,R}$ we conclude that $f(V) \in \mathcal{C}_Y$. □

If f is Lipschitz with Lipschitz constant C, then we can take $R := CS$. In particular, Lipshitz maps are controlled. More generally maps which satisfy

$$d(f(x), f(x')) \leq Cd(x, x') + C'$$

(quasi-Lipschitz) for some C, C' and all x, x' in X are controlled.

The map

$$\mathbb{R} \to \mathbb{Z} , \quad t \mapsto \text{nearest integer to } t$$

satisfy this with $C' = 1$ and $C = 1$.

The map $x \mapsto x + 1$ on $\mathbb{R}$ is controlled.

The map $x \mapsto -x$ on $\mathbb{R}$ is controlled.

The map

$$\mathbb{N} \to \mathbb{N} , \quad n \mapsto n^2$$

is not controlled. □

Example 2.27. Let X be a coarse space and $f : Y \to X$ be a map of sets. Then $\mathcal{C}\langle f^{-1}(\mathcal{C}_X)\rangle$ is the induced coarse structure on Y. It is the largest coarse structure on Y for which f becomes controlled map. □

Example 2.28. Let X be a coarse space and $f : X \to Y$ be a map of sets. Then $\mathcal{C}\langle f(\mathcal{C}_X)\rangle$ is the coinduced coarse structure on Y. It is the smallest coarse structure on Y such that f becomes a controlled map. □

We now study the category **Coarse** and its relation with the category **Set** of sets.

Let S : **Coarse** → **Set** be the forgetful functor.

Lemma 2.29. *We have adjunctions*

$$S : \mathbf{Coarse} \leftrightarrows \mathbf{Set} : (X \mapsto X_{max})$$

and

$$(X \mapsto X_{min}) : \mathbf{Set} \leftrightarrows \mathbf{Coarse} : S \ .$$

Proof. We have equalities for all sets X and coarse spaces Y

$$\mathtt{Hom}_{\mathbf{Coarse}}(X_{min}, Y) = \mathtt{Hom}_{\mathbf{Set}}(X, S(Y))$$

and

$$\mathtt{Hom}_{\mathbf{Coarse}}(Y, X_{max}) = \mathtt{Hom}_{\mathbf{Set}}(S(Y), X) \ .$$

□

By Lemma 2.29 the underlying set of a limit or colimit is the limit or colimit of the underlying sets.

Proposition 2.30. *The category* **Coarse** *admits all limits and colimits.*

Proof. We start with colimits. Let $X : \mathbf{I} \to \mathbf{Coarse}$ be a diagram. We consider the colimit of sets $Y := \mathtt{colim}_{\mathbf{I}} S(Y)$ with the family of structure maps $(e_i : X_i \to Y)_{i \in \mathbf{I}}$. We equip Y with the smallest coarse structure such that e_i is controlled for all i, i.e., with $\mathcal{C}_Y := \mathcal{C}\langle \bigcup_{i \in I} e_i(\mathcal{C}_{X_i}) \rangle$.

We claim that $(Y, (e_i)_{i \in \mathbf{I}})$ is the colimit of the diagram X. Let T be a coarse space. Then we have a bijection

$$\mathtt{Hom}(S(Y), S(T)) \stackrel{\cong}{\to} \lim_{\mathbf{I}^{\mathrm{op}}} \mathtt{Hom}(S(X), S(T)) \ .$$

It restricts to an injective map

$$\mathtt{Hom}_{\mathbf{Coarse}}(Y, T) \to \lim_{\mathbf{I}^{\mathrm{op}}} \mathtt{Hom}_{\mathbf{Coarse}}(X, T) \ .$$

We must show that it is surjective. Let $(f_i : X \to T)_{i \in I}$ represent an element in the limit and assume that $g : Y \to T$ is the corresponding map of underlying sets. We must show that g is controlled. It is enough to consider generating entourages. Let U be an entourage of Y. Let U_i be an entourage of X_i and $e_i(U_i)$ be the corresponding generating entourage of Y. Then $g(e_i(U)) = f_i(U_i)$ is a coarse entourage of T since f_i is controlled.

We now consider limits. We form the limit $Z := \mathtt{lim}_{\mathbf{I}} S(X)$ of underlying sets with canonical projections $p_i : Z \to S(X_i)$. We equip Z with the maximal coarse structure such that all p_i become controlled. This is the intersection of the coarse structures induced by the p_i. We claim that $(Z,(p_i)_{i\in \mathbf{I}})$ has the required universal property of a limit of X. For T in **Coarse** we have a bijection

$$\mathtt{Hom}(S(T),S(Z)) \stackrel{\cong}{\to} \lim_{\mathbf{I}} \mathtt{Hom}(S(T),S(Z)) \ .$$

It restricts to an injective map

$$\mathtt{Hom}_{\mathbf{Coarse}}(T,Z) \to \lim_{\mathbf{I}} \mathtt{Hom}(T,X) \ .$$

In order to show that this map is surjective we consider a family $(f_i : T \to X_i)_{i\in\mathbf{I}}$ in the limit and the corresponding map of underlying sets $g : T \to Z$. We must show that g is controlled. Let U be an entourage of T. Then $f_i(U) = p_i(g(U))$ is a coarse entourage of X_i for every i in $\mathbf{I}$. This implies by construction of the coarse structure of Y that $g(U)$ is a coarse entourage of Z. □

Example 2.31. Let $(X_i)_{i\in I}$ be a family of coarse spaces. Then we can describe its coproduct $\coprod_{i\in I} X_i$ explicitly. The underlying set of the coproduct is the disjoint union $S(X) := \bigsqcup_{i\in I} S(X_i)$ union of the underlying sets. An entourage U of X is coarse in X for all i in I, and if $U \cap (X_i \times X_i)$ is coarse in X_i and $U \cap (X_i \times X_i) = \mathtt{diag}(X_i)$ for all but finitely many i in I. □

Example 2.32. Let $(X_i)_{i\in I}$ be a family of coarse spaces. Then we can describe its cartesian product $\prod_{i\in I} X_i$ explicitly. The underlying set of the product is the cartesian product $S(X) := \prod_{i\in I} S(X_i)$ union of the underlying sets. The coarse structure on X is generated by entourages $(U_i)_{i\in I}$ of X where U_i is a coarse entourage of X_i for every i in I. □

In the following we will introduce various concepts of coarse geometry.

Let $f,g : X \to Y$ be two maps between sets and U be an entourage of Y.

Definition 2.33. *We say that f and g are U-close to each other if $(f,g)(\mathtt{diag}(X)) \subseteq U$.*

We also write $f \sim_U g$.

Let $f,g : X \to Y$ be two maps into a coarse space.

Definition 2.34. *f and g are close to each other if $f \sim_U g$ for some U in $\mathcal{C}_Y$.*

Remark 2.35. Let X be a set and Y be a coarse space. The condition that $f,g : X \to Y$ are close to each other is equivalent to the condition that map $h : \{0,1\}_{max} \times X_{min} \to Y$ with $h(0,x) := f(x)$ and $h(1,x) := g(x)$ is a morphism of coarse spaces. □

Example 2.36. The map

$$\mathbb{R} \to \mathbb{R} \ , \quad x \mapsto x+1$$

is close to $\mathtt{id}_{\mathbb{R}}$. The map

$$\mathbb{R} \to \mathbb{R} \ , \quad x \mapsto -x$$

is not close to $\mathtt{id}_{\mathbb{R}}$. □

Lemma 2.37.

1. *Closeness is an equivalence relation on morphisms of* **Coarse**.

2. *Closeness is compatible with composition.*

Proof. We use the symbol $\sim$ in order to denote the relation of closeness. We consider maps from X to Y. We first show that closeness is an equivalence relation.

1. Since $(f,f)(\mathtt{diag}(X)) \subseteq \mathtt{diag}(Y)$ it is clear that $f \sim f$.

2. We have $(f,g)(\mathtt{diag}(X)) = (g,f)(\mathtt{diag}(X))^{-1}$. Then the coarse structure $\mathcal{C}_Y$ is closed under taking inverses conclude that $f \sim g$ if and only $g \sim f$.

3. For three maps f, g, h from X to Y we have
$$(f,h)(\mathtt{diag}(X)) \subseteq (f,g)(\mathtt{diag}(X)) \circ (g,h)(\mathtt{diag}(X)) \ .$$
Since the coarse structure $\mathcal{C}_Y$ is closed under forming compositions we conclude that $f \sim g$ and $g \sim h$ implies $f \sim h$.

We next show that closeness is compatible with compositions. Let $h : Y \to Z$ be a morphism. If $f \sim g$, then using that h is controlled we see that
$$(h \circ f, h \circ g))(\mathtt{diag}(X)) = h((f,g)(\mathtt{diag}(X)))$$
is a coarse entourage of Z. Hence $h \circ f \sim h \circ g$.

Let $l : W \to X$ be a morphism. Then
$$(f \circ l, g \circ l)(\mathtt{diag}(W)) \subseteq (g,f)(l(\mathtt{diag}(W)) \subseteq (f,g)(\mathtt{diag}(X))$$
is a coarse entouarge of Y. Hence $f \circ l \sim g \circ l$. $\square$

We can form the category $\overline{\mathbf{Coarse}}$ with the same objects as **Coarse** and closeness classes of maps.

Definition 2.38. *A morphism $f : X \to Y$ in* **Coarse** *is a coarse equivalence if it is invertible in* $\overline{\mathbf{Coarse}}$. *Two coarse spaces are called coarsely equivalent if they are isomorphic in* $\overline{\mathbf{Coarse}}$.

Remark 2.39. Explicitly, $f : X \to Y$ is a coarse equivalence if and only if there exists a morphism $g : Y \to X$ such that $f \circ g \sim \mathtt{id}_Y$ and $g \circ f \sim \mathtt{id}_X$. $\square$

It is an overall idea in coarse geometry that one should study coarse spaces up to coarse equivalence.

Example 2.40. The embedding $\mathbb{Z} \to \mathbb{R}$ (where $\mathbb{Z}$ has the induced coarse structure) is a coarse equivalence. An inverse up to closeness is the map $x \mapsto [x]$ (integer part of x) $\square$

Example 2.41. Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$ be a map between the underlying sets.

Definition 2.42. *The map f is a quasi-isometry if there are constants C, C' in $(0, \infty)$ such that*

$$C^{-1}d_X(x', x) - C' \leq d_Y(f(x), f(x')) \leq Cd(x, x') + C' \tag{2.1}$$

for all x, x' in X.

We have already seen in Example 2.26 that a quasi-isometry induces a morphism $f : X_d \to Y_d$ in **Coarse**. This only uses the second inequality in (2.1). Let r be in $(0, \infty)$.

Definition 2.43. *We say that $f(X)$ is r-dense if $\bigcup_{x \in X} B(f(x), r) = Y$.*

Lemma 2.44. *If f is a quasi-isometry and $f(X)$ is r-dense for some r in $(0, \infty)$, then f is a coarse equivalence.*

Proof. We define $g : Y \to X$ by choosing for every y in Y some x in X such that y in $B(f(x), y)$. Then g is controlled. Note that $d_Y(f(g(y)), y) \leq r$ for all y in Y. By the first inequality in (2.1) we have

$$\begin{aligned} d_X(g(y), g(y')) &\leq Cd_Y(f(g(y)), f(g(y'))) + CC' \\ &\leq Cd_Y(y, y) + C(2r + C') \end{aligned}$$

for all y, y'.

Furthermore $f \circ g$ is close to $\mathtt{id}_Y$ since $d_Y(f(g(x)), y) \leq r$ for all y in Y and $g \circ f$ is close to $\mathtt{id}_X$ since

$$d_X(g(f(x)), x) \leq Cd_Y(f(g(f(x), f(x)) + CC' \leq Cr + CC'$$

for all x in X. $\square$

Let (X, d) be a metric space. Then we can define a new metric by

$$d'(x, y) := \ln(1 + d(x, y)) \ .$$

The identity of the underlying sets is a coarse equivalence between X_d and $X_{d'}$. But if X is unbounded, then it is not a quasi-isometry provided X.

So in general the condition of being quasi-isometric is stronger than the condition of being coarsely equivalent. $\square$

Example 2.45. The embedding $\mathbb{Z}^n \to \mathbb{R}^n$ is an isometry, hence in particular a quasi isometry. Furthermore, $\mathbb{Z}^n$ is $\sqrt{n/2}$-dense in $\mathbb{R}^n$. $\square$

The coarse spaces $\mathbb{Z}$ and $\mathbb{R}$ are equivalent, but $\mathbb{Z}$ is much smaller. It is often convenient to represent coarse spaces up to equivalence but small models. To this end the notion of a dense subset of a coarse space is useful. This notion extends the notion of r-density in the metric case.

Let X be a set and L be a subset. Let U be an entourage of X.

Definition 2.46. *L is U-dense if $U[L] = X$.*

Explicitly this means that for every point x of X there exists a point l in L such that $(x, l) \in U$.

Let X be a coarse space and L be a subset of X.

Definition 2.47. *L is dense in X if it is U-dense for some coarse entourage U of X.*

Example 2.48. The subset $\mathbb{Z}$ is U_1-dense in $\mathbb{R}$.

The subset $\{n^2 \mid n \in \mathbb{N}\}$ is not dense in $\mathbb{N}$. $\square$

Let X be a coarse space and L be a subset equipped with the induced coarse structure.

Lemma 2.49. *If L is dense, then the inclusion $f : L \to X$ is a coarse equivalence.*

Proof. Let U be a coarse entourage of X such that L is U-dense. In order to construct an inverse define a map $g : X \to L$ by choosing for every x in X the point $g(x)$ in L such that $x \in U[\{g(x)\}]$. Then $f \circ g$ and $\mathtt{id}_X$ are U-close and $g \circ f$ and $\mathtt{id}_L$ are $U \cap (L \times L)$-close to each other.

It remains to check that g is controlled. Let W be in $\mathcal{C}_X$. If (x, y) is in W, then $(g(x), g(y)) \in U^{-1} \circ W \circ U$ since $g(x) \sim_{U^{-1}} x \sim_W y \sim_U g(y)$. Hence $g(W) \subseteq (U^{-1} \circ W \circ U) \cap (L \times L)$.

This shows that $g(W) \in \mathcal{C}_L$. $\square$

Let X be a set, U be an entourage of X containing the diagonal and L be a subset.

Definition 2.50. *We say that L is U-separated, if for every l, l' with $l \neq l'$ we have $l' \notin U[\{l\}]$.*

In other words, we have $U \cap (L \times L) \subseteq \mathtt{diag}(L)$.

Lemma 2.51. *If U is symmetric, then X admits a U-dense and U-separated subset.*

Proof. We consider the poset of U-separated subsets with respect to inclusion. We check that any totally ordered chain $(L_\alpha)_{\alpha\in A}$ in this poset is bounded by $\bigcup_{\alpha\in A} L_\alpha$. Indeed, this subset is also U-separated.

By the lemma of Zorn there exists a maximal U-separated subset $\bar{L}$. We claim that it is U-dense. Assume that it is not. Then there exists x in X with $x \notin U[\bar{L}]$. But then $\bar{L} \cup \{x\}$ is still U-separated. In order to see that $U[\{x\}] \cap L = \emptyset$ we use that U is symmmetric. We obtain a contradiction to the maximality of $\bar{L}$. □

Let X be a coarse space.

Definition 2.52.

1. *X is uniformly locally finite if for every entourage U of X we have $\sup_{x\in X} |U[\{x\}]| < \infty$.*

2. *X has bounded geometry if it is coarsely equivalent to a uniformly locally finite coarse space.*

Bounded geometry is preserved under coarse equivalences by definition. This is not true for uniform local finiteness.

Example 2.53. The inclusion $\mathbb{Z} \to \mathbb{R}$ is a coarse equivalence. $\mathbb{Z}$ is uniformly locally finite, but $\mathbb{R}$ is not.

Let X be a set and U be in $\mathcal{P}_{X\times X}$. Let B be a subset of X.

Definition 2.54. *The subset B is called U-bounded if $B \times B \subseteq U$.*

Example 2.55. If X is a metric space, then B is U_r-bounded if and only if $\mathrm{diam}(B) \leq r$. □

Example 2.56. For a set X let $P(X)$ denote the space of finitely supported functions $\mu : X \to [0,1]$ such that $\sum_{x\in X} \mu(x) = 1$. If $f : X \to X'$ is a map of sets, then we define the map $P(f) : P(X) \to P(X')$ by

$$P(f)(\mu(x')) := \sum_{x\in f^{-1}(x')} \mu(x) \ .$$

We have

$$P(X) \cong \operatorname*{colim}_{F\subseteq X} P(F)$$

as sets where F runs over all finite subsets of X. Note that the structure maps are injective so that this can also be interpreted as forming a union of subets.

As a topological space we identify $P(F)$ with a subspace of $\mathbb{R}^F$. Then we equip $P(X)$ with the topology of the colimit. The map $P(f) : P(X) \to P(X')$ is then continuous.

Note that $P(F)$ is a standard simplex of dimension $|F| - 1$, and $P(X)$ is a simplicial complex. The map $P(f)$ is a morphism of simplicial complexes.

Let now U be an entourage of X. Then we consider the closed subspace $P_U(X)$ of $P(X)$ of functions which have U-bounded support. Note that $P_U(X)$ is a subcomplex of $P(X)$. If X_U has bounded geomety, then $P_U(X)$ is finite-dimensional. If U' is an entourage of X' and $f(U) \subseteq U'$, then by restriction we get a continuous map $P(f) : P_U(X) \to P_{U'}(X')$. Indeed, if (x', y') are in $\mathtt{supp} f_*(\mu)$, then there exist x, y in X such that $f(x) = x'$ and $f(y) = y'$. Then $(x, y) \in U$ and hence $(x', y') = (f(x), f(y)) \in f(U) \subseteq U'$.

This construction is functorial on the category of pairs (X, U) (the Grothendieck construction of the functor $\mathbf{Coarse} \to \mathbf{Set}$, $X \mapsto \mathcal{C}_X$).

Assume now that $f, g : X \to X'$ are morphisms such that $f(U) \subseteq U'$, $g(U) \subseteq U'$ and f and g are V'-close to each other for some further entourage V'. Then $P(f)$ is homotopic to $P(g)$ as maps from $P_U(X)$ to $P_{W'}(X')$ for any entourage W' of X' such that $U' \subseteq W'$ and $VU'^{-1} \subseteq W'$. The homotopy is given by convex interpolation:

$$h(u, \mu) := (1 - u)P(f)(\mu) + uP(g)(\mu) \ .$$

One first checks continuity by observing that for every finite subset F of X the restriction of this map to $P_U(F)$ factors over the obviously continuous map $P_U(F) \to P(f(F) \cup g(F))$ given by the same formula. It remains to check that the support of $h(u, \mu)$ is U'-bounded for every u in $[0, 1]$.

If x', y' are in $\mathtt{supp}(h(u, \mu))$ then we have one of the following cases:

1. x', y' are in $\mathtt{supp} P(f)(\mu)$: In this case $(x', y') \in U'$ since the support of $P(f)(\mu)$ is U'-bounded.

2. x', y' are in $\mathtt{supp} P(g)(\mu)$: In this case $(x', y') \in U'$ since the support of $P(g)(\mu)$ is U'-bounded.

3. x' is in $\mathtt{supp} P(f)(\mu)$ and y' is $P(g)(\mu)$: Then $x' = f(x)$ and $y' = g(y)$ for some x, y in $\mathtt{supp}(\mu)$. Then $(x, y) \in U$. This implies that $(x', y') \in V \circ U'$. To this end we consider the chain $x' = f(x) \sim_V g(x) \sim_{U'} g(y) = y'$

4. x' is in $\mathtt{supp} P(g)(\mu)$ and y' is $P(g)(\mu)$: This is analoguous to the previous case.

Let $F : \mathbf{Top} \to \mathbf{M}$ be any functor to a cocomplete target. Then we can define a functor

$$F\mathbf{P} : \mathbf{Coarse} \to \mathbf{M} \ , \quad F\mathbf{P}(X) := \operatorname*{\mathtt{colim}}_{U \in \mathcal{C}_X} F(P_U(X)) \ .$$

If F is homotopy invariant, then $F\mathbf{P}$ is coarsely invariant and hence factorizes over a functor

$$\overline{F\mathbf{P}} : \overline{\mathbf{Coarse}} \to \mathbf{M} \ .$$

For example, we can consider $\pi_0 : \mathbf{Top} \to \mathbf{Set}$. Then

$$\pi_0^{coarse}(X) \cong \pi_0 \mathbf{P}(X) \ .$$

Another example of a homotopy invariant functor is $H_n^{\mathtt{sing}}(-) : \mathbf{Top} \to \mathbf{Ab}$. But observe (Excercise!) that for $n \geq 1$ we have $H_n^{\mathtt{sing}}\mathbf{P}(X) \cong 0$ for every coarse space X. We will learn later how to modify the functor $H_n^{\mathtt{sing}}$ to get non-trivial answers.

Using this construction we can use homotopy invariant functors from algebraic topology to get invariants of coarse spaces up to equivalence.

Example 2.57. This example is the combinatorial version of Example 2.56. Let X be a coarse space. Let U be an entourage containing the diagonal. Then we can define a simplicial set $P_U^\bullet(X)$ as follows. A point $(x_0, \ldots, x_n)$ in $\prod_{i=0}^n X$ is called U-bounded if $(x_i, x_{i'}) \in U$ for all i, i' in $\{0, \ldots, n\}$.

We consider the simplicial set $X^\bullet$ with the set of n-simplices $X^n = \prod_{i=0}^n X$. The faces are the projections, and the degenerations are diagonal insertions. Thus

$$\partial_i(x_0, \ldots, x_n) := (x_0, \ldots, \hat{x}_i, \ldots, x_n)$$

and

$$s_i(x_0, \ldots, x_n) := (x_0, \ldots, x_i, x_i, \ldots x_n) \ .$$

The complex $P_U^\bullet(X)$ is the simplicial subset of $X^\bullet$ consisting of all U-bounded simplices. It is clear that it is preserved by the faces and degenerations.

Definition 2.58. *The simplicial set $P_U^\bullet(X)$ is called the Rips complex of X for U.*

If $f : X \to X'$ is a map and $f(U) \subseteq U'$, then we get an induced map $P(f) : P_U^\bullet(X) \to P_{U'}^\bullet(X')$.

The dimension of a simplicial set is the supremum of the dimensions of its non-degenerated simplices.

If X has uniformly locally bounded geometry, then $P_U^\bullet(X)$ is finite-dimensional for every coarse entourage U containing $\mathtt{diag}(X)$.

Seien $f, g : X \to X'$ be morphisms of coarse spaces such that $f(U) \subseteq U'$ and $g(U) \subseteq U'$ and f and g are V'-close. Let W' be an entourage of X' such that $U' \subseteq W'$ and $U'V' \subseteq W'$.

Lemma 2.59. *$P^\bullet(f)$ and $P^\bullet(g)$ are simplicially homotopic as morphisms $P^\bullet_U(X) \to P^\bullet_{W'}(X')$.*

Proof. For all n in $\mathbb{N}$ and i in $\{0,\dots,n\}$ we define maps $h_i : P^n_U(X) \to P^{n+1}_{W'}(Y)$ as follows:

$$h_i(x_0,\dots,x_n) := (f(x_0),\dots,f(x_i),g(x_i),\dots,g(x_n)) \ .$$

One checks the following defining relations:

$$d_i h_j = \begin{cases} h_{j-1}d_i & i<j \\ d_i h_{i-1} & i=j\neq 0 \\ h_j d_{i-1} & i>j+1 \end{cases}$$

and

$$s_i h_j = \begin{cases} h_{j+1}s_i & i\le j \\ h_j s_{i-1} & i>j \end{cases} \ .$$

□

Consider a functor $F : \mathbf{sSet} \to \mathbf{M}$ to some target. Then we can consider

$$F\mathbf{P}^\bullet(X) := \operatorname*{colim}_{U\in\mathcal{C}_X} F(P^\bullet_U(X)) \ .$$

Then $X \mapsto F\mathbf{P}^\bullet(X)$ is a functor

$$F\mathbf{P}^\bullet : \mathbf{Coarse} \to \mathbf{M} \ .$$

If F is homotopy invariant, then $F\mathbf{P}^\bullet$ sends coarse equivalences to equivalences (equalities) and hence factorizes over a functor

$$\overline{F\mathbf{P}^\bullet} : \overline{\mathbf{Coarse}} \to \mathbf{M} \ .$$

3 Bornological coarse spaces

Let X be a set with a coarse structure $\mathcal{C}$ and a (generalized) bornological structure $\mathcal{B}$.

Definition 3.1. *$\mathcal{C}$ and $\mathcal{B}$ are compatible if for every B in $\mathcal{B}$ and U in $\mathcal{C}$ we have $U[B]\in\mathcal{B}$.*

Compatibility means that the bornology $\mathcal{B}$ is stable under U-thickening for all coarse entourages U of X.

Let X be a coarse space. Let $\mathcal{B}$ be a subset of $\mathcal{P}_X$.

Lemma 3.2. *The following are equivalent.*

1. *$\mathcal{B}$ is the minimal bornology compatible with $\mathcal{C}_X$.*

2. *$\mathcal{B} = \mathcal{B}\langle\{U[\{x\}] \mid x \in X \text{ and} U \in \mathcal{C}_X\}\rangle$.*

3. *$\mathcal{B} = \mathcal{B}\langle\{B \mid B \text{ is } U\text{-bounded for some } U \text{ in } \mathcal{C}_X\}\rangle$*

Proof. $1 \Leftrightarrow 2$: Let $\mathcal{B}$ as in 1. and $\mathcal{B}' := \mathcal{B}\langle\{U[\{x\}] \mid x \in X \& U \in \mathcal{C}_X\}\rangle$. Since $\{x\}$ is bounded in any bornology we have $\{x\} \in \mathcal{B}$. Since $\mathcal{B}$ is compatible with $\mathcal{C}_X$ we have $U[\{x\}] \in \mathcal{B}$ for all U in $\mathcal{C}_X$ and x in X. Hence $\mathcal{B}' \subseteq \mathcal{B}$. We show that $\mathcal{B}'$ is compatible with $\mathcal{C}_X$ and conclude $\mathcal{B}' = \mathcal{B}$ by minimality of $\mathcal{B}$. Let A be in $\mathcal{B}'$ and $U \in \mathcal{C}_X$. Then there exist finite families $(U_i)_{i\in I}$ in $\mathcal{C}_X$ and $(x_i)_{i\in I}$ such that $A \subseteq \bigcup_{i\in I} U_i[x_i]$ (see Example 1.6). We conclude that $U[A] \subseteq \bigcup_{i\in I}(U \circ U_i)[x_i]$. Since $U \circ U_i \in \mathcal{C}_X$ for all i in I this implies that $U[A] \in \mathcal{B}'$.

$2 \Leftrightarrow 3$: Let $\mathcal{B}'' := \mathcal{B}\langle\{B \mid B \text{ is } U\text{-bounded for some } U \text{ in } \mathcal{C}_X\}$

We first show that $\mathcal{B}' \subseteq \mathcal{B}''$. Let x be in X and U be in $\mathcal{C}_X$. Then we have $[U\{x\}] \times U[\{x\}] \subseteq U \circ U^{-1}$. Since for U in $\mathcal{C}_X$ also $U \circ U^{-1} \in \mathcal{C}_X$ we conclude that $[U\{x\}] \times U[\{x\}]$ is bounded by a coarse entourage of X. Hence $[U\{x\}] \times U[\{x\}] \in \mathcal{B}''$. This implies that $\mathcal{B}' \subseteq \mathcal{B}''$.

We now show that $\mathcal{B}'' \subseteq \mathcal{B}'$. Let B be in $\mathcal{B}''$ and not empty. Then there exists a finite family $(B_i)_{i\in I}$ of non-empty subsets of X such that $B = \bigcup_{i\in I} B_i$ and $U_i := B_i \times B_i \in \mathcal{C}_X$. Then for every i in I we have $B_i = U_i[\{b_i\}]$ for some point b_i of B_i. This implies $B_i \in \mathcal{B}'$ for all i in I and hence $B \in \mathcal{B}'$. □

Example 3.3. The minimal generalized bornology compatible with a coarse structure is the empty bornology.

Every (generalized) bornology is compatible with the minimal coarse structure.

The maximal bornology is compatible with any coarse structure. □

Example 3.4. Show by example that in general $\{B \mid B \text{ is } U\text{-bounded for some } U \text{ in } \mathcal{C}_X\}$ is not a bornology. □

Remark 3.5. Let X be a bornological space and $\mathcal{A}$ be a subset of $\mathcal{P}_{X\times X}$. In order to check that $\mathcal{C}_X := \mathcal{C}\langle\mathcal{A}\rangle$ is compatible with $\mathcal{B}_X$ it suffices to show that $A[B] \in \mathcal{B}_X$ and $A^{-1}[B] \in \mathcal{B}_X$ for all A in $\mathcal{A}$ and B in $\mathcal{B}_X$.

In order to see this we will use the notation from Lemma 2.12. If V is in $\mathcal{C}_X$, then $V \subseteq \bigcup_{j\in J} \bigcup_{i=1,\dots,n_j} A_{j,i}$. We then use that $(W \circ W')[B] = W[W'[B]]$ and $(W \cup W')[B] \subseteq W[B] \cup W'[B]$ for any two entourages W and W' of X. □

Definition 3.6. *A bornological coarse space is a triple $(X, \mathcal{C}, \mathcal{B})$ of a set with a coarse and a bornological structure which are compatible.*

Example 3.7. This example generalizes 1.4 and 2.7 at the same time. Let X be a bornological coarse space. We assume that $\mathcal{B}_{min}$ is compatible with $\mathcal{C}_X$. This means that $U[\{x\}]$ is finite for every x in X and U in $\mathcal{C}_X$.

We consider the R-module

$$\mathbf{A}(X) := \{A \in R^{X\times X} \mid \mathtt{supp}(A) \in \mathcal{C}_X\} \ .$$

This R-module has an associative algebra structure defined by matrix multiplication:

$$A''(x,y) := (A' \circ A)(x,y) := \sum_{z\in X} A'(x,z)A(z,y) \ .$$

Indeed, the sum runs over the finite set $\mathtt{supp}(A')^{-1}[\{x\}] \cap \mathtt{supp}(A)[\{y\}]$. Note that $\mathtt{supp}(A) \subseteq \mathtt{supp}(A') \circ \mathtt{supp}(A)$.

We consider the R-module $C_{\mathrm{lf}}(X,R)$ of functions $f : X \to R$ whose support is locally finite (Definition 1.17). Thus for f in $C_{\mathrm{lf}}(X,R)$ we have $|\mathtt{supp}(f) \cap B|$ is finite for every B in $\mathcal{B}_X$.

We can define an action of $\mathbf{A}(X)$ on $C_{\mathrm{lf}}(X,R)$ as follows:

$$(Af)(x) = \sum_{y\in X} A(x,y)f(y) \ .$$

This sum is finite. Let B be bounded in X. Then $B \cap \mathtt{supp}(Af) \subseteq A[\mathtt{supp}(A)^{-1}[B] \cap \mathtt{supp}(f)]$. Since $[\mathtt{supp}(A)^{-1}[B]$ is bounded the intersection with $\mathtt{supp}(f)$ is finite, and hence $A[\mathtt{supp}(A)^{-1}[B] \cap \mathtt{supp}(f)]$ is finite. This shows that Af belongs to $C_{\mathrm{lf}}(X,R)$. □

Example 3.8. If (X,d) is a metric space, then $\mathcal{B}_d$ and $\mathcal{C}_d$ are compatible. We get a bornological coarse space $X_d := (X, \mathcal{C}_d, \mathcal{B}_d)$.

Example 3.9. Let X be a Hausdorff space and A be a subset. We set $X := X \setminus A$. Then the continuously controlled coarse structure $\mathcal{C}$ and the bornology $\mathcal{B}$ of subsets B with $\bar{B} \cap A = \emptyset$ are compatible.

We check the compatibility. Let B be in $\mathcal{B}$ and U in $\mathcal{C}$. Assume that $U[B] \notin \mathcal{B}$. Then $\overline{U[B]} \cap A$ contains a point a. Then there exists a net $(x_i, b_i)_{i\in I}$ in U such that $\mathtt{lim}_{i\in I}\, x_i = a$. Then also $\mathtt{lim}_{i\in I}\, b_i = a$ and hence $\bar{B} \cap A \neq \emptyset$. This is a contradiction.

We call the structure $(\mathcal{C}, \mathcal{B})$ the continuously controlled bornological coarse structure on X.

If we omit the Hausdorff assumption then the same works for generalized bornologies. □

By **BornCoarse** we denote the category of bornological coarse spaces and proper and controlled maps. We can apply the above definitions to generalized bornologies and obtain the notion of a generalized bornological coarse space. We get the category $\widetilde{\mathbf{BornCoarse}}$ of generalized bornological coarse spaces and a fully faithful inclusion

$$\mathbf{BornCoarse} \to \widetilde{\mathbf{BornCoarse}} \ .$$

Example 3.10. If X is a generalized bornological coarse space, then we have a canonical coarsely disjoint decomposition $X = X_b \sqcup X_u$ into the subsets of bounded and unbounded points. Assume that b is in X_b and u is in X_u and b and u belong to same coarse component, then $\{(u,b)\}$ would be a coarse entourage of X. But then $u \in U[\{b\}]$ and hence $\{u\}$ would be bounded, which is a contradiction. □

Example 3.11. Let $f : X \to Y$ be a map of sets and assume that Y has a bornological coarse structure $(\mathcal{C}_Y, \mathcal{B}_Y)$. Then we can equip X with the maximal coarse structure $\mathcal{C}_X$ such that $f : X \to Y$ is controlled and the minimal bornology $\mathcal{B}_X$ such that $f : X \to Y$ is proper.

We check that these structures are compatible. Let U be in $\mathcal{C}_X$ and B be in $\mathcal{B}_X$. Then there exists a finite family $(B_i)_{i\in I}$ in $\mathcal{B}_Y$ such that $B \subseteq \bigcup_{i\in I} f^{-1}(B_i)$. We then have $U[B] \subseteq \bigcup_{i\in I} U[f^{-1}(B_i)]$. We now check that $U[f^{-1}(B_i)] \subseteq f^{-1}((f\times f)(U)[B_i])$. By construction of $\mathcal{C}_X$ we know that $(f\times f)(U)$ is controlled in Y and hence $(f\times f)(U)[B_i]$ is bounded in Y. But then $f^{-1}((f\times f)(U)[B_i])$ is bounded in X for all i in I and hence also $U[B]$ is bounded.

We call the bornological coarse structure on X the induced structure. Note that $f : X \to Y$ is then a morphism in **BornCoarse**. □

Example 3.12. Let G be a group. we equip G with the minimal bornology $\mathcal{B}_{min}$ consisting of the finite subsets. Furthermore we consider the canonical coarse structure $\mathcal{C}_{can} := \mathcal{C}\langle\{G(B\times B) \mid B \in \mathcal{B}_{min}\}\rangle$. Then $\mathcal{C}_{can}$ and $\mathcal{B}_{min}$ are compatible. Indeed, if A is in $\mathcal{B}_{min}$ and $U = G(B\times B)$, then

$$U[A] \subseteq \bigcap_{\{g\in G \mid gB\cap A\neq\emptyset\}} gB \cap A \ .$$

This set is finite since $\{g \in G \mid gB \cap A \neq \emptyset\}$ is finite and $gB \cap A$ is finite for every g in G.

Definition 3.13. *We write $G_{can,min}$ for the bornological coarse space G with the structures $\mathcal{C}_{can}$ and $\mathcal{B}_{min}$.*

Note that G acts on $G_{can,min}$ by automorphisms of bornological coarse spaces from the left. Furthermore the set of G-invariant entourages $\mathcal{C}^G_{can}$ is cofinal in $\mathcal{C}_{can}$. This condition characterizes G-bornological coarse spaces among bornological coarse spaces with G-action.

Many groups admit a finite description by generators and relations $G \cong \langle S|R\rangle$. Going over to $G_{can,min}$ we obtain a description of interesting bornological coarse spaces (with high symmetry) in finite terms.

Note that $\mathbb{Z}^n_{can,min}$ is equivalent to $\mathbb{Z}^n$ with the metric structures from $\mathbb{R}^n$. The group $\mathbb{Q}^n_{can,min}$ is completely different from $\mathbb{Q}^n$ with the metric structure. □

Let X, Y be in **BornCoarse**.

Definition 3.14. *A morphism $f : X \to Y$ is called an equivalence if there exists a morphism $g : Y \to X$ such that $f \circ g$ is close to $\mathtt{id}_Y$ and $g \circ f$ is close to $\mathtt{id}_X$.*

Example 3.15. We consider $\mathbb{R}$ with the standard bornological coarse structure from the metric. We consider $\mathbb{Z}$ with the bornological coarse structure induced from $\mathbb{R}$. Then $\mathbb{Z} \to \mathbb{R}$ is a coarse equivalence. If we equip $\mathbb{Z}$ with the maximal bornology, then this map is no longer an equivalence in **BornCoarse** since any potential inverse $\mathbb{R} \to \mathbb{Z}$ is not proper. □

The category **BornCoarse** has a symmetric monoidal structure $\otimes$. Let X and Y be in **BornCoarse**.

Definition 3.16.

We define $X \otimes Y$ as follows:

1. *The underlying set of $X \otimes Y$ is $X \times Y$.*
2. *The coarse structure on $X \otimes Y$ is generated by the entourages $U \times V$ for all U in $\mathcal{C}_X$ and $V \in \mathcal{C}_Y$.*
3. *The bornology of $X \otimes Y$ is generated by the subsets $A \times B$, where A is in $\mathcal{B}_X$ and B is in $\mathcal{B}_Y$.*

One checks that $\mathcal{B}_X$ is compatible with $\mathcal{C}_X$. The unit, associativity and symmetry constraints are induced from the cartesian symmetric monoidal structure on **Set**.

Example 3.17. The underlying coarse space of $X \otimes Y$ is the cartesian product of the underlying coarse spaces. The underlying bornological space of $X \otimes Y$ is the bornological tensor product from Definition 1.31. □

Example 3.18. We have an equivalence $\mathbb{R}^{n+m} \simeq \mathbb{R}^n \otimes \mathbb{R}^m$. □

Example 3.19. There are functors

$$(-)_{min,max}, (-)_{max,max} : \mathbf{Set} \to \mathbf{BornCoarse}$$

which send a set X to the bornological coarse space $X_{min,max}$ (or $X_{max,max}$) with the minimal (or maximal) coarse and the maximal bornological structure.

There are no such functors which equip X with the minimal bornology. □

Lemma 3.20. *We have an adjunction*

$$(X \mapsto (X, \mathcal{C}_{min}, \mathcal{B}_X)) : \mathbf{Born} \leftrightarrows \mathbf{BornCoarse} : \mathrm{forget} \ .$$

Proof. We write $X_{min,\mathcal{B}_X} := (X, \mathcal{C}_{min}, \mathcal{B}_X)$. For every T in **BornCoarse** we have an equality

$$\mathtt{Hom}_{\mathbf{BornCoarse}}(X_{min,\mathcal{B}_X}, T) = \mathtt{Hom}_{\mathbf{Born}}(X, T) \ .$$

□

The same argument gives:

Lemma 3.21. *We have an adjunction*

$$(X \mapsto (X, \mathcal{C}_{min}, \mathcal{B}_X)) : \widetilde{\mathbf{Born}} \leftrightarrows \mathbf{Born}\widetilde{\mathbf{Coarse}} : \mathrm{forget} \ .$$

Remark 3.22. Note that there is no adjunction

$$\mathrm{forget} : \mathbf{Born}\widetilde{\mathbf{Coarse}} \leftrightarrows \widetilde{\mathbf{Born}} : (X \mapsto (X, \mathcal{C}_{max}, \mathcal{B}_X))$$

generalizing the adjunction from Lemma 2.29. The problem is that $\mathcal{C}_{\max}$ is not compatible with a general bornology.

Lemma 3.23. *We have an adjunction*

$$(X \mapsto (X, \mathcal{C}_X, \mathcal{B}_{max})) : \mathbf{Coarse} \leftrightarrows \mathbf{BornCoarse} : \mathrm{forget} \ .$$

Proof. We write $X_{\mathcal{C}_X,max} := (X, \mathcal{C}_X, \mathcal{B}_{max})$. For every T in **BornCoarse** we have an equality

$$\mathtt{Hom}_{\mathbf{BornCoarse}}(X_{\mathcal{C}_X,max}, T) = \mathtt{Hom}_{\mathbf{Coarse}}(X, T) \ .$$

□

Lemma 3.24. *We have an adjunctions*

$$(X \mapsto (X, \mathcal{C}_X, \mathcal{B}_{max})) : \mathbf{Coarse} \leftrightarrows \mathbf{Born}\widetilde{\mathbf{Coarse}} : \mathrm{forget}$$

and

$$\mathrm{forget} : \mathbf{Born}\widetilde{\mathbf{Coarse}} \leftrightarrows \mathbf{Coarse} : (X \mapsto (X, \mathcal{C}_X, \emptyset))$$

Proof. The first case is as in Lemma 3.23. For the second we write $X_{\mathcal{C}_X,\emptyset} := (X, \mathcal{C}_X, \emptyset)$. For every T in $\mathbf{Born}\widetilde{\mathbf{Coarse}}$ we have an equality

$$\mathtt{Hom}_{\mathbf{Born}\widetilde{\mathbf{Coarse}}}(T, X_{\mathcal{C}_X,\emptyset}) = \mathtt{Hom}_{\mathbf{Coarse}}(X, T) \ .$$

□

Proposition 3.25. *The category* $\widetilde{\mathbf{BornCoarse}}$ *is complete and cocomplete.*

Proof. It is clear from the adjunctions above that the underlying coarse spaces of colimits and limits in $\widetilde{\mathbf{BornCoarse}}$ are the limits and colimits of the underlying diagrams in **Coarse**. The same applies to the underlying bornological space of a limit. As suggested by the observation in Remark 3.22 the bornology of a colimit is more complicated.

Let $X : \mathbf{I} \to \widetilde{\mathbf{BornCoarse}}$ be a diagram. Then we equip the limit Y of the diagram of underlying sets with the coarse structure and generalized bornology such that the resulting coarse and generalized bornological space represent the limit in **Coarse** and $\widetilde{\mathbf{Born}}$. We first check that the coarse structure and the generalized bornology on Y are compatible so that Y becomes an object of $\widetilde{\mathbf{BornCoarse}}$. We check the compatibility on the generators of the coarse structure with the bornology.

For i in $\mathbf{I}$ let $p_i : Y \to X_i$ be the canonical projection. By construction p_i is a morphism in $\widetilde{\mathbf{BornCoarse}}$ for every i in $\mathbf{I}$. An entourage U of X is coarse if $p_i(U)$ is coarse in X_i for all i in $\mathbf{I}$. The generators of the bornology of X are the subsets $p_i^{-1}(B)$ for i in $\mathbf{I}$ and bounded subsets B in X_i. For such a generator we have $U[p^{-1}(B)] \subseteq p_i^{-1}(p_i(U)[B])$. Since $p_i(U)[B]$ is again bounded by the compatibility of structures on X_i we conclude that $U[p^{-1}(B)]$ is bounded in X.

We now show that $(Y, (p_i)_{i\in\mathbf{I}})$ is a limit of the diagram X. Let T be in $\widetilde{\mathbf{BornCoarse}}$. By construction we have bijections

$$\mathtt{Hom}_{\widetilde{\mathbf{Born}}}(T, Y) \stackrel{\cong}{\to} \lim_{\mathbf{I}} \mathtt{Hom}_{\widetilde{\mathbf{Born}}}(T, X) \ , \quad \mathtt{Hom}_{\mathbf{Coarse}}(T, Y) \stackrel{\cong}{\to} \lim_{\mathbf{I}} \mathtt{Hom}_{\mathbf{Coarse}}(T, X)$$

which immediately implies $\mathtt{Hom}_{\widetilde{\mathbf{BornCoarse}}}(T, Y) \to \lim_{i\mathbf{I}} \mathtt{Hom}_{\widetilde{\mathbf{BornCoarse}}}(T, X)$.

For cocompleteness we show the existence of coproducts and coequalizers.

Let $(X_i)_{i\in I}$ be a family in $\widetilde{\mathbf{BornCoarse}}$. We consider the coproduct of sets $X := \coprod_{i\in I} X_i$ and the embeddings $e_i : X_i \to X$. We equip X with the minimal coarse structure such that e_i is controlled for all i in I and the maximal generalized bornology such that e_i is proper for all i in I. This the coarse structure generated by the entourages $e_i(U)$ for i in I and coarse entourages U of X_i, and a subset B is bounded if $e_i^{-1}(B)$ is bounded for every i in I. We check compatibility on generators. We have

$$e_i^{-1}(e_j(U)[B]) \subseteq \begin{cases} \emptyset & i \neq j \\ U[e_i^{-1}(B)] & i = j \end{cases} .$$

This set is bounded for every i by the compatibility of structures on X_i. Hence $e_j(U)[B]$ is bounded in X.

We claim that the generalized bornological coarse space X with the family of embeddings $(e_i)_{i\in I}$ represents the coproduct in $\widetilde{\mathbf{BornCoarse}}$. Let T be in $\widetilde{\mathbf{BornCoarse}}$ together with a family of morphisms $t_i : X_i \to T_i$. There exists a unique map of sets $t : X \to T$ such that $t \circ e_i = t_i$ for all i in I. This map is controlled and proper. Indeed, for a bounded B in T the set $e_i^{-1}(t^{-1}(B)) = t_i^{-1}(B)$ is bounded for every i in I. Hence $t^{-1}(B)$ is bounded in X. For a coarse entourage U of X_i the entourage $t(e_i(U)) = t_i(U)$ is coarse in X_i. This implies that t is controlled.

We now show the existence of coequalizers. Let $f, g : X \to Y$ be two morphisms in $\widetilde{\mathbf{BornCoarse}}$. Then we consider the coequalizer $q : Y \to Q$ of the two maps on the level of underlying sets. We equip Y with the minimal coarse structure structure such that this map is controlled and the maximal compatible genberalized bornology such that this map is proper. If U is an entourage of Y and B is a subset of Q, then we have $U(q^{-1}[B]) \subseteq q^{-1}(q(U)[B])$. Even if $q^{-1}(B)$ is bounded this does in general not imply that $q^{-1}(q(U)[B])$ is bounded. The generalized bornology of Q is given by subsets B such that $q^{-1}(q(U)[B])$ is bounded for all U in $\mathcal{C}_Y$.

We claim that $Y \to X$ represents the coequalizer of f and g. Let $t : Y \to T$ be a morphism such that $t \circ f = t \circ g$. Then there is a unique factorization over a map of sets $c : Q \to T$. One checks that this map is a morphism. The coarse structure of Q is generated by the entourages $q(U)$ for coarse entouragesq of Y. Since $c(q(U)) = t(U)$ we conclude that t is controlled. Let now B be bounded in T. Then $t(U)[B]$ is also bounded for every coarse entourage U of Y. This implies that $t^{-1}(t(U)[B])$ is bounded. Now

$$q^{-1}(q(U)[c^{-1}(B)]) \subseteq q^{-1}c^{-1}(c(q(U))[B]) = t^{-1}(t(U)[B])$$

implies that $q^{-1}(q(U)[c^{-1}(B)])$ is bounded. We conclude that $c^{-1}(B)$ is bounded in Q.

□

Example 3.26. Consider the two projections $\mathtt{pr}_0, \mathtt{pr}_1 (X \times X)_{min,max} \to X_{min,min}$. If X is infinite, then the coequalizer of this diagram is the unbounded point. □

Note that **BornCoarse** is a full subcategory of $\widetilde{\mathbf{BornCoarse}}$. It therefore inherits all limits and colimits taken in $\widetilde{\mathbf{BornCoarse}}$ of diagrams in **BornCoarse** which are represented by objects of **BornCoarse**.

Proposition 3.27. *The category* **BornCoarse** *has all non-empty limits.*

Proof. If $X_- : \mathbf{I} \to \mathbf{BornCoarse}$ is a diagram such that $\mathbf{I}$ is non-empty, then $\mathtt{lim}_{\mathbf{I}} X_i$ consists of bounded points. Indeed let x be such a point. Then $p_i(x)$ is bounded for every i in I. Hence x is a point in the bounded subset $\bigcap_{i \in I} p_i^{-1}(\{x\})$. □

Proposition 3.28. *The category* **BornCoarse** *has all coproducts.*

Proof. Exercise. □

Example 3.26 shows that **BornCoarse** does not have all colimits.

Example 3.29. The square

$$\begin{array}{ccc} \{*\} & \longrightarrow & \mathbb{N} \\ \downarrow & & \downarrow \\ -\mathbb{N} & \longrightarrow & \mathbb{Z} \end{array}$$

is a push-out in **BornCoarse**. □

Example 3.30. We consider subsets of $\mathbb{N}$ given by

$$Y := \bigcup_{n\in\mathbb{N}} [(2n)^2, (2n+1)^2] \ , \quad X := \bigcup_{n\in\mathbb{N}} [(2n+1)^2, (2n+2)^2] \ .$$

The square

$$\begin{array}{ccc} Y\cap Z & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & \mathbb{N} \end{array}$$

(all subspaces have the induced structure) is not a push-out. The coarse structure of $\mathbb{N}$ is generated by a single entourage U_1 and $|\pi_0^{coarse}(\mathbb{N})| = 1$. But the coarse structure of the push-out is not generated by a single entourage. Indeed, for any entourage U of the push-out structure we have $|\pi_0^{coarse}(\mathbb{N}_U)| = \infty$. □

4 Coarse homology theories

Let $E : \mathbf{BornCoarse} \to \mathbf{M}$ be a functor.

Definition 4.1. *E is coarsely invariant if E sends coarse equivalences to equivalences (isomorphisms).*

Lemma 4.2. *The following are equivalent.*

1. *E is coarsely invariant.*

2. *E sends pairs of close map to pairs of equivalent (equal) maps.*

3. *$E(\{0,1\}_{max,max} \otimes X) \to X$ is an equivalence (isomorphism) for all X in $\mathbf{C}$.*

Proof.

$1 \Rightarrow 2$

We observe that $p : \{0,1\}_{max,max} \otimes X \to X$ is coarse equivalence. Inverses are the inclusion $i_0, i_1 : X \to \{0,1\}_{max,max} \otimes X$ given by $i_0(x) := (0,x)$ and $i_1(x) := (1,x)$. Indeed,

$p \circ i_0 = \mathtt{id}_X$ and $i_0 \circ p$ is close to $\mathtt{id}$ since $(i_0 \circ p, \mathtt{id})(x,x) \in \{0,1\} \times \{0,1\} \times \mathtt{diag}(X)$ for all x in X, and $\{0,1\} \times \{0,1\} \times \mathtt{diag}(X)$ is a coarse entourage of $\{0,1\}_{max,max} \otimes X$. A similar argument applies to i_1.

Now assume that $f, g : X \to Y$ are morphisms which are close to each other. Then we define

$$h : \{0,1\} \otimes X \to Y \ , \quad h(i,x) := \begin{cases} f(x) & i = 0 \\ g(x) & i = 1 \end{cases} \ .$$

This is a morphism. If B is bounded in Y, then $h^{-1}(B) \subseteq \{0,1\} \times (f^{-1}(B) \cup g^{-1}(B))$. This shows that h is proper.

Assume that $f \sim_V g$ for some entourage V of Y. Let U be an entourage of X and set $W := \{0,1\} \times \{0,1\} \times U$. Then $h(W) \subseteq V \cup f(U) \cup g(U)$.

Since E is coarsely invariant we have $E(i_0) = E(i_1)$ since both are inverse to $E(p)$. We have $f = h \circ i_0$, $g = h \circ i_1$. By functoriality, $E(f) = E(h)E(i_0) = E(h)E(i_1) = E(g)$.

$2 \Rightarrow 1$

Let $f : X \to Y$ be a coarse equivalence. Then there exists a morphism $g : Y \to X$ such that $f \circ g \sim \mathtt{id}_Y$ and $g \circ f \sim \mathtt{id}_X$. We have $E(f) \circ E(g) = E(\mathtt{id}_Y) = \mathtt{id}_{E(Y)}$ and $E(g) \circ E(f) = E(\mathtt{id}_X) = \mathtt{id}_{E(X)}$.

$1 \Rightarrow 3$

We have already seen that p is a coarse equivalence. Hence $E(p) : E(\{0,1\} \otimes X) \to X$ is an equivalence.

$3 \Rightarrow 2$

Let $f, g : X \to Y$ be two morphisms which are close to each other. Then we form $h : \{0,1\}_{max,max} \otimes X \to Y$ as above. Since $E(p)$ is an equivalence and $p \circ i_0 = p \circ i_1 = \mathtt{id}_X$ we conclude that $E(i_0) = E(i_1)$ since both are right inverse to $E(p)$. Then we calculate, using functoriality, that $E(f) = E(h)E(i_0) = E(h)E(i_1) = E(g)$

□

Example 4.3. The functor $X \mapsto \pi_0^{coarse}(X)$ is coarsely invariant. Indeed

$$\pi_0^{coarse}(\{0,1\}_{max,max} \otimes X) \to \pi_0^{coarse}(X)$$

is isomorphism. In order to see this note that $[i,x] \simeq [j,y]$ in $\pi_0^{coarse}(\{0,1\}_{max,max} \otimes X)$ if and only if $[x] = [y]$ in $\pi_0^{coarse}(X)$. □ □

Example 4.4. For every Y in **BornCoarse** the functor

$$X \mapsto \mathtt{Hom}_{\overline{\mathbf{Coarse}}}(Y, X)$$

is coarsely invariant. $\square$

Definition 4.5. *A bornological coarse space is flasque if it admits an endomorphisms $f : X \to X$ satisfying:*

1. *f is close to $\mathtt{id}_X$.*
2. *For every U in $\mathcal{C}_X$ we have $\bigcup_{n\in\mathbb{N}} f^n(U) \in \mathcal{C}_X$.*
3. *For every B in $\mathcal{B}_X$ there exists n in $\mathbb{N}$ such that $f^n(X) \cap B = \emptyset$.*

We say that f witnesses flasqueness of X.

Example 4.6. For X in **BornCoarse** the space $\mathbb{N}\otimes X$ is flasque with $f : \mathbb{N}\otimes X \to X$given by $f(n, x) := (n + 1, x)$.

We check the axioms.

1. We have $(\mathtt{id}, f)(\mathtt{diag}(\mathbb{N} \otimes X) \subseteq U_1 \times \mathtt{diag}(X)$.
2. We have for an entourage $U_r \times V$ of $\mathbb{N} \otimes X$ that $\bigcup_{n\in\mathbb{N}} f^n(U_r \times V) \subseteq U_r \times V$.
3. Finally, if B is bounded in $\mathbb{N} \times X$, then $B \subseteq [0, n] \otimes X$ for some n in $\mathbb{N}$. Since $f^{n+1}(\mathbb{N} \times X) \subseteq [n + 1, \infty) \times X$ we conclude that $f^{n+1}(\mathbb{N} \times X) \cap B = \emptyset$.

Note that also $[0, \infty) \otimes X$ is flasque.

On the other hand $\mathbb{Z} \times X$ is not flasque. The map $(n, x) \mapsto (n + 1, x)$ does not work since the third axiom is not fullfilled. We will use coarse homology theories to see that there is no other map implementing flasqueness.

$\square$

Definition 4.7. *X is flasque in the generalized sense if $X \to \mathbb{N} \otimes X$, $x \mapsto (0, x)$ has a retract $r : \mathbb{N} \otimes X \to X$.*

Lemma 4.8. *If X is flasque, then it is flasque in the generalized sense.*

Proof. Let $f : X \to X$ witnesses flasqueness of X. We define $r : \mathbb{N} \otimes X \to X$ by $r(n, x) := f^n(x)$. Indeed, if U is an entourage of X, then $r(\mathtt{diag}(\mathbb{N}) \times U) \subseteq \bigcup_{n\in\mathbb{N}} f^n(U)$ is coarse in X. Furthermore, if $\mathtt{id}_X \sim_V f$, then $r(U_n \times \mathtt{diag}(X)) \subseteq V^n$. If B is bounded in X, then there exists n in $\mathbb{N}$ such that $f^n(X) \cap B = \emptyset$. Then $r^{-1}(B) \subseteq \bigcup_{i=0}^{n-1} (f^i)^{-1}(B)$ is bounded in X.

We have $r \circ i_0 = \mathtt{id}_X$. □

A pointed category is a category in which initial and final objects coincide. We write 0 for such objects. A morphism in a pointed category is a zero morphism if it admits a factorization over a zero object. The composition of a zero morphism with any morphism is again a zero morphism. Between any two objects there exists a unique zero morphism.

Example 4.9. The category of pointed sets $\mathbf{Set}_*$ and base-point preserving maps is pointed with zero object $*$.

Ab is pointed by 0. □

Let $E : \mathbf{C} \to \mathbf{M}$ be a functor to a pointed category.

Definition 4.10. *E vanishes in flasques if for every flasque X the canonical map $0 \to E(X)$ is an equivalence (isomorphism).*

Lemma 4.11. *The following assertions are equivalent.*

1. *E vanishes on generalized flasques.*
2. *E vanishes in flasques.*
3. *$0 \to E(\mathbb{N} \otimes X)$ is an equivalence for every X in* **BornCoarse**.

Proof.

$1 \Rightarrow 2$

This is clear since clear since flasques are generalized flasques by Lemma 4.8.

$2 \Rightarrow 3$

This is cleat since since $\mathbb{N} \otimes X$ is flasque by Example 4.6.

$3 \Rightarrow 1$

By assumption we have a retract $r : \mathbb{N} \otimes X \to X$ of $i_0 : X \to \mathbb{N} \otimes X$. Then $E(\mathtt{id}_X) = E(r) \circ E(i_0)$. $E(X)$ is retract of 0 and hence 0 since $\mathbb{N}$

□

Recall that for X in **BornCoarse** and U in $\mathcal{C}_X$ we write X_U for the bornological space X equipped with the coarse structure $\mathcal{C}_U := \mathcal{C}\langle\{U\}\rangle$. Since $\mathcal{C}_U \subseteq \mathcal{C}_X$ this structure is compatible with the bornology.

If U' is in $\mathcal{C}_X$ and $U \subseteq U'$, then the identity of X induces a map $X_U \to X_{U'}$. We have a map $X_U \to X$.

Definition 4.12. *We say that E is u-continuous if the canonical map $\mathtt{colim}_{U\in\mathcal{C}_X} E(X_U) \to E(X)$ is an equivalence.*

Example 4.13. The functor

$$P : \mathbf{C} \to \mathbf{sSet}\ , \quad X \mapsto \mathop{\mathtt{colim}}_{U\in\mathcal{C}_X} P_U(X)$$

is u-continuous. The composition of P with any filtered colimit-preserving functor is again u-continuous, e.g. $\mathbb{Z}[P] : \mathbf{BornCoarse} \to \mathbf{sAb}$. □

Example 4.14. The functor $X \mapsto \pi_0^{coarse}(X)$ is u-continuous. It is clear that the canonical maps $\pi_0^{coarse}(X_U) \to \pi_0^{coarse}(X_{U'}) \to \pi_0^{coarse}(X)$ are surjective for every U, U' in $\mathcal{C}_X$ with $U \subseteq U'$. This implies that $\mathtt{colim}_{U\in\mathcal{C}_X} \pi_0^{coarse}(X_U) \to \pi_0^{coarse}(X)$ is surjective. In order to show injectivity let x, x' be in X and assume that $[x]_X = [x']_X$ in $\pi_0^{coarse}(X)$. Then $x \sim x'$ for some U in $\mathcal{C}_X$. But then $[x]_{X_U} = [x']_{X_U}$ in $\pi_0^{coarse}(X_U)$. □

Example 4.15. Let Y be in **BornCoarse** such that $\mathcal{C}_Y = \mathcal{C}_V$ for some entourage V of Y. Then the corepresented functor

$$\mathtt{Hom}_{\mathbf{BornCoarse}}(Y, -) : \mathbf{BornCoarse} \to \mathbf{Set}$$

is u-continuous. Indeed, $\mathtt{Hom}_{\mathbf{BornCoarse}}(Y, X_U) \to \mathtt{Hom}_{\mathbf{BornCoarse}}(Y, X)$ is obviously the inclusion of a subset for every U in $\mathcal{C}_Y$. If f is in $\mathtt{Hom}_{\mathbf{BornCoarse}}(Y, X)$, then we have $f \in \mathtt{Hom}_{\mathbf{BornCoarse}}(Y, X_{f(V)})$. This shows that

$$\mathop{\mathtt{colim}}_{U\in\mathcal{C}_X} \mathtt{Hom}_{\mathbf{BornCoarse}}(Y, X_U) = \mathtt{Hom}_{\mathbf{BornCoarse}}(Y, X)\ .$$

Without the condition of the coarse structure of Y this assertion is wrong in general. For example, let $X := \{n^2 \mid n \in \mathbb{N}\}$ with the structures induced from the inclusion into $\mathbb{N}$. Then $\mathtt{id}_X \notin \mathtt{colim}_{U\in\mathcal{C}_X} \mathtt{Hom}_{\mathbf{BornCoarse}}(X, X_U)$ since otherwise $X = X_U$ for some entourage U of X. But one can show that the coarse structure on X is not generated by any finite set of entourages. □

Let X be in **BornCoarse**.

Definition 4.16. *A big family in X is a filtered subposet $\mathcal{Y}$ of $\mathcal{P}_X$ such that for every member Y in $\mathcal{Y}$ and U in $\mathcal{C}_X$ there exists a member Y' in $\mathcal{Y}$ such that we have $U[Y] \subseteq Y'$.*

We say that $\mathcal{Y}$ is complete if Y in $\mathcal{Y}$ and $Y' \subseteq Y$ imply that $Y' \in \mathcal{Y}$. We can form the completion $\bar{\mathcal{Y}}$ by adding all subsets of members of $\mathcal{Y}$.

Example 4.17. The bornology of X is a complete big family. □

Example 4.18. Let Y be a subset of X in **BornCoarse**. Then we can form the big family $\{Y\} := \{U[Y] \mid U \in \mathcal{C}_X\}$ generated by Y. □

For a map $f : X \to X'$ we write $f(\mathcal{Y}) = \{f(Y) \mid Y \in \mathcal{Y}\}$. In general this is not a big family.

We consider pairs $(X, \mathcal{Y})$ of X in **BornCoarse** and a big family $\mathcal{Y}$ on X.

Definition 4.19. *A morphism $f : (X, \mathcal{Y}) \to (X', \mathcal{Y}')$ is a morphism $f : X \to X'$ such that $f(\mathcal{Y}) \subseteq \bar{\mathcal{Y}}'$.*

We get the category $\mathbf{BC}^{\mathrm{pair}}$ of pairs and morphisms.

We have a functor

$$\mathbf{BornCoarse} \to \mathbf{BC}^{\mathrm{pair}} \ , \quad X \mapsto (X, \emptyset) \ .$$

If $E : \mathbf{BC}^{\mathrm{pair}} \to \mathbf{M}$ is a functor, then we can restrict E to a functor

$$uE : \mathbf{BornCoarse} \to \mathbf{M} \ , \quad X \mapsto E(X, \emptyset) \ ,$$

called the underlying functor. We will use the notation uE in order to denote the functor, but if we insert an argument, then we will omit $u$since it is clear from that fact that the argument has one entry that we mean the underlying functor. So we write $E(X)$ instead of $uE(X)$.

We have a functor

$$\mathbf{BC}^{\mathrm{pair}} \to \mathbf{BornCoarse} \ , \quad (X, \mathcal{Y}) \mapsto X \ .$$

Any functor $E : \mathbf{BornCoarse} \to \mathbf{M}$ extends to a functor

$$E : \mathbf{BC}^{\mathrm{pair}} \to \mathbf{M} \ , \quad E(X, \mathcal{Y}) := E(X) \ .$$

Extending and then restricting reproduces the initial functor.

If $E : \mathbf{BC}^{\mathrm{pair}} \to \mathbf{M}$ is a functor to a target which admits filtered colimits, then we define a functor

$$\partial E : \mathbf{BC}^{\mathrm{pair}} \to \mathbf{M} \ , \quad (X, \mathcal{Y}) := \operatorname*{colim}_{Y \in \bar{\mathcal{Y}}} E(Y) \ .$$

On morphisms $f : (X, \mathcal{Y}) \to (X', \mathcal{Y}')$ the inclusion $f_{|Y} : E(Y) \to E(f(Y))$ induces a compatible family of morphisms

$$E(Y) \to E(f(Y)) \to \operatorname*{colim}_{Y' \in \bar{\mathcal{Y}}'} uE(Y')$$

for all Y in $\mathcal{Y}$. We finally get a morphism

$$\partial E(\mathcal{Y}) = \operatorname{colim}_{Y \in \bar{\mathcal{Y}}} E(Y) \to \operatorname{colim}_{\bar{\mathcal{Y}}'} E(Y') = \partial E(\mathcal{Y}') \ .$$

This finishes the construction of ∂E up to straightforward verifications.

The inclusion morphisms $Y \to X$ for all Y in $\mathcal{Y}$ induce compatible morphisms $uE(Y) \to uE(X)$ and finally a morphism

$$\partial E(\mathcal{Y}) \to E(X) \ .$$

These morphisms for all $(X, \mathcal{Y})$ in $\mathbf{BC}^{\mathrm{pair}}$ fit into a natural transformation of functors

$$\partial E \Rightarrow uE : \mathbf{BC}^{\mathrm{pair}} \to \mathbf{M} \ .$$

Let $(X, \mathcal{Y})$ be in $\mathbf{BC}^{\mathrm{pair}}$ and Z be a subset of X.

Definition 4.20. *We say that $(Z, \mathcal{Y})$ is a complementary pair on X if $X \setminus Z \in \bar{\mathcal{Y}}$.*

Note that $Z \cap \mathcal{Y} = \{Z \cap Y \mid Y \in \mathcal{Y}\}$ is a big family on Z. This follows from the inclusion $U_Z[Y \cap Z] \subseteq U[Y] \cap Z$ for all U in $\mathcal{C}_X$, where $U_Z = U \cap (Z \times Z)$. We get an canonical morphism $(Z, Z \cap \mathcal{Y}) \to (X, \mathcal{Y})$.

We consider functor $F : \mathbf{BC}^{\mathrm{pair}} \to \mathbf{M}$.

Definition 4.21. *The functor F is called excisive if for every complementary pair $(Z, \mathcal{Y})$ on X in* **BornCoarse** *the canonical functor induces an equivalence (isomorphism)*

$$E(Z, Z \cap \mathcal{Y}) \to E(X, \mathcal{Y}) \ .$$

Example 4.22. Let R be a ring. Then we have a functor **BornCoarse** $\to \mathrm{Mod}(R)^{\mathrm{op}}$ given by

$$X \mapsto C_{\mathrm{lf}}(X, R) \ , \quad (f : X \to X') \mapsto (f^* : C_{\mathrm{lf}}(X', R) \to C_{\mathrm{lf}}(X, R)) \ .$$

Here f^* preserves local finiteness since f preserves locally finite subsets by properness (see Example 1.21). We have a functor

$$(X, \mathcal{Y}) \mapsto C_{\mathrm{lf}}(X, \mathcal{Y}, R) := \operatorname{colim}_{Y \in \mathcal{Y}} C_{\mathrm{lf}}(X, R)/C_{\mathrm{lf}}(Y, R) \ .$$

We claim that this functor is excisive.

The map $C_{\mathrm{lf}}(X, R) \to C_{\mathrm{lf}}(Z, R)$ is surjective since we can extend any ϕ in $C_{\mathrm{lf}}(Z, R)$ by zero in order to obtain a preimage in $C_{\mathrm{lf}}(X, R)$. This implies that $C_{\mathrm{lf}}(X, \mathcal{Y}, R) \to C_{\mathrm{lf}}(Z, Z \cap \mathcal{Y}, R)$ is surjective. We now show injectivity. Assume that $[\phi]$ is in $C_{\mathrm{lf}}(X, \mathcal{Y}, R)$ such that $[\phi_{|Z}] = 0$. Then $\mathtt{supp}(\phi_{|Z}) \in \overline{Z \cap \mathcal{Y}}$. Hence $\mathtt{supp}(\phi) \subseteq (X \setminus Z) \cup \mathtt{supp}(\phi_{|Z})$ belongs to $\overline{\mathcal{Y}}$. Hence $[\phi] = 0$. □

We let $\mathbf{Ab}^{\mathbb{Z}\mathrm{gr}}$ denote the category of $\mathbb{Z}$-graded abelian groups. Let G be in $\mathbf{Ab}^{\mathbb{Z}\mathrm{gr}}$. Then we write G_n for the degree-n-component. Let $[k] : \mathbf{Ab}^{\mathbb{Z}\mathrm{gr}} \to \mathbf{Ab}^{\mathbb{Z}\mathrm{gr}}$ be the shift functor given by $(G[k])_n = G_{n+k}$. given by

Definition 4.23. *A coarse δ-functor is a pair (E, δ) of a functor $E : \mathbf{BC}^{\mathrm{pair}} \to \mathbf{Ab}^{\mathbb{Z}\mathrm{gr}}$ and a natural transformation $\delta : uE \to \partial E[-1]$ such that for every pair $(X, \mathcal{Y})$ the sequence*

$$\partial E(\mathcal{Y}) \to E(X) \to E(X, \mathcal{Y}) \xrightarrow{\delta} \partial E(\mathcal{Y})[-1]$$

is exact.

Definition 4.24. *A coarse δ-functor (E, δ) is a linear coarse homology theory if*

1. *E is excisive.*
2. *The underlying functor $E : \mathbf{BornCoarse} \to \mathbf{M}$ is coarsely invariant.*
3. *The underlying functor $E : \mathbf{BornCoarse} \to \mathbf{M}$ vanishes on flasques.*
4. *The underlying functor $E : \mathbf{BornCoarse} \to \mathbf{M}$ is u-continuous.*

Example 4.25. We calculate that $E_n(\mathbb{Z}^k) \cong E_{n+k}(*)$ by induction.

The case $k = 0$ is clear.

Assume for $k - 1$. Then consider pair $(Z, \mathcal{Y})$ on $\mathbb{Z}^k$ where

$$Z := \{(n_1, \ldots, n_k) \in \mathbb{Z}^k \mid n_1 \geq 0\}$$

and

$$\mathcal{Y} := \{Y_r := \{(n_1, \ldots, n_k) \in \mathbb{Z}^k \mid n_1 \leq r\} \mid r \in \mathbb{N}\} \ .$$

We have $Z \cong \mathbb{N} \otimes \mathbb{Z}^{k-1}$ and also $Y_r \cong \mathbb{N} \otimes \mathbb{Z}^{k-1}$. These spaces are flasque. Furthermore $Z \cap Y_r = [0, r] \times \mathbb{Z}^{k-1}$ is coarsely equivalent to $\mathbb{Z}^{k-1}$.

The long exact sequence for $(\mathbb{Z}^k, \mathcal{Y})$ and $\partial E(\mathcal{Y}) \simeq 0$ gives the first isomorphism in

$$E(\mathbb{Z}^k) \overset{\cong}{\to} E(\mathbb{Z}^k, \mathcal{Y}) \overset{excision}{\to} E(Z, Z \cap \mathcal{Y}) \overset{\cong, \delta}{\to} \partial E(Z \cap \mathcal{Y})[-1] = E(\mathbb{Z}^{k-1})[-1] \ .$$

Hence

$$E_n(\mathbb{Z}^k) \cong E_{n-1}(\mathbb{Z}^{k-1}) \ .$$

□