Products and push-forwards in parametrised cohomology theories

Robert Waldmüller

November 8, 2006

\(^1\)Partially supported by the DFG
Introduction

It is well known that generalised cohomology theories defined on topological spaces correspond to spectra. Under this correspondence, the cohomology groups $E^*(X)$ of a space $X$ arise as maps in the stable homotopy category from the suspension spectrum $\Sigma^\infty X$ of $X$ to $E$, the spectrum representing the cohomology theory. If one assumes that $X$ is a CW-complex and $E$ is an $\Omega$-spectrum (that is, the maps $E(n) \to \Omega_mE(n+m)$ are weak homotopy equivalences), elements of the $n$th group can be realised as maps from $X$ to $E(n)$. Maps from $X$ to $E(n)$ are the same as sections in the trivial bundle over $X$ with fibre $E(n)$. The idea to define twisted cohomology groups is to replace the trivial $E(n)$-bundle over $X$ by a a non-trivial $E(n)$-bundle $M$ over $X$ and define the twisted groups as homotopy classes of sections of $M$. Assuming that $\mathcal{E}(n) \to B$ is the universal $E(n)$-bundle, there is a map $q : X \to B$ such that $M \cong q^* \mathcal{E}(n)$. Sections of $M$ are nothing but maps $u : X \to \mathcal{E}(n)$ such that

$$
\begin{array}{c}
X \\
\downarrow q \\
\mathcal{E}(n) \\
\downarrow \\
B
\end{array}
$$

commutes, i.e. maps from $X$ to $M$ in the category of spaces over $B$. This leads to the definition of a twisted or parametrised cohomology theory as a cohomology theory on the category of spaces over a fixed base space $B$. In this picture of twisted cohomology theories, the set of twists on a given space $X$ corresponds to the set of maps $X \to B$; we will see later that a homotopy of twists defines an isomorphism of the corresponding twisted cohomology groups.

Since a cohomology theory should have suspension isomorphisms relating the groups in different degrees, the spaces $\mathcal{E}(m)$ should be interrelated. More precisely, there should be maps $\Sigma^m \mathcal{E}(n) \to \mathcal{E}(n+m)$, where $\Sigma$ denotes fibre-wise suspension, i.e., $\mathcal{E}$ should be what is called a parametrised spectrum or ex-spectrum. Ex-spectra were first studied in [13, 14]. The category of ex-spectra $\mathcal{S}_B$ over a fixed base $B$ is similar to the category of non-parametrised spectra $\mathcal{S}$. In particular, one can define the stable parametrised homotopy category $Ho\mathcal{S}_B$ in which the suspension functor is invertible. Moreover, just
as generalised cohomology theories on spaces correspond to spectra, cohomology theories on the category of spaces over a fixed base correspond to ex-spectra [32].

In [31], the authors discuss $HoS_B$ in the language of model categories. Using the concept of orthogonal spectra, they are able to define a symmetric monoidal structure which exists already before passage to the stable homotopy category.

An important point in the application of twisted cohomology theories is the existence of a product. In the examples, there is frequently the notion of the sum $\alpha + \beta$ of twists $\alpha, \beta$. The products in the examples of multiplicative twisted cohomology theories and the sum of twists are related in that the product adds the twists, i.e. if $u$ is an $\alpha$-twisted class and $v$ is a $\beta$-twisted class, $u \cup v$ is usually $\alpha + \beta$ twisted. The fact that multiplicative non-parametrised cohomology theories correspond to ring spectra suggests that multiplicative parametrised cohomology theories should correspond to parametrised ring spectra. However, the symmetric monoidal structure on $HoS_B$ constructed in [13, 14, 31] is not suitable for this purpose, because the product would not add twists. In addition, if one thinks of twists as maps to some space $B$, it is for a general $B$ not clear what the sum of twists should be. If $B$ is an $H$-space, one can use the product on $B$ to define the sum $f + g$ of maps $f, g : X \to B$. To obtain our symmetric monoidal structure on $HoS_B$, an $H$-space structure on $B$ is not enough. We need better homotopical control of the product on $B$. In particular, the product should be not only homotopy associative, but the associativity homotopy should be unique up to higher homotopies. This is exactly what is captured by the notion of an action of an $E_\infty$-operad on $B$. Such an action defines products on $B$ in which all commutativity and associativity diagrams commute up to homotopy, and these homotopies are all unique up to higher homotopies. We define symmetric monoidal structures $p$ on $HoS_B$ for base spaces $B$ with an action of an $E_\infty$-operad. In fact, we only need up to four-fold associativities. Therefore, a truncated action would be sufficient. We use the symmetric monoidal structures $p$ to define multiplicative parametrised cohomology theories on the category of spaces over such a $B$.

In the past few years, a twisted cohomology theory of particular interest was $H^3$-twisted $K$-theory [2, 3, 17–19, 34, 35]. There are various pictures of twisted $K$-theory, corresponding to different pictures of $H^3$. One way to interpret elements of $H^3(X, \mathbb{Z})$ is as principal $PU(H)$-bundles over $X$, where $H$ denotes some fixed separable Hilbert space. In [2, 3], the authors associate to a $PU(H)$-bundle a $Fred(H)$-bundle, using the conjugation action of $PU(H)$ on $Fred(H)$, and define twisted $K$-theory in terms of homotopy classes of sections of the $Fred(H)$-bundle. Another interpretation
of $H^3$ (which readily carries over to the equivariant setting) is used in [34,35]. An $H^3$-class of a topological stack corresponds to a central $S^1$-extension of a groupoid representing the stack, and twisted $K$-theory is defined as the $K$-theory of the reduced $C^*$-algebra of this extension.

Recall that a a class in $H^3(X,\mathbb{Z})$ can be realised as a continuous map $X \to K(\mathbb{Z}, 3)$. Therefore, the category of ”spaces with $H^3$-twists” may be identified with the category of spaces over $K(\mathbb{Z}, 3)$. In consequence, to define $H^3$ twisted $K$-theory as a multiplicative parametrised cohomology theory amounts to constructing a parametrised ring spectrum over $BPU(H) \cong K(\mathbb{Z}, 3)$ representing twisted $K$-theory, and we do precisely that. The properties of twisted $K$-theory such as homotopy invariance, Mayer-Vietoris sequences, excision and so forth follow from the general statement that it is a multiplicative cohomology theory on spaces over $BPU(H)$. In particular, the Thom isomorphisms and push-forwards in twisted $K$-theory which were geometrically constructed in [12] follow from the corresponding construction for multiplicative twisted cohomology theories.

The text is organised as follows. In the first chapter, we summarise the construction of $HoS_B$ and some of its properties from [31]. We define the symmetric monoidal structure $p$ in Proposition 32. If a group $G$ acts on a spectrum $F$, an operad acts on both $G$ and $F$ and the actions are compatible, we construct a parametrised ring spectrum with fibre $F$ over $BG$ in Lemma 35.

In the second chapter, we define twisted cohomology theories and products in twisted cohomology theories. We prove a Thom isomorphism theorem for generalised twisted cohomology theories and define push-forward homomorphisms for proper oriented maps. In addition, we prove functoriality of push-forwards and a projection formula.

In the last chapter, we construct parametrised ring spectra representing twisted $K$-theory and twisted $Spin^c$-cobordism.

Throughout the text, we use Quillen's notion of a model category as defined in [36]. A concise introduction to model categories is [15], a detailed account with an eye to examples is [20]. All model categories in our text are cofibrantly generated and therefore admit cofibrant and fibrant replacement functors which we denote by $Q$ and $R$. For a topologically enriched category $\mathcal{C}$ and $X,Y \in Ob(\mathcal{C})$, we denote both the morphism set and the morphism space by $\mathcal{C}(X,Y)$. 
Chapter 1

Basic definitions and constructions

1.1 Ex-spaces

**Definition 1** An ex-space over a topological space $B$ is a space $X$ with continuous maps $q : X \rightarrow B$ and $s : B \rightarrow X$ such that $q \circ s = id_B$.

The point-set topology to set up convenient categories of ex-spaces is quite tedious. We refrain ourselves from a lengthy discussion, but rather refer the interested reader to [5–9,22,31] and give a short account of the results we need. Recall that a topological space $X$ is called a $k$-space if a subset $A \subset X$ is closed if $f^{-1}(A)$ is closed for all compact Hausdorff spaces $K$ and all continuous maps $f : K \rightarrow X$. $X$ is called weak Hausdorff if $f(K)$ is closed for all compact Hausdorff spaces $K$ and all continuous maps $f : K \rightarrow X$. Let $\mathcal{U}$ denote the category of weak Hausdorff $k$-spaces and $\mathcal{K}$ the category of $k$-spaces.

**Definition 2** For $B \in \mathcal{U}$, define $\mathcal{K}_B$ to be the category of ex-spaces $X$ over $B$ with $X \in \text{Ob}(\mathcal{K})$. Define $\mathcal{K}/B$ to be the category of $k$-spaces over $B$. The functor $+ : \mathcal{K}/B \rightarrow \mathcal{K}_B$ is given by adjoining a disjoint basepoint, i.e. a disjoint copy of $B$ to spaces $X$ over $B$.

We topologize the morphism sets with the subspace topology, where we use the compact-open topology for $\mathcal{K}$. To improve legibility, we will omit the maps $q : X \rightarrow B$ and $s : B \rightarrow X$ from our notation most of the time and only include them when necessary. For $K \in \mathcal{K}$ and $(X, q) \in \mathcal{K}/B$ we define

$$\text{Map}_B(K, X) := \{f : K \rightarrow X | \exists b \in B \text{ s.t. } f(K) \subset q^{-1}(b)\} \subset \text{Map}(K, X),$$

again using the compact-open topology on $\text{Map}(K, X)$ and the obvious map to $B$. Together with $X \times_B K := (X \times K, q \circ pr_X)$ this defines a tensor
and cotensor structur of the topological category $\mathcal{K}/B$ over $\mathcal{K}$, i.e. there are natural homeomorphisms

$$\mathcal{K}/B(X \times_B K, Y) \cong \mathcal{K}(K, \mathcal{K}/B(X, Y)) \cong \mathcal{K}/B(X, \text{Map}_B(K, Y))$$

Similarly, the pointed category $\mathcal{K}_B$ is tensored and cotensored as topological category over $\mathcal{K}_{+}$.

**Definition 3** For ex-spaces $X$ and $Y$ over $B$, define the smash product $X \land_B Y$ by the following pushout in $\mathcal{K}/B$.

$$\begin{array}{c}
X \lor_B Y \\
\downarrow \\
* \rightarrow
\end{array} \begin{array}{c}
X \times_B Y \\
\downarrow \\
X \land_B Y.
\end{array}$$

There are internal mapping spaces $\text{Map}_B(.,.)$ in $\mathcal{K}/B$ and $\text{F}_B(.,.)$ in $\mathcal{K}_B$, giving $\mathcal{K}/B$ the structure of a closed cartesian category and $\mathcal{K}_B$ the structure of a closed symmetric monoidal category [5–7].

A continuous map $f : A \rightarrow B$ defines functors

$$f_! : \mathcal{K}_A \leftarrow \mathcal{K}_B : f^*$$

by setting

$$\begin{array}{c}
\begin{array}{c}
A \rightarrow B \\
\downarrow \downarrow
\end{array} \\
\begin{array}{c}
f^*Y \rightarrow Y \\
\downarrow \downarrow
\end{array} \\
\begin{array}{c}
X \rightarrow f_*X \\
\downarrow \downarrow
\end{array} \\
\begin{array}{c}
A \rightarrow B \\
\downarrow \downarrow
f
\end{array}
\end{array}$$

for $(X, q, s) \in \mathcal{K}_A$, $(Y, q', s') \in \mathcal{K}_B$. The first diagram is a pushout and we define the missing map $r : f_*X \rightarrow B$ to be the pushout of $\text{id}_B : B \rightarrow B$ and $f \circ q : X \rightarrow B$; the second diagram is a pullback and $t' : A \rightarrow f^*Y$ is defined to be the pullback of $\text{id}_A : A \rightarrow A$ and $s' \circ f : A \rightarrow Y$. These functors are adjoint as functors of topologically enriched categories, i.e. they are continuous and the adjunction isomorphisms

$$\mathcal{K}_B(f_!X, Y) \cong \mathcal{K}_A(X, f^*Y)$$

are homeomorphisms [31, 2.1.2].

Furthermore, there are external products $\sqcap : \mathcal{K}/A \times \mathcal{K}/B \rightarrow \mathcal{K}/(A \times B)$ and $\sqcap : \mathcal{K}_A \times \mathcal{K}_B \rightarrow \mathcal{K}_{A \times B}$ and mapping spaces $\text{Map}$ and $\text{F}$ with natural isomorphisms

$$\mathcal{K}/(A \times B)(X \sqcap Y, Z) \cong \mathcal{K}/A(X, \text{Map}(Y, Z))$$

and

$$\mathcal{K}_{A \times B}(X \sqcap Y, Z) \cong \mathcal{K}_A(X, \text{F}(Y, Z)).$$
1.1 Ex-spaces

The base change functors can be used to derive the external products from the internal ones; for example, $X \otimes Y := \pi^*_AX \otimes_{A \times B} \pi^*_BY$. A pointed space $K \in \mathcal{K}_*$ is just an ex-space over a point. Using the projection $\pi : B \to \ast$, the trivial ex-space with fibre $K$ is

$$K_B := \pi^*K = K \times B \in \mathcal{K}_B.$$ 

Notice that there are natural isomorphisms

$$X \otimes K \cong X \otimes_B K \cong X \otimes K_B,$$

relating the tensor structure of $\mathcal{K}_B$ over $\mathcal{K}$, the external and the internal smash product; see [31] for a more thorough discussion of the functors $\otimes, \otimes, f^*, f_!, \otimes_B, Map_B, F_B, F$ and their relations.

**Proposition 4** [31, 2.2.1] For a map $f : A \to B$, the functor $f^* : \mathcal{K}_B \to \mathcal{K}_A$ is closed symmetric monoidal.

In [31], the authors develop model structures on $\mathcal{K}/B$ and $\mathcal{K}_B$. We will now give a short summary of their results, again omitting the proofs.

**Definition 5** $f \in \mathcal{K}/B(X, Y)$ is a fibrewise cofibration if it has the left lifting property with respect to the maps

$$\pi_0 : Map_B(I, Z) \to Z \forall Z \in \mathcal{K}/B,$$

where $\pi_0$ is given by evaluation at zero.

Denote the upper (lower) hemisphere of a $n$-disc by $S_{+}^{n-1}$ ($S_{-}^{n-1}$) with the inclusion $i_{\pm} : S_{\pm}^{n-1} \to D^n$. Similarly, we write

$$i : S^{n-1} \to D^n \quad \text{and} \quad j_+ : S^{n-1} \to S_+^{n}$$

for the inclusions.

**Definition 6**

$$I_B := \{ i : (S^{n-1}, d \circ i) \to (D^n, d) \mid i \text{ is a fibrewise cofibration} \}$$

$$J_B := \{ i_+ : (S_+^{n}, d \circ i_+) \to (D^{n+1}, d) \mid i : (S^n, d \circ i) \to (D^{n+1}, d) \text{ and } j_- : (S_-^{n-1}, d \circ i_- \circ j_-) \to (S^n, d \circ i_-) \text{ are fibrewise cofibrations} \}$$

**Theorem 7** [31, 6.2.5] $\mathcal{K}/B$ is a cofibrantly generated model category, with weak equivalences given by weak homotopy equivalences of total spaces and $I_B$ ($J_B$) as set of generating (acyclic) cofibrations.
Theorem 8 [31, 6.2.6] $\mathcal{K}_B$ is a cofibrantly generated model category, with weak equivalences given by weak homotopy equivalences of total spaces and $(I_B)_+ ((J_B)_+)$ as set of generating (acyclic) cofibrations.

Proposition 9 [31, 7.3.4, 7.3.5] For a continuous map $f : A \to B$, the adjunction $(f_!, f^*)$ is a Quillen adjunction between $\mathcal{K}_A$ and $\mathcal{K}_B$. It is a Quillen equivalence if $f$ is a weak homotopy equivalence.

The internal smash product $\wedge : \mathcal{K}_B \times \mathcal{K}_B \to \mathcal{K}_B$ is not a Quillen bifunctor. Nonetheless, the authors of [31] construct a "derived" internal smash product on the homotopy category. However, the external smash product $\wedge : \mathcal{K}_A \times \mathcal{K}_B \to \mathcal{K}_A \times \mathcal{K}_B$ is a Quillen bifunctor [31, 7.3] and thus can be derived to obtain $L\wedge : Ho\mathcal{K}_A \times Ho\mathcal{K}_B \to Ho\mathcal{K}_{A \times B}$.

For an ex-space $E \in \mathcal{K}_A$, there is a natural map $i_0! QE \cong QE \coprod_{A \times \{0\}} A \times I \to RE \times I \cong \pi_A^* RE$ given by the extension of $Q E \to R Q E \to RE$. This is a weak equivalence, hence it induces a natural isomorphism of total derived functors $Li_0! \cong R \pi_A^*$. Similarly, $Li_1! \cong R \pi_A^*$ and therefore we get a natural isomorphism $\rho : Li_0! \sim Li_1!$.

Proposition 10 A homotopy $h : A \times I \to B$ between $f, g : A \to B$ gives rise to a natural isomorphism $\phi : Lf_! \sim Lg_!$.

If two homotopies $h_0, h_1$ are homotopic (as homotopies) via $H$, the isomorphisms $\phi_0$ and $\phi_1$ coincide.

Proof: The natural isomorphisms $h_0 i_0! \cong f_!, h_1 i_1! \cong g_!$ between left Quillen functors yield isomorphisms between the derived functors [20]. We combine these and the natural isomorphism $\rho : Li_0! \sim Li_1!$ to obtain the desired isomorphism $Lf_! \cong Lh_! Li_0! \overset{Lh_!(\rho)}{\longrightarrow} Lh_! Li_1! \cong Lg_!$.

Let us now discuss the case in which there are two homotopic homotopies $h_0, h_1$ from $f$ to $g$. For the inclusions $j_0, j_1 : A \times I \to A \times I \times I$, and $E \in \mathcal{K}_A$, we have again natural weak equivalences in $\mathcal{K}_{A \times I \times I}$

$$j_0! Qi_0! Q E \overset{\alpha_0}{\sim} \pi_{A \times I}^* Ri_0! Q E \overset{\alpha_1}{\sim} j_1! Qi_0! Q E$$
Note that $\pi^*_{A \times I} \cong \tau S^0_I$, hence it preserves cofibrant objects. For $E \in \mathcal{K}_A$, commutativity of the diagram

\[
\begin{array}{c}
\pi^*_{A \times I} \times I \sim \cong I, \\
\pi^*_{A \times I} \times I \to \pi^*_{A \times I} \times I \to \pi^*_{A \times I} \times I \to \pi^*_{A \times I} \times I.
\end{array}
\]

in $Ho \mathcal{K}_B$ gives the result, so let us first show that the middle square commutes. A lift of the square to $\mathcal{K}_B$ is $H_!$ applied to the following diagram in $\mathcal{K}_{A \times I \times I}$.

The top and bottom horizontal maps stem from the natural maps

\[
i_0QE \to \pi^*_A RE, \quad i_1QE \to \pi^*_A RE.
\]

Since $i_0QE$, $i_1QE$ are cofibrant, these maps can be lifted to maps

\[
i_0QE \to Q\pi^*_ARE, \quad i_1QE \to Q\pi^*_ARE
\]

which yield the middle horizontal arrows. The above diagram commutes in $\mathcal{K}_{A \times I \times I}$ and all maps are weak equivalences. Since all objects in the diagram are cofibrant and $H_!$, being a left Quillen functor, preserves all weak equivalences between cofibrant objects, the middle square commutes in $Ho \mathcal{K}_B$.

To show commutativity of the left triangle, we choose again a lift to $\mathcal{K}_B$, namely

\[
H_!(\alpha_1)
\]

in $Ho \mathcal{K}_B$. The result follows, so let us first show that the middle square commutes.
with \( q \) being the natural map \( Q_{i_0}E \to i_0!QE \). Observe that we have factorisations

\[
\alpha_0 : j_0!Q_{i_0}E \to \pi^*_{A \times I} i_0!QE \to \pi^*_{A \times I} Ri_0!QE
\]

\[
\alpha_1 : j_1!Q_{i_0}QE \to \pi^*_{A \times I} i_0!QE \to \pi^*_{A \times I} Ri_0!QE.
\]

We use these to see that \( H(\alpha_0) \) and \( H(\alpha_1) \) factor through \( H!\pi^*_{A \times I} i_0!QE \) and we get a commutative diagram

\[
\begin{array}{ccc}
Q_{i_0} & \xrightarrow{h_0 !(q)} & H!\pi^*_{A \times I} i_0!QE \\
\downarrow \hspace{1cm} & & \downarrow H(\alpha_0) \\
\downarrow \hspace{1cm} & & \downarrow H(\alpha_1) \\
Q_{i_0} & \xrightarrow{h_1 !(q)} & H!\pi^*_{A \times I} i_0!QE
\end{array}
\]

Furthermore, there are maps

\[
\xi_s : j_s!i_0!QE \to \pi^*_{A \times I} i_0!QE \quad \forall \ s \in I
\]

given by extension of the identity \( i_0!QE \to i_0!QE \). The triangles

\[
\begin{array}{ccc}
Q_{i_0} & \xrightarrow{h_0 !(q)} & H!\pi^*_{A \times I} i_0!QE \\
\downarrow \hspace{1cm} & & \downarrow H(\xi_0) \\
\downarrow \hspace{1cm} & & \downarrow \hspace{1cm} \\
Q_{i_0} & \xrightarrow{h_1 !(q)} & H!\pi^*_{A \times I} i_0!QE
\end{array}
\]

and

\[
\begin{array}{ccc}
Q_{i_0} & \xrightarrow{h_0 !(q)} & H!\pi^*_{A \times I} i_0!QE \\
\downarrow \hspace{1cm} & & \downarrow H(\xi_1) \\
\downarrow \hspace{1cm} & & \downarrow \hspace{1cm} \\
Q_{i_0} & \xrightarrow{h_1 !(q)} & H!\pi^*_{A \times I} i_0!QE
\end{array}
\]

commute. Since \( H \circ j_s \circ i_0 = f \quad \forall \ s \in I, \ H(\xi_0) \) and \( H(\xi_1) \) are homotopic over \( B \). Thus, the left triangle commutes in \( HoK_B \).

The argument for the right triangle is similar. \( \square \)

In addition, the correspondence between homotopies and natural equivalences is functorial in the following sense.

**Proposition 11** Let \( h_0 \) be a homotopy between \( f_0, f_1 : A \to B \) and \( h_1 \) be one connecting \( f_1 \) and \( f_2 \), and denote by \( h \) the homotopy from \( f_0 \) to \( f_2 \) given by composing \( h_0 \) and \( h_1 \). Then the corresponding isomorphisms \( \phi_0 : Lf_0 ! \to Lf_1 !, \phi_1 : Lf_1 ! \to Lf_2 ! \) and \( \phi : Lf_0 ! \to Lf_2 ! \) satisfy \( \phi_1 \circ \phi_0 = \phi : Lf_0 ! \to Lf_2 ! \).
1.1 Ex-spaces

Proof: Consider the double interval $J := [0, 2]$ and the maps

$$\pi : A \times J \to A, \ j_0 : A \times I \to A \times J, \ j_1 : A \times I \to A \times J, \ i_{kl} := j_k \circ i_l$$

and $h$ as a map $A \times J \to B$. There are identities

$$f_{k+l} = h_k \circ i_l = h \circ i_{kl} \ \forall k, l \in \{0, 1\}.$$

Consider the diagram

in which all maps except $\alpha_i$ are either natural "extensions by zero" or of the form $QF \to F$, and $\alpha_i$ are obtained by lifting the respective maps to the cofibrant replacements. Since maps of the form $QF \to F$ are weak equivalences, the diagram commutes in $HoK_{A \times J}$. Similarly, let $\alpha_3$ be defined to be the lift in

and observe that

$$j_0 ! i_1 ! QE \to j_0 ! Q\pi_A^* RE \to \pi^* RE$$

commutes in $HoK_{A \times J}$. Putting everything together, applying $h_!$ and using the natural identifications

$$h_! \circ j_! \circ i_! \cong f_{k+l},$$
we obtain

\[
\begin{array}{c}
h_0 Q_{i_0!} \to h_0 Q_{\pi_*} \to h_1 Q_{i_1!} \\
h_{(\alpha_2)} \quad h_{(\alpha_2)} \quad h_{(\alpha_3)} \\
h_{i_0!} \quad h_{i_0!} \quad h_{i_1!} \\
f_0! \quad f_0! \quad f_1! \\
\end{array}
\]

One can repeat the construction with \( h_1 \) instead of \( h_0 \) and the second half of \( J \). This completes the previous diagram to

\[
\begin{array}{c}
h_1 Q_{i_0!} \to h_1 Q_{\pi_*} \to h_0 Q_{i_1!} \\
h_{(\alpha_2)} \quad h_{(\alpha_2)} \quad h_{(\alpha_3)} \\
h_{i_0!} \quad h_{i_1!} \quad h_{i_2!} \\
f_0! \quad f_1! \quad f_2! \\
\end{array}
\]

The natural transformation corresponding to \( h \) is obtained by the left and right vertical arrows and the top horizontal ones whereas the \( \phi_i \) correspond to the left and right 'hats'. □

## 1.2 Ex-spectra

**Definition 12** Let \( \mathcal{J} \) denote the category of finite dimensional real inner product spaces and linear isometric isomorphisms. A \( \mathcal{J} \)-space over \( B \in \mathcal{U} \) is a continuous functor \( E : \mathcal{J} \to \mathcal{K}_B \); \( \mathcal{J}\mathcal{K}_B \) is the category of \( \mathcal{J} \)-spaces over \( B \).

The external smash product of \( X, Y \in \mathcal{J}\mathcal{K}_B \) is given by

\[
X \wedge_B Y := \wedge \circ (X \times Y) : \mathcal{J} \times \mathcal{J} \to \mathcal{K}_B.
\]

Define the smash product \( X \wedge_B Y \) to be the topological left Kan extension of \( X \wedge_B Y \) along \( \oplus : \mathcal{J} \times \mathcal{J} \to \mathcal{J} \).
Theorem 13 \cite[11.1.3, 11.1.6]{31} \((JK_B \wedge B)\) is a closed symmetric monoidal category. It is tensored and cotensored over \(K_B\).

The tensor and cotensor structure are defined levelwise. Moreover, for a map \(f : A \to B\) of base spaces, levelwise application of the adjunction \((f_! : f^*)\) defines an adjunction

\(f_! : JK_A \xrightarrow{\simeq} JK_B : f^*\).

For a real inner product space \(V\), let \(S^V\) denote the one-point compactification. Define the sphere \(S_B\) over \(B\) to be the \(J\)-space sending \(V\) to \(S^V_B\). This is a commutative monoid in \(JK_B\), so we can define \(S_B\), the ex-spectra over \(B\), to be the \(S_B\)-modules. The smash product \(X \wedge Y\) of ex-spectra \(X, Y\) is the coequalizer (in \(JK_B\)) of

\[ X \wedge_B S_B \wedge_B Y \xrightarrow{\simeq} X \wedge_B Y. \]

Proposition 14 \cite[11.2.5]{31} \((S_B, \wedge)\) is a closed symmetric monoidal category with unit \(S_B\).

Again, levelwise application of the functors \(f_!, f^*\) gives an adjunction \(f_! : S_A \xrightarrow{\simeq} S_B : f^*\) and the functor \(f^* : S_B \to S_A\) is closed symmetric monoidal.

Definition 15 \cite{31} For base spaces \(A, B\), the external smash product is

\[ \wedge := \wedge \circ \left( \pi_A^\ast \times \pi_B^\ast \right) : S_A \times S_B \to S_{A \times B}. \]

There is an obvious pair of adjoint functors

\[ \Sigma^\infty : K_B \to S_B \quad \text{and} \quad \Omega^\infty : S_B \to K_B, \]

given by \(\Sigma^\infty(X)(V) = X \wedge_B S_B(V)\) and \(\Omega^\infty(E) := E(0)\). More generally, the functor \(\Omega^\infty_V\) given by evaluation at \(V\) has a left adjoint \(\Sigma^\infty_V\) \cite{31}.

Proposition 16 \cite[11.2.5]{31} \(S_B\) is tensored and cotensored over \(K_B\). For \(X \in S_B\) and \(K \in K_B\), there are natural isomorphims

\[ X \wedge_B K \cong X \wedge_B \Sigma^\infty K. \]

1.3 Model structures on ex-spectra

In this section, we give a short summary of the model structures on ex-spectra developed in \cite{31}. First, a level model structure is defined which is then utilised to obtain a stable model structure, where the adjectives "level" and "stable" refer to the respective classes of weak equivalences.
Definition 17 [31] A map \( f : X \to Y \) of spectra over \( B \) is called a level weak equivalence if \( f(V) : X(V) \to Y(V) \) is a weak homotopy equivalence for all \( V \in \mathcal{J} \). It is called a level fibration if each \( f(V) \) is a fibration in the model structure on \( \mathcal{K}_B \) defined in Theorem 8.

Fix a skeleton \( \text{skel}(\mathcal{J}) \) of \( \mathcal{J} \).

Definition 18 [31]

\[
FI_B := \{ \Sigma^\infty i \mid i \in I_B, \ V \in \text{skel}(\mathcal{J}) \}
\]

\[
FJ_B := \{ \Sigma^\infty j \mid j \in J_B, \ V \in \text{skel}(\mathcal{J}) \}
\]

Theorem 19 [31, 12.1.7] \( \mathcal{S}_B \) is a cofibrantly generated topological model category with level weak equivalences as weak equivalences, level fibrations as fibrations and \( FI_B(FJ_B) \) as generating (acyclic) cofibrations.

Recall that the homotopy groups \( \pi_q(X) \) of an (unparametrised) spectrum \( X \) are defined as the colimits of the groups \( \pi_q(\Omega^V(X(V))) \).

Definition 20 [31] A map \( f : X \to Y \) of parametrised spectra is called a stable weak equivalence if, after level fibrant approximation \( R^l \), it induces an isomorphism \( i^*_b R^l f : i^*_b R^l X \to i^*_b R^l Y \) on homotopy groups of (unparametrised) spectra for all points \( b \in B \).

A map of ex-spectra is called s-cofibration if it is a cofibration in the level model structure.

Theorem 21 [31, 12.3.10] \( \mathcal{S}_B \) is a cofibrantly generated model category with the stable weak equivalences as weak equivalences and the s-cofibrations as cofibrations.

As in the unparametrised case, any level weak equivalence is a stable weak equivalence. The stable model structure from the above theorem is indeed stable in the sense that the suspension functor \( \Sigma : \mathcal{S}_B \to \mathcal{S}_B, \Sigma(E) = E \wedge_{B} S^1 \) is invertible in the homotopy category.

Definition 22 [31] An \( \Omega \)-spectrum over \( B \) is a level fibrant spectrum over \( B \) such that each of its adjoint structure maps is a weak equivalence.

It is shown in [31, 12.3.10,12.3.14] that the \( \Omega \)-spectra over \( B \) are the fibrant objects in the stable model structure. Moreover, the sets of level and stable weak equivalences between \( \Omega \)-spectra coincide.

Proposition 23 [31, 12.6.2] The adjunction \( \Sigma^\infty : \mathcal{K}_B \rightleftarrows \mathcal{S}_B : \Omega^\infty \) is a Quillen adjunction for all \( V \in \mathcal{I} \).
Proposition 24 [31, 12.6.7] For a continuous map \( f : A \to B \), the adjunction \((f_! , f^*)\) is a Quillen adjunction. It is a Quillen equivalence if \( f \) is a weak homotopy equivalence.

The external smash product \( \wedge : \mathcal{S}_A \times \mathcal{S}_B \to \mathcal{S}_{A \times B} \) is homotopically well-behaved in the stable model structure and compatible with the base change functors; more precisely, we have the following proposition.

Proposition 25 [31, 12.6.5, 13.7.2] \( \wedge : \mathcal{S}_A \times \mathcal{S}_B \to \mathcal{S}_{A \times B} \) is a Quillen bifunctor. For maps \( f : A \to B \) and \( g : A' \to B' \), there are natural equivalences
\[
L(\wedge \circ (f_! \times g_!)) \cong L((f \times g)_! \circ \wedge).
\]

For a spectrum \( E \) over \( A \), there is a natural map \( i_0! Q E \to i_0! R Q E \to \pi_A^* R E \) of spectra over \( A \times I \). The first map is a stable weak equivalence since \( i_0! \) is a Quillen left adjoint, and \( R Q E \to R E \) is a stable weak equivalence between fibrant spectra, hence it is a level weak equivalence which implies that the second map is a (level and therefore stable) weak equivalence as well. Moreover, since the base change functors are defined levelwise, the same proofs as for Proposition 10 and 11 give the corresponding statements for \( Ho\mathcal{S}_B \).

Proposition 26 A homotopy \( h : A \times I \to B \) between \( f, g : A \to B \) gives rise to a natural isomorphism \( \phi : Lf_! \sim Lg_! \) of functors \( Ho\mathcal{S}_A \to Ho\mathcal{S}_B \). If two homotopies \( h_0, h_1 \) are homotopic (as homotopies) via \( H \), the isomorphisms \( \phi_0 \) and \( \phi_1 \) coincide.

Proposition 27 Let \( h_0 \) be a homotopy between \( f_0, f_1 : A \to B \) and \( h_1 \) be one connecting \( f_1 \) and \( f_2 \), and denote by \( h \) the homotopy from \( f_0 \) to \( f_2 \) given by composing \( h_0 \) and \( h_1 \). Then the corresponding isomorphisms \( \phi_0 : Lf_0_! \sim Lf_1 !, \phi_1 : Lf_1 ! \sim Lf_2 ! \) and \( \phi : Lf_0_! \sim Lf_2 ! \) satisfy \( \phi_1 \circ \phi_0 = \phi : Lf_0_! \sim Lf_2 ! \).

1.4 Operads

Definition 28 Let \( \Delta \) denote the category with the sets \( (n) = \{0, ..., n\}, n \geq 0 \) as objects and the non-decreasing functions as morphisms. The category of simplicial spaces \( \mathcal{SU} \) is defined as the category of functors \( \Delta^{op} \to \mathcal{U} \).

Using the functor \( \Delta : \Delta \to \mathcal{U} \) mapping \( (n) \) to the topological n-simplex
\[
\Delta^n := \{(t_0, ..., t_n) \in \mathbb{R}^{n+1} | \Sigma t_i = 1, t_i \geq 0\}
\]
we define the geometric realisation \( |\cdot| : \mathcal{SU} \to \mathcal{U} \) as
\[
|X_\bullet| := \int_{(n) \in \Delta} X_n \times \Delta^n,
\]
where the coend is defined as the initial object of $X_\bullet \times \Delta^\bullet / \mathcal{U}$, the category whose objects are spaces $Y \in \mathcal{U}$ together with maps $X_n \times \Delta^n \to Y$ for all $n$ such that

\[
\begin{array}{ccc}
X_k \times \Delta^l & \longrightarrow & X_k \times \Delta^k \\
\bigg\downarrow & & \bigg\downarrow \\
X_l \times \Delta^l & \longrightarrow & Y
\end{array}
\]

commutes for all morphisms in $\Delta((l), (k))$.

**Proposition 29** [28, 11.5] Geometric realisation preserves finite products, i.e. there are natural homeomorphisms $\xi : |X_\bullet \times Y_\bullet| \cong |X_\bullet| \times |Y_\bullet|$ for all simplicial spaces $X_\bullet, Y_\bullet \in \mathcal{S}_\mathcal{U}$.

**Definition 30** [27, 28] An operad (in $\mathcal{U}$) is a sequence of spaces $\mathcal{P}(j) \in \mathcal{U}$, $j \in \mathbb{N}$ with right $\Sigma_j$-actions on $\mathcal{P}(j)$, continuous $\Sigma_j$-equivariant maps

$$
\gamma : \mathcal{P}(k) \times \mathcal{P}(j_1) \times \ldots \times \mathcal{P}(j_k) \to \mathcal{P}(j_1 + j_2 + \ldots j_k)
$$

satisfying associativity and a unit $1 \in \mathcal{P}(1)$. A morphism of operads $\theta : \mathcal{P} \to \mathcal{P}'$ is a sequence of continuous $\Sigma_j$-equivariant maps $\theta(j) : \mathcal{P}(j) \to \mathcal{P'}(j)$ such that

$$
\theta(j_1 + \ldots + j_k) \circ \gamma = \gamma' \circ (\theta(k) \times \theta(j_1) \times \ldots \theta(j_k)).
$$

An $E_\infty$-operad is an operad $\mathcal{P}$ such that all spaces $\mathcal{P}(j)$ are contractible.

An operad is called pointed if it is indexed on $\mathbb{N}_0$ and $\mathcal{P}(0)$ consists of a single point.

**Example:** Let $(\mathcal{C}, \wedge)$ be a symmetric monoidal category enriched over $\mathcal{U}$. For any $X \in \mathcal{C}$, the endomorphism operad $\text{End}_X$ is defined as

$$
\text{End}_X(j) := \mathcal{C}(X^j, X)
$$

with the identity map as unit.

**Definition 31** An action of an operad $\mathcal{P}$ on $X \in \mathcal{C}$ is a morphism of operads $\theta : \mathcal{P} \to \text{End}_X$. $(X, \theta)$ is called an algebra over $\mathcal{P}$. We denote the category of algebras over $\mathcal{P}$ in $\mathcal{C}$ by $\mathcal{P}[\mathcal{C}]$.

**Proposition 32** If a pointed $E_\infty$-operad $\mathcal{P}$ acts on a space $B$, any $p \in \mathcal{P}(2)$ endows $\text{HoSh}_B$ with the structure of a symmetric monoidal category.

**Proof:** For any $q \in \mathcal{P}(j)$, the $\mathcal{P}$-algebra structure yields a map $\theta(q) : B^j \to B$ and any path in $\mathcal{P}(j)$ from $q$ to $q'$ yields a homotopy from $\theta(q)$ to $\theta(q')$. By Proposition [26] this homotopy yields a natural transformation

$$
L\theta(q) \sim L\theta(q'),
$$
and since $\mathcal{P}(j)$ is contractible by assumption, any two such paths are homotopic and thus define the same natural transformation. We define the bifunctor associated to $p \in \mathcal{P}(2)$ to be

$$p := L(\theta(p) \circ \kappa) : \text{Ho}S_B \times \text{Ho}S_B \to \text{Ho}S_B.$$ 

and the unit by $1_B := i_\ast S$ with

$$i_\ast := \theta(\mathcal{P}(0)) : \ast \to B$$

and $S$ the sphere spectrum. Observe that $S$ and hence also $1_B$ is cofibrant. The idea to obtain unit, symmetry and associativity isomorphisms for $p$ is very simple: the things one wants to relate are obtained as push-forwards along maps corresponding to different points in the operad. These points in $\mathcal{P}(j)$ can be connected by paths in $\mathcal{P}(j)$ and we use the natural transformations corresponding to these paths to construct the isomorphisms.

Throughout the proof, we suppress the associativity, unit and commutativity isomorphisms for $\wedge$ from the notation.

To get the (left) unit isomorphism, observe that the map $p_e : B \to B, b \mapsto \theta(p)(\ast, b)$ is given by $\theta(\gamma(p; \ast, id))$. Since $\mathcal{P}(1)$ is connected, $p_e$ is homotopic to the identity [29]. Combining Propositions 23 and 26 we obtain

$$p(1_B, X) \cong \theta(p)_!(((i_\ast S)\wedge QX) \cong \theta(p)_!(i_\ast \times id)_!(\wedge QX) \cong p_e! QX \cong X.$$ 

Similarly, we make use of the fact that the maps $p_{12} := \theta(\gamma(p; p, 1)) : B^3 \to B$ and $p_{21} := \theta(\gamma(p; 1, p)) : B^3 \to B$ are homotopic.

$$p(p(X, Y), Z) \cong \theta(p)_!(\theta(p)_!(QX\wedge QY)\wedge QZ) 
\cong \theta(p)_!(\theta(p) \times id)_!(\wedge QX\wedge QY)\wedge QZ) 
\cong p_{12}((QX\wedge QY)\wedge QZ) \cong p_{21}((QX\wedge QY)\wedge QZ) 
\cong p(X, p(Y, Z))$$

which yields the associativity transformation. For the non-trivial element $\tau \in \Sigma_2$, the maps $\theta(p)$ and $\theta(\tau p)$ are homotopic, hence Proposition 26 gives the commutativity isomorphism $T$.

What remains to be checked are the usual coherence diagrams which can be found for example in [24].

For spectra $X, Y, W, Z \in S_B$ we need to show that

$$p(X, p(Y, p(Z, W))) \to p(p(X, Y), p(Z, W)) \to p(p(X, Y), Z, W)$$

$$p(X, p(p(Y, Z), W)) \to p(p(X, p(Y, Z)), W)$$
commutes. Using Propositions \[25\] and \[26\] and writing

\[ E := QX\pi QY\pi QZ\pi QW, \]

this reduces to the commutativity of

\[
\begin{array}{cccc}
p_1! E & p_2! E & p_3! E & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \q
1.4 Operads

\[ \theta_j(p)^q : B_q(*, G, *)^j \cong (G^q)^j \cong (G^j)^q \to G_q \cong B_q(*, G, *) \] and
\[ \theta_j(p)^{q+1} : B_q(*, G, G)^j \cong (G^{q+1})^j \cong (G^j)^{q+1} \to G^{q+1} \cong B_q(*, G, G) \]. Compatibility of \( \theta \) with the group structure of \( G \) implies that these are maps of simplicial spaces. Since
\[
\theta_j : \mathcal{P}(j) \to \mathcal{U}(G^j, G) \quad \text{and} \quad (.)^q : \mathcal{U}(G^j, G) \to \mathcal{U}((G^j)^q, G^q)
\]
are continuous, \( \mathcal{B}(*, G, *) \) and \( \mathcal{B}(*, G, G) \) are indeed in \( \mathcal{P}[SU] \). Proposition 33 gives \( \mathcal{P} \)-algebra structures on \( \theta^B \) on \( BG \) and \( \theta^E \) on \( EG \). The commutativity of the diagrams

\[
\begin{array}{ccc}
B_q(*, G, G)^j & \xrightarrow{\theta^E(p)} & B_q(*, G, *)^j \\
\downarrow \theta^B(p) & & \downarrow \theta^B(p) \\
B_q(*, G, G) & \xrightarrow{\theta^E(p) \times \theta_j(p)} & B_q(*, G, G) \\
\end{array}
\]
implies that \( EG \to BG \) and \( EG \times G \to EG \) are maps of \( \mathcal{P} \)-algebras. \( \square \)

If we have an action \( m : G_+ \wedge F \to F \) of \( G \) on a spectrum \( F \in \mathcal{S} \), we can define an ex-spectrum \( BGF \) over \( BG \) by setting \( BGF(V) := EG \times_G F(V) \) with the evident projection and section. The \( O(V) \) action is the one induced by the action on \( F(V) \) and since \( (EG \times_G F(V)) \wedge S(W) \cong EG \times_G (F(V) \wedge S(W)) \), the structure maps of \( F \) give rise to structure maps \( \sigma_{V,W} : BGF(V) \wedge S(W) \to BGF(V \oplus W) \).

**Lemma 35** Let \((F, \phi) \in \mathcal{P}[S], (G, \theta) \) a group in \( \mathcal{P}[U] \) with an action \( m : G_+ \wedge F \to F \) in \( \mathcal{P}[S] \) and \( \mathcal{P} \) a pointed \( B_\infty \) operad. Then for each \( p \in \mathcal{P}(2) \), \( BGF \) is a monoid in \((HoS_{BG}, p)\), i.e. there is a map
\[
\mu(p) : p(BGF, BGF) \to BGF \quad \text{and} \quad \text{a unit } \eta : 1_{BG} \to BGF \text{ satisfying the usual conditions.}
\]

**Proof:** To construct \( \mu(p) \), we have to define \( O(V_1) \times O(V_2) \)-equivariant maps
\[
\mu(p)_{V_1, V_2} : BGF(V_1) \wedge BGF(V_2) \to BGF(V_1 \oplus V_2)
\]
for all \( V_1, V_2 \in \mathcal{I} \). Recall the construction of the \( \mathcal{P} \)-actions \( \theta^B \) on \( BG \) and \( \theta^E \) on \( EG \) from Lemma 33. The \( \mathcal{P} \)-structure of \( F \) gives
\[
\phi(p)_{V_1, V_2} : F(V_1) \wedge F(V_2) \to F(V_1 \oplus V_2).
\]
Define
\[
\tilde{\mu}(p)_{V_1, V_2} : BGF(V_1) \wedge BGF(V_2) \to \theta^B(p)^* BGF(V_1 \oplus V_2)
\]
with \([e_1, f_1], [e_2, f_2] \mapsto [\theta^E(p)(e_1, e_2), \phi(p)_{V_1, V_2}(f_1, f_2)] \).
These maps are by Proposition 34 well defined and since they are compatible with the structure maps as well as $O(V) \times O(W)$ equivariant, we obtain

$$\bar{\mu}(p) : BGF \wedge BGF \to \theta^B(p) \ast BGF.$$ 

Letting $\mu(p)'$ denote the adjoint of $\bar{\mu}(p)$, $\mu(p)$ is given by (the homotopy class of)

$$p(BGF, BGF) = \theta^B(p)_!(QBGF \wedge QBGF) \to \theta^B(p)_!(BGF \wedge BGF) \mu(p)' \to BGF.$$

Notice that the same procedure yields maps

$$\mu(q)' : \theta^B(q)_! BGF \wedge BGF \to BGF$$

for each $q \in \mathcal{P}(j)$. Similarly, $\eta : 1_{BG} = i_0 S \to BGF$ is the adjoint of the unit $1_F : S \to F \cong i^*_b BGF$. What remains to be shown is the compatibility of $\mu(p)$ and $\eta$ and the associativity and unit isomorphisms of $p$, i.e. commutativity of the diagrams

and

Recall that those isomorphisms were constructed using homotopies given by paths in the operad. Using the construction above, we define maps of spectra over these homotopies which we use to prove the result.

The diagram for the unit transformation is the following, where $h$ is a path in $\mathcal{P}(1)$ connecting $p_* := \gamma(p; \ast, id)$ and 1 and $E := BGF$. 

"
The commutativity I, II, III follows from the definition of the maps of the definition of the maps 

\[ (f) \circ (g) = (g) \circ (f) \]

and the definition of an action of an operad. Similarly, the compatibility with the associativity transformations is shown from the definition of the maps of the operad. The commutativity I, II, III follows from the definition of the maps of the operad.
Chapter 2

Twisted cohomology theories

2.1 Definition and basic properties

Recall that generalised cohomology theories on \( \mathcal{U}_* \) correspond to spectra, where the theory corresponding to a spectrum \( E \) can be defined as

\[
\tilde{E}^n(X) := [(L\Sigma^\infty_{\mathbb{R}^n})X, E] = [\Sigma^\infty_{\mathbb{R}^n}QX, E] \cong [X, (RE)((\mathbb{R}^n))].
\]

By Brown's representability theorem, any cohomology theory on \( \mathcal{U}_* \) arises in this way. The situation for twisted cohomology theories is very similar, where we replace \( \mathcal{U}_* \) by the category of ex-spaces \( \mathcal{K}_B \) and \( \mathcal{S} \) by the category of ex-spaces \( \mathcal{S}_B \). Just as in the unparametrised case, the morphism sets in \( \text{Ho}\mathcal{S}_B \) can be equipped with an abelian group structure as follows. \( S^2 \) is a homotopy commutative H-cogroup with comultiplication given by the 'pinch map' \( S^2 \rightarrow S^2 \vee S^2 \). Hence \( \Sigma^2 E \cong E \wedge S^2 \) is an H-cogroup as well for all \( E \in \mathcal{S}_B \). This comultiplication defines a commutative product on \( [\Sigma^2 E, F] \) for any \( F \) and since \( \text{Ho}\mathcal{S}_B \) is stable,

\[
[E, G] \cong [\Sigma E, \Sigma G] \cong [\Sigma^2 E, \Sigma^2 G]
\]

is an abelian group for any \( E, G \in \mathcal{S}_B \).

We will now define reduced twisted cohomology theories and then use these to define the unreduced versions.

Recall that in pointed model categories the cofiber of a map \( f : X \rightarrow Y \) is defined to be the coequalizer \( g : Y \rightarrow Z \) of \( f \) and the zero map. If \( f \) is a cofibration of cofibrant objects,

\[
X \xrightarrow{f} Y \xrightarrow{g} Z
\]

is called cofiber sequence.
Definition 36 A (reduced) generalised cohomology theory on \( K_B \) consists of contravariant functors \( \tilde{H}^* : K_B \to \text{Ab} \) indexed on \( \mathbb{Z} \) and natural isomorphisms \( \sigma^n : \tilde{H}^{n+1} \circ \Sigma \to \tilde{H}^n \) such that

- \( \tilde{H}^* \) factors through \( \gamma : K_B \to \text{Ho}K_B \);
- If \( X \to Y \to Z \) is a cofiber sequence, \( \tilde{H}^n(Z) \to \tilde{H}^n(Y) \to \tilde{H}^n(X) \) is exact for all \( n \);
- \( \tilde{H}^*(\coprod_{i \in I} X_i) \cong \prod_{i \in I} \tilde{H}^n(X_i) \).

Lemma 37 The generalised cohomology theory on \( K_B \) corresponding to an ex-spectrum \( E \in S_B \) defined by

\[
\tilde{E}^n(X) := [(L\Sigma^\infty_\mathbb{R}_n)(X), E] \\
\tilde{E}^{-n}(X) := \tilde{E}^0(\Sigma^n X)
\]

is indeed a cohomology theory.

Proof: Left derived functors preserve cofiber sequences, hence

\[
\tilde{E}^n(Z) \to \tilde{E}^n(Y) \to \tilde{E}^n(X)
\]

is exact for all cofiber sequences \( X \to Y \to Z \). For the suspension isomorphism, recall that the adjoints of the structure maps \( E(V) \to \Omega W E(V \oplus W) \) are weak equivalences of ex-spaces for all fibrant ex-spectra \( E \), hence

\[
[(L\Sigma^\infty_\mathbb{R}_{n+1}) \Sigma X, E] \cong [\Sigma X, (RE)(\mathbb{R}^{n+1})] \cong [X, \Omega(RE)(\mathbb{R}^{n+1})] \cong [X, (RE)(\mathbb{R}^n)].
\]

Lemma 38 [32] For every generalised cohomology theory \( \tilde{H} \), there is an ex-spectrum \( E \) such that \( \tilde{H}^n(X) = [(L\Sigma^\infty_\mathbb{R}_n)X, E] \)

Proof: In [10], the author proves a general representability theorem for functors satisfying the wedge and the Mayer-Vietoris axiom on what he calls homotopy categories. In [31, 7.5], it is shown that \( \text{Ho}K_B \) is a homotopy category in the sense of [10], thus the functors \( H^n \) are representable by \( E(\mathbb{R}^n) \in K_B \). The transformations \( \sigma^n \) yield the maps \( \Sigma E(\mathbb{R}^n) \to E(\mathbb{R}^{n+1}) \).

To obtain the unreduced theories, we proceed exactly as in the unparametrised case.

Definition 39 Let \( K_B^2 \) be the category of pairs in \( K_B \). We define the cone \( C : K_B^2 \to K_B \) by

\[
C(X, A) := QX \coprod_{QA \times \{0\}} QA \times I \coprod_{QA \times \{1\}} B \\
C(X, \emptyset) := QX_+.
\]
2.1 Definition and basic properties

The unreduced cohomology groups are defined by
\[ E^n(X, A) := \widetilde{E}^n(C(X, A)), \quad E^n(X) := E^n(X, \emptyset). \]

Note that there are natural maps
\[ C(X, A) \to C(X, A)/_BQX_+ \cong \Sigma QA_+. \]
Combining these with the suspension isomorphism, we obtain natural maps
\[ \delta^n : E^{n-1}(A) \cong \widetilde{E}^n(\Sigma QA_+) \to \widetilde{E}^n(C(X, A)) = E^n(X, A). \]

Lemma 40 \[31, 5.6.2\] The sequence
\[ \ldots \to E^n(X) \to E^n(A) \xrightarrow{\delta^{n+1}} E^{n+1}(X, A) \to E^{n+1}(X) \to \ldots \]
is exact.

The same proof as in the unparametrised case shows that \( E^* \) satisfies excision.

Lemma 41 If \( X = A \cup D \), the inclusion \( i : (A, A \cap D) \to (X, D) \) induces an isomorphism \( i^* : E^n(X, D) \to E^n(A, A \cap D) \).

**Proof:** First, let us assume that \( X \) is a CW-complex and \( A, D \) are subcomplexes. The natural maps
\[ C(X, D) \to X \cup D \times I \cup B \quad C(A, A \cap D) \to A \cup (A \cap D) \times I \cup B \]
are weak equivalences by the 5-lemma and \[31, 5.6.2\]. We use a representation of \((D, A \cap D)\) as NDR-pair to construct a homotopy inverse of the inclusion \( i : A \cup (A \cap D) \times I \cup B \to X \cup D \times I \cup B. \)

Recall that a representation as NDR-pair is given by maps
\[ u : D \to I \quad h : D \times I \to D \]
such that
\[ u^{-1}(0) = A \cap D, \quad h_1(u^{-1}[0, 1]) = A \cap D, \quad h_0 = id, \quad h_{|A \cap D} = id. \]

Now, a homotopy inverse of \( i \) is
\[
\begin{array}{ccc}
(A \cup D) \cup D \times I \cup B & \to & A \cup (A \cap D) \times I \cup B \\
d & \mapsto & (h_1(d), u(d)) \\
(d, t) & \mapsto & (h_1(d), \max(t, u(d))) \\
b & \mapsto & b.
\end{array}
\]
This shows that $\iota$ is a weak equivalence of $\text{ex}$-spaces and therefore induces an isomorphism. To give the proof for general $X$, one uses CW-substitutes. The crucial observation is that if $X = A \cup D$ and one has CW-substitutes $A'$ of $A$ and $D'$ of $D$ which are obtained as extensions of a CW-substitute $(A \cap D)'$ of $A \cap D$, then $A' \cup D'$ is a CW-substitute of $X$. \qed

Thus, $E^*$ does indeed define a cohomology theory on pairs.

**Definition 42** A triad $(X; A, D)$ over $B$ is called proper with respect to $E$ if the inclusions 

\[ i_A : (A, A \cap D) \to (A \cup D, D) \quad \text{and} \quad i_D : (D, A \cap D) \to (A \cup D, A) \]

induce isomorphisms 

\[ i_A^* : E^*(A, A \cap D) \to E^*(A \cup D, D) \quad \text{and} \quad i_D^* : E^*(D, A \cap D) \to E^*(A \cup D, A). \]

The Mayer-Vietoris sequence, the exact sequence of a triple and the exact sequence of a proper triad follow formally from Lemma 40 and 41.

**Lemma 43** If $A \subset D \subset X$, the sequence 

\[ \ldots \to E^{n-1}(A, D) \xrightarrow{\Delta} E^n(X, A) \to E^n(X, D) \to E^n(A, D) \to \ldots \]

where $\Delta$ is the composition 

\[ E^{n-1}(A, D) \to E^{n-1}(A) \xrightarrow{\delta_n} E^n(X, A) \]

is exact.

**Lemma 44** If $X = A \cup D$, the sequence 

\[ \ldots \to E^{n-1}(A \cup D) \xrightarrow{\Delta} E^n(X) \to E^n(A) \oplus E^n(D) \to E^n(A \cup D) \to \ldots \]

where $\Delta$ is the composition 

\[ E^{n-1}(A \cup D) \xrightarrow{\delta_n} E^n(A, A \cup D) \cong E^n(X, D) \to E^n(X) \]

is exact.

**Lemma 45** If $(X; A, D)$ is a proper triad, the sequence 

\[ \ldots \to E^{n-1}(A, A \cap D) \xrightarrow{\Delta} E^n(X, A \cap D) \to E^n(X, D) \to E^n(A, A \cap D) \to \ldots \]

where $\Delta$ is the composition 

\[ E^{n-1}(A, A \cap D) \cong E^{n-1}(A \cup D, D) \to E^{n-1}(A \cup D) \xrightarrow{\delta_n} E^n(X, A \cup D) \]

is exact.
2.1 Definition and basic properties

There is an additional feature of parametrised cohomology theories which does not show up in the classical ones, namely the following.

**Definition 46** If \( q, r : X \to B \) are homotopic via \( h : X \times I \to B \), the isomorphism induced by \( h \) is

\[
\psi(h) := i_1^*(i_0^*)^{-1} : E^n(X, A, q) \to E^n((X, A) \times I, h) \to E^n(X, A, r).
\]

In particular, maps \( h : X \times S^1 \to B \) yield isomorphisms

\[
\psi(h) : E^n((X, A, h|_{X \times \{1\}}) \to E^n((X, A, h|_{X \times \{1\}}).
\]

**Lemma 47** If two homotopies \( h_0, h_1 \) are homotopic through \( H \), they induce the same isomorphism. If \( h \) is the composition of the homotopies \( h_0 \) and \( h_1 \), \( \psi(h) = \psi(h_1)\psi(h_0) \).

**Proof:** Consider the diagram

\[
\begin{array}{ccc}
((X, A), q) & \xrightarrow{i_0} & ((X, A) \times I, h_0) \\
\downarrow{j_0} & & \xrightarrow{i_1} \downarrow{j_1}
\end{array}
\]

\[
((X, A) \times I^2, H)
\]

Observe that

\[
j_0 \circ i_0, j_1 \circ i_0 : ((X, A), q) \to ((X, A) \times I^2, H)
\]

as well as

\[
j_0 \circ i_1, j_1 \circ i_1 : ((X, A), r) \to ((X, A) \times I^2, H)
\]

are homotopic over \( B \) and therefore

\[
(j_0 \circ i_0)^* = (j_1 \circ i_0)^* \quad \text{and} \quad (j_0 \circ i_1)^* = (j_1 \circ i_1)^*.
\]

Thus,

\[
\psi(h_0) = i_1^*(i_0^*)^{-1} = (j_0 \circ i_1)^*((j_0 \circ i_0)^*)^{-1} = (j_1 \circ i_1)^*(j_1 \circ i_0)^* = \psi(h_1).
\]

The proof for the second statement is similar. \( \square \)
2.2 Product structures

Recall the monoidal structure \( p \) on \( \text{Ho}\mathcal{S}_B \) for base spaces with an action by an \( E_\infty \)-operad from Proposition 32. Note that the same construction can be carried through with \( \mathcal{K}_B \) instead of \( \mathcal{S}_B \). This defines a monoidal structure on \( \text{Ho}\mathcal{K}_B \), which we again denote by \( p \). Observe that there are natural isomorphisms

\[
L\Sigma^\infty_{V\otimes W} \circ p \cong p \circ (L\Sigma^\infty_V \times L\Sigma^\infty_W),
\]

justifying the use of the same notation for the monoidal structures on spaces and spectra.

**Definition 48** A parametrised ring spectrum over \( B \) is a monoid in \((\text{Ho}\mathcal{S}_B, p)\).

We use parametrised ring spectra to define products in parametrised cohomology theories, mimicking the use of ring spectra in the construction of multiplicative theories as in [1, 33]. There are three different products, the wedge product for the reduced theories and the external and internal (or cup) product for the unreduced ones. All of these are very similar to their counterparts in the unparametrised case. Since the latter are discussed extensively in the literature and the proofs carry over to our setting, we will make it short and give little more than the definitions. The formal reason why the proofs carry over is the following. In [30], the definition when a triangulation and a closed symmetric monoidal structure on a category are compatible is given. Moreover, he shows how to prove compatibility if the category in question is the stable homotopy category of a topological model category and the symmetric monoidal structure is obtained from a Quillen bifunctor. The parametrised stable homotopy category \( \text{Ho}\mathcal{S}_B \) is triangulated [31, 13.1.5] and the symmetric monoidal structure \( p \) on \( \text{Ho}\mathcal{S}_B \) is obtained from a Quillen bifunctor, hence one can follow [30] to proof compatibility.

**Definition 49** Let \((E, \mu)\) be a parametrised ring spectrum. The product

\[
\wedge_p : \tilde{E}^n(X) \times \tilde{E}^m(Y) \to \tilde{E}^{n+m}(p(X,Y))
\]

in \( \tilde{E} \) is given by

\[
\wedge_p : [L\Sigma^\infty_{R^n} X, E] \times [L\Sigma^\infty_{R^m} Y, E] \xrightarrow{\mu} [p(L\Sigma^\infty_{R^n} X, L\Sigma^\infty_{R^m} Y), p(E, E)] \cong [L\Sigma^\infty_{R^{n+m}} p(X,Y), p(E, E)] \xrightarrow{\mu} [L\Sigma^\infty_{R^{n+m}} p(X,Y), E].
\]

The wedge product has the same properties as its non-parametrised analogue, i.e. it is associative, additive in each variable, commutes with suspension and
Since $\theta(p)_!$ is a left Quillen functor, the natural weak equivalences of cofibrant objects
\[ C(X, A) \to C(Y, D) \mapsto C(X \times Y, X \times D \cup A \times Y) \]
yield weak equivalences
\[ \underline{p}(C(X, A), C(Y, D)) \mapsto \theta(p)_! C(X \times Y, X \times D \cup A \times Y). \]
We abuse notation and write \((X, A) \times (Y, D) := (X \times Y, A \times Y \cup X \times D)\). Using the above equivalences, we obtain the external product
\[ \times_p : E^n(X, A) \times E^m(Y, D) \to E^{n+m}(X, A)(Y, D)) \]
and again, this product shares the main features of its non-parametrised analogue. Moreover, the product is compatible with the isomorphisms induced by homotopies of twists in the following sense. If \(r, r' : Y \to B\) are homotopic via \(h : Y \times I \to B\),
\[ \theta(p) \circ (q \times h) : X \times Y \times I \to B \]
is a homotopy from \(\theta(p) \circ (q \times r)\) to \(\theta(p) \circ (q \times r')\) and we have the following commutative diagram:
\[ \begin{array}{ccc}
E^n((X, A), q) \times E^m((Y, D), r) & \xrightarrow{\times_p} & E^{n+m}(\theta(p)_!((X, A) \times (Y, D), q \times r)) \\
\downarrow^{1 \times \psi(h)} & & \downarrow^{\psi(\circ (q \times h))} \\
E^n((X, A), q) \times E^m((Y, D), r') & \xrightarrow{\times_p} & E^{n+m}(\theta(p)_!((X, A) \times (Y, D), q \times r')). \\
\end{array} \]
Finally, we turn to the parametrised analogue of internal products. Note that the \(P\)-action equips \([X, B]\) with the structure of an abelian group, where the sum is given by
\[ [q] + [r] = [q +_p r] := [p \circ (q \times r) \circ \Delta] \]
and the neutral element is the constant map \(pr_0 : X \to * \xrightarrow{i} B\) onto the homotopy unit. To define internal products, we need parametrised analogues of the diagonal
\[ \Delta : (X, A \cup D) \to (X, A) \times (X, D) \]
and the sum of twists given by the \(P\)-action is exactly what is needed to make this map a morphism over \(B\). If \(q, r : X \to B\) are two twists, we equip \((X, A \cup D)\) with the twist \(q +_p r\), obtaining
\[ \Delta : ((X, A \cup D), q +_p r) \to \theta(p)_!((X, A) \times (X, D), q \times r) \]
which yields the internal product
\[ \cup_p := \Delta^* \circ \times_p : E^n((X, A), q) \times E^m((X, D), r) \to E^{n+m}((X, A \cup D), q + p r). \]

The \( \mathcal{P} \)-action can be used to define associativity and unit homotopies \( h_{ass} \) and \( h_1 \), connecting \( q + p (r + p s) \) to \( (q + p r) + p s \) and \( q + p pr_0 \) to \( q \). These are unique up to higher homotopies, so induce unique isomorphisms

\[ \psi(h_{ass}) : E^*((X, A), q + p (r + p s)) \to E^*((X, A), (q + p r) + p s) \]

and

\[ \psi(h_1) : E^*((X, A), q + p pr_0) \to E^*((X, A), q). \]

We will henceforth make frequent use of these isomorphisms without further mention. For example, combining the unit isomorphism and the cup product makes the untwisted groups \( E^*((X, A), pr_0) \) a ring. Similarly, the twisted groups \( E^*((X, A), q) \) are \( E^*((X, A), pr_0) \)-modules for any twist \( q : X \to B \).

## 2.3 Orientation

Let \( \pi : X \to M \) be a vector bundle. We denote the restriction of \( X \) to \( U \subset M \) by \( X_U \) and the complement of the zero section by \( X' \).

**Definition 50** Let \( \pi : X \to M \) be a vector bundle over \( M \xrightarrow{q} B \) and \( \|\| \) a fibrewise norm on \( X \). The parametrised Thom space is

\[ Th_B(X, q) := X/B(X \setminus D), \]

where \( D \subset X \) is the open unit disc bundle.

We will frequently omit the twist from our notation. Since all bundles are assumed to be bundles over \( B \), we use the same letter for the twist on \( M \) and for the corresponding twist on \( X \). Of course, \( Th_B(X) \) is weakly equivalent to the cone \( C(X, X') \) which was defined in Definition 39. From now on, we assume that the base space \( B \) is connected, i.e. that any constant map is homotopic to the constant map \( pr_0 \) onto the homotopy unit of \( B \). This implies that

\[ E^{i+n}(X_m, X'_m, q_m) \cong \tilde{E}^{i+n}(S^n, pr_{q(m)}) \cong \tilde{E}^i(S^0, pr_0) \]

for any \( n \)-dimensional vector bundle \( X \to M \) and any \( m \in M \). In particular, \( E^*((X_m, X'_m, q_m)) \) is a free \( \tilde{E}^*(S^0, pr_0) \)-module with one generator of degree \( n \). Note that the above isomorphism depends on the choice of a path between \( q(m) \) and the homotopy unit.
2.3 Orientation

**Definition 51** A vector bundle \( X^n \to M \) over \( q : M \to B \) is called \((E, q)\)-oriented by \( u \in E^n(X, X') \) if \( i_m^* u \) is a generator of \( E^*(X_m, X'_m) \) as \( \widetilde{E}^*(S^0, pr_0) \)-module.

**Theorem 52** If \( u \) is an \((E, q)\)-orientation of \( X^n \to M \), and \( M \) is a finite CW-complex, the homomorphism

\[
Th(u) : E^i(M, r) \xrightarrow{\pi'^*} E^i(X, r) \xrightarrow{\cup \eta u} E^{i+n}((X, X'), r+p q) \cong \widetilde{E}^{i+n}(Th_B(X, r+p q))
\]

is an isomorphism for all \( r : M \to B \).

**Proof:** We examine the spectral sequences of the fibrations

\[
M \xrightarrow{id} M \quad \text{and} \quad X \to M
\]

and observe that \( Th(u) \) induces a morphism of spectral sequences which is an isomorphism on the first pages. Let us first recall the construction of the spectral sequence. Denote the preimage of the \( s \)-skeleton \( M^s \) by \( X^s \). The exact sequence of the triad \((X^s; X^s-1, X'^s)\) is

\[
\cdots \to E^{n-1}(X^{s-1}, X'^{s-1}) \to E^n(X^s, X'^s \cup X^{s-1}) \to E^n(X^s, X'^s) \to E^n(X^s-1, X'^{s-1}) \to E^{n+1}(X^s, X'^s \cup X^{s-1}) \to \cdots
\]

and thus, the groups

\[
A^{s,t} := E^{s+t}(X^s, X'^s), \quad C^{s,t} := E^{s+t}(X^s, X'^s \cup X^{s-1})
\]

form an exact couple. The standard machinery constructs a spectral sequence with \( E_1 \)-page equal to \( C^{s,t} \) and converging to \( E^* (X, X') \).

For each \( s \)-cell \( j : D^s \to M \), we denote the bundle \( j^* X \) by \( X_{D^s} \) and regard \( D^s \) as a disc over \( B \) with the twist given by the pull-back of the twist of \( M \). The map \( j \) yields a map of ex-spaces

\[
C(X_{D^s}, X_{S^{s-1}} \cup X'_{D^s}) \to C(X^s, X^{s-1} \cup X'^s).
\]

Taking the coproduct over all cells, we obtain

\[
\coprod_{D^s \subset M^s} C(X_{D^s}, X_{S^{s-1}} \cup X'_{D^s}) \to C(X^s, X^{s-1} \cup X'^s).
\]

To show that this map is a weak equivalence we use a representation of \((X^s, X^{s-1})\) as an NDR-pair, i.e. maps

\[
u : X^s \to I \quad h : X^s \times I \to X^s
\]
such that
\[ u^{-1}(1) = X^{s-1}, \quad h_4|_{X^{s-1}} = id, \quad h_0 = id, \quad h_1(u^{-1}(0,1)) = X^{s-1}. \]

We use this representation to define
\[ X^{s} \cup (X^{s-1} \cup X^{t^s}) \times I \cup B \to \coprod_{D^s, X_{D^s+1} \cup X'_{D^s}} (X_{S^{s-1}} \cup X'_{D^s}) \times I \cup B \]

\[ (x, t) \mapsto (h_1(x), u(x)), \quad b \mapsto (h_1(x), \max(t, u(x))) \]

Since for \( x \in X^s \setminus X^{s-1} \) there is a unique cell such that \( x \in X_{D^s} \) and \( X^{s-1} \times I \)

is mapped to \( B \), this is well-defined. Note that for general twists, the map is not (and cannot be chosen to be) a morphism of ex-spaces. Nonetheless, it is

a homotopy inverse of the map induced by

\[ \coprod_{D^s \subset M^s} (X_{D^s}, X_{S^{s-1}} \cup X'_{D^s}) \to (X^s, X^{s-1} \cup X^{t^s}) \]

which is a morphism of ex-spaces. Therefore, we get an isomorphism

\[ E^*(X^s, X^{s-1} \cup X^{t^s}) \cong \bigoplus_{D^s \subset M^s} E^*(X_{D^s}, X_{S^{s-1}} \cup X'_{D^s}). \]

To analyse the groups \( E^*(X_{D^s}, X_{S^{s-1}} \cup X'_{D^s}) \), denote the upper half disc by \( D^s_+ \subset S^{s-1} \). For any cell \( D^s \subset M^s \), the exact sequence of the triple

\[ (X_{D^s}, X_{S^{s-1}} \cup X'_{D^s}, X_{D^s+1} \cup X'_{D^s}) \]

is

\[ \ldots \to E^{n-1}(X_{S^{s-1}} \cup X'_{D^s}, X_{D^s+1} \cup X'_{D^s}) \to E^n(X_{D^s}, X_{S^{s-1}} \cup X'_{D^s}) \to \]

\[ E^n(X_{D^s}, X_{D^s+1} \cup X'_{D^s}) \to E^n(X_{S^{s-1}} \cup X'_{D^s}, X_{D^s+1} \cup X'_{D^s}) \to \ldots \]

Since the inclusion

\[ X_{D^s+1} \cup X'_{D^s} \to X_{D^s} \]

is a weak equivalence, we obtain isomorphisms

\[ E^{i-1}(X_{S^{s-1}} \cup X'_{D^s}, X_{D^s+1} \cup X'_{D^s}) \cong E^i(X_{D^s}, X_{S^{s-1}} \cup X'_{D^s}), \]

while excision yields

\[ E^{i-1}(X_{S^{s-1}} \cup X'_{D^s}, X_{D^s+1} \cup X'_{D^s}) \cong E^{i-1}(X_{D^s+1}, X_{D^s+1} \cup X'_{D^s}). \]

Combining these isomorphisms, we get

\[ E^{i+s}(X_{D^s}, X_{S^{s-1}} \cup X'_{D^s}) \cong E^{i+s-1}(X_{D^s+1}, X_{D^s+1} \cup X'_{D^s+1}) \cong \ldots \cong E^i(X_{v_s}, X'_{v_s}) \]
with $v_s$ being the south pole of $D^s$. Summarising the above discussion,

$$C^{s,t} \cong \bigoplus_{D^s \subset M^s} E^s(X_{D^s}, X_{S^s-1} \cup X_{D^s}') \cong \bigoplus_{D^s \subset M^s} E^t(X_{v_s}, X_{v_s}').$$

Similarly, the spectral sequence of the fibration $M \to M$ is constructed from the exact couple

$$A^{s,t}_M := E^{s+t}(M^s), \quad C^{s,t}_M := E^{s+t}(M^s, M^{s-1}),$$

and we have isomorphisms

$$C^{s,t}_M \cong \bigoplus_{D^s \subset M^s} E^t(v_s).$$

The restriction of $Th(u)$ to the respective groups defines a morphism of exact couples. Now, for any $r : M \to B$, we have

$$
\begin{array}{ccc}
C^{s,t}_M & \xrightarrow{Th(u)} & C^{s,t} \\
\bigoplus_{D^s} E^t(v_s) & \xrightarrow{\pi^*} & \bigoplus_{D^s} E^t(X_{v_s}) \\
\bigoplus_{D^s} E^t(v_s) & \xrightarrow{\cup v'_s} & \bigoplus_{D^s} E^{t+n}(X_{v_s}, X_{v_s}').
\end{array}
$$

The bottom horizontal line is an isomorphism by assumption. The vertical isomorphisms were constructed using only excision and the exact sequence of a triple, thus the diagram commutes.

Notice that the only part of the proof where we used the finiteness assumption on $M$ was the convergence of the spectral sequence. If the spectral sequence converges for other reasons (for example, if the coefficients of the cohomology theory are bounded), the theorem is true for arbitrary $M$.

The notion of orientation is stable in the sense that if $u \in E^n(X, X')$ is a $q$-orientation, then $\sigma^m(u) \in E^{n+m}(X \oplus \mathbb{R}^m, (X \oplus \mathbb{R}^m)')$ is a $q$-orientation as well. Sums and pull-backs of oriented bundles are again oriented in the following way. If $u \in E^n((X, X'), q)$ is an $(E, q)$-orientation of a bundle $X \to M$, $f^*u$ is an $(E, f^*q)$-orientation of $f^*X$ for all $f : N \to M$. If $Y' \to M$ is another bundle over $M$ and $v \in E^m((Y, Y'), r)$ is its orientation, $\Delta^*(u \times_p v)$ is an $(E, q+p r)$-orientation of $X \oplus Y \cong \Delta^*(X \times_p Y)$ with the diagonal $\Delta : M \to M \times M$.

2.4 Integration

We will now discuss parametrised push-forward homomorphisms for oriented maps. Let us first recall the construction of push-forwards in the non-parametrised setting. Starting with a smooth proper map $f : (M, f^*q) \to (N, q)$,
we choose a closed embedding

\[
\tilde{f} : M \xrightarrow{f \times \iota} N \times \mathbb{R}^m \xrightarrow{\pi_N} N
\]

for some large \(m\). We will frequently assume that the image of \(\tilde{f}\) is contained in \(N \times D^m\). Note that any two such embeddings are stably isotopic. This implies that the normal bundle \(\nu(\tilde{f})\) is, up to stable isomorphism, independent of the chosen map \(\iota\). The stable isomorphism class of \(\nu(\tilde{f})\) is called stable normal bundle. \(f\) is called oriented if the stable normal bundle of \(f\) is oriented; since an orientation class of a bundle defines orientation classes on the stabilisations of the bundle, this notion does make sense. Now, let \(\phi : \nu(\tilde{f}) \sim \to U \subset N \times \mathbb{R}^m\) be a tubular neighbourhood. Using \(\phi\), one constructs a collapsing map

\[
\hat{f} : N_+ \wedge S^m \to Th(\nu(\tilde{f}))
\]

and defines

\[
f_! : E^\ast(M) \xrightarrow{Th(\mu)} \tilde{E}^\ast(Th(\nu(f))) \xrightarrow{\hat{f}_\ast} \tilde{E}^\ast(N_+ \wedge S^m) \cong E^\ast(N).
\]

We will now mimic the construction of \(f_!\) in the parametrised setting.

**Definition 53** A smooth proper map \(f : M \to N\) is called \(q\)-oriented for a twist \(q : N \to B\) if (a representative of) the stable normal bundle of \(f\) is \(f^\ast(q)\)-oriented.

Note that we demand that the twist of \(\nu(\tilde{f})\) is pulled back from \(N\). The main difficulty lies in the fact that for general twists \(q\), \(\phi\) can’t be chosen as a map over \(B\). However, the choice of a fibrewise norm \(\|\cdot\|\) on \(\nu(\tilde{f})\) enables us to deform \(q : N \times \mathbb{R}^n \to B\) so that at least the restriction of \(\phi\) to the unit disc bundle is indeed a map over \(B\). The idea is to make \(q\) constant along the fibres of the embedded normal bundle. More precisely, define \(\tilde{q}^\| : N \times \mathbb{R}^m \to B\) by

\[
\tilde{q}^\|(n, x) := \begin{cases} 
q(f(m)) & \text{if } (n, x) = (\phi(m, v), \|v\| \leq 1) \\
q(\pi_N \phi(m, (\|v\| - 1)v) & \text{if } (n, x) = (\phi(m, v), 1 \leq \|v\| \leq 2) \\
q(n) & \text{else.}
\end{cases}
\]

If we compose \(\phi\) with a homeomorphism from \(\nu(\tilde{f})\) to the open unit disc bundle \(D \subset \nu(\tilde{f})\), we obtain a tubular neighbourhood over \(B\). Note that we have a homotopy \(h : N \times \mathbb{R}^m \times I \to N \times \mathbb{R}^m\) such that \(h_1\) is the identity and \(q \circ h_0 = \tilde{q}^\|\). We define \(\tilde{r}^\| := r \circ h_0\) for any twist \(r : N \to B\). Observe that

\[
\tilde{r}^\| +_p \tilde{r}'^\| = \tilde{r}^\| +_p \tilde{r}'^\|.
\]
2.4 Integration

Write \( Y := N \times \mathbb{R}^m, Y_{c\|\|} := N \times \mathbb{R}^m \setminus \phi(D) \) and define the parametrised Pontrjagin-Thom map

\[
\tilde{f}_{c\|\|} : (Y/_{B}Y_{c\|\|}, \tilde{q}_{c\|\|}) \to (\text{Th}_{B}(\nu(\tilde{f}), f^*q)), \quad y \mapsto \begin{cases} 
\tilde{q}_{c\|\|}(y) & y \notin \phi(D) \\
\phi^{-1}(y) & y \in \phi(D)
\end{cases}
\]

By the construction of \( \tilde{q}_{c\|\|} \), this is a map of ex-spaces. Now, we define

\[
f_1 : E^*(M, f^*r) \xrightarrow{\text{Th}(u)} \tilde{E}^* \text{Th}_{B}(\nu(\tilde{f}), f^*r + p f^*q) \xrightarrow{\tilde{f}_{c\|\|}^*} \tilde{E}^*(Y/_{B}Y_{c\|\|}, r + p \tilde{q}_{c\|\|})
\]

\[
\cong E^*(Y, Y_{c\|\|}, r + p \tilde{q}_{c\|\|}) \xrightarrow{\psi(h)} E^*(Y, Y_{c\|\|}, r + p q)
\]

\[
\cong \tilde{E}^*(\Sigma^m N_+, r + p q) \cong E^*(N, r + p q).
\]

**Proposition 54** \( f_1 \) is independent of the choice of the fibrewise norm, the embedding and the tubular neighbourhood.

**Proof:** Let us first show that it is independent of the fibrewise norm. If \( \| \cdot \|_0, \| \cdot \|_1 \) are two fibrewise norms on \( \nu(\tilde{f}) \), we can find a path \( G \) in the space of norms connecting \( \| \cdot \|_0 \) and \( \| \cdot \|_1 \). We use \( G \) to show that the morphism

\[
E^*(N \times \mathbb{R}^m, N \times (\mathbb{R}^m \setminus D^m)) \xrightarrow{} E^*(Y, Y_{c\|\|}, \| \cdot \|_0) \xrightarrow{\psi(h_{\|\|,0})} E^*(Y, Y_{c\|\|,1})
\]

\[
\cong \tilde{E}^*(Y/_{B}Y_{c\|\|}, \| \cdot \|_0) \xrightarrow{\tilde{f}_{c\|\|}^*} \tilde{E}^*(\text{Th}_{B}(\nu(\tilde{f}), f^*r + p f^*q)) \xleftarrow{\tilde{\psi}(h_{\|\|,1})} E^*(\nu(\tilde{f}), \nu(\tilde{f}'))
\]

is the same for \( \| \cdot \|_0 \) and \( \| \cdot \|_1 \). Set

\[
D^G := \{(v, t) \mid \|v\|_{G(t)} < 1\} \subset \nu(\tilde{f}) \times I
\]

and

\[
(Y \times I)^{G^G} := N \times \mathbb{R}^m \times I \setminus (\phi \times id_I)(D^G).
\]

\( \| \cdot \|_0 \) corresponds to the upper path in the following diagram, whereas the bottom path corresponds to \( \| \cdot \|_1 \). The middle squares commute and the left and right triangle commute up to homotopy, thus the two morphisms agree in \( HoK_B \).
The proof that different embeddings and tubular neighbourhoods yield the
2.4 Integration

same \( f \) is similar. If

\[ t_0 : M \to D^{m_0}, \quad t_1 : M \to D^{m_1} \]

are two embeddings and

\[ \phi_0 : \nu(f^0) \to U_0 \subset N \times D^{m_0}, \quad \phi_1 : \nu(f^1) \to U_1 \subset N \times D^{m_1} \]

tubular neighbourhoods, they are (stably) isotopic, i.e. after stabilising, there are \( t : M \times I \to D^m \times I, \Phi : \nu(\hat{f}) \times I \to U \subset N \times D^m \times I \) from \((t_0, \phi_0)\) to \((t_1, \phi_1)\). As in the proof for the different norms, these isotopies are used to show that the different embeddings give the same \( f \).

Since the push-forward is independent of the chosen norm, we will henceforth only include the norm in the notation if it is relevant. Otherwise, we write \( \hat{f}, Y^c, \tilde{q}, h \) instead of \( \hat{f}^{\|\|}, Y^{c \|\|}, \tilde{q}^{\|\|}, h^{\|\|} \).

**Theorem 55** If \( f \) is \( q \)-oriented,

\[ f_!(f^*(x) \cup y) = x \cup f_!y \]

for all \( x \in E^*(N, r), y \in E^*(M, f^*s) \).

**Proof** The proof is very similar to the one in the unparametrised case, see e.g [16]. Let us first show that

\[ \hat{f}^*(\pi_M^* f^* \cup y') = \psi(h)^{-1} \pi_N^* x \cup \hat{f}^* y' \quad \forall \ x \in \tilde{E}^*(N, r), y' \in \tilde{E}^*(Th_B(\nu(\hat{f}), f^* s)) \]

where \( h \) is the homotopy needed to make the tubular neighbourhood a map over \( B \). To see this, observe that the following diagram commutes.

\[
\begin{array}{ccc}
E^*(N, r) \times \tilde{E}^*(Th_B(\nu(\hat{f}), f^* s)) & \xrightarrow{\pi_N^* \times 1} & E^*(Y, r) \times \tilde{E}(Th_B(\nu(\hat{f}), f^* s)) \\
\downarrow f^* \times 1 & & \downarrow \psi(h)^{-1} \times 1 \\
E^*(M, f^* r) \times \tilde{E}(Th_B(\nu(\hat{f}), f^* s)) & \xrightarrow{\pi_M^* \times 1} & E^*(Y, \tilde{r}) \times \tilde{E}^*(Th_B(\nu(\hat{f}), f^* s)) \\
\downarrow 1 \times \tilde{f}^* & & \downarrow 1 \times \tilde{f}^*
\end{array}
\]

\[
\begin{array}{ccc}
E^*(\nu(\hat{f}), f^* r) \times \tilde{E}^*((\nu(\hat{f}), \nu(\hat{f}) \setminus D), f^* s) & \xrightarrow{\phi^* \times \phi^*} & E^*(Y, \tilde{r}) \times \tilde{E}^*((Y, Y^c), \tilde{s}) \\
\downarrow \cup & & \downarrow \cup \\
E^*((\nu(\hat{f}), \nu(\hat{f}) \setminus D), f^*(r + p s)) & & E^*((Y, Y^c), \tilde{r} + p \tilde{s}) \\
\downarrow \sim & & \downarrow \sim \\
E^*(Th_B(\nu(\hat{f}), f^*(r + p s))) & \xrightarrow{\hat{f}^*} & E^*((Y, \tilde{r} + p \tilde{s}) / B Y^c)
\end{array}
\]
Starting with \( x \in \tilde{E}^*(N, r), y' \in \tilde{E}^*(Th_B(\nu(\tilde{f}), f^*s)) \), following the arrows on the left it is mapped to \( \hat{f}^*(\pi_M^*f^*x \cup y') \) whereas it is mapped to \( \psi(h)^{-1}\pi_M^*x \cup \hat{f}^*y' \) by the arrows on the right, proving the equality

\[
\hat{f}^*(\pi_M^*f^*x \cup y') = \psi(h)^{-1}\pi_M^*x \cup \hat{f}^*y'.
\]

Recall that

\[
f_i(x') := \sigma^{-n}i^*\psi(h)\hat{f}^*(\pi_M^*x \cup u)
\]

with \( i : (N \times \mathbb{R}^m, N \times (\mathbb{R}^m - D^m)) \rightarrow (Y, \text{y}^{cs}) \). Using the above equality,

\[
f_i(f^*x \cup y) = \sigma^{-n}i^*\psi(h)\hat{f}^*(\pi_M^*(f^*x \cup y) \cup u)
= \sigma^{-n}i^*\psi(h)\hat{f}^*(\pi_M^*f^*x \cup (\pi_M^*y \cup u))
= \sigma^{-n}(\pi_M^*x \cup i^*\psi(h)\hat{f}^*(\pi_M^*y \cup u))
= x \cup \hat{f}y.
\]

\[\square\]

Let us now discuss the functoriality of the push-forward, so take

\[f : M \rightarrow N, \ g : N \rightarrow L\]

smooth and assume that \( f \) is \( g^*q \)-oriented by \( u \in E^*((\nu(\tilde{f}), \nu(\tilde{f}')), f^*g^*q) \) and \( g \) is \( r \)-oriented by \( v \in E^*((\nu(\tilde{g}), \nu(\tilde{g}')), g^*r) \). We choose tubular neighbourhoods

\[\phi^M : \nu(\tilde{f}) \rightarrow N \times \mathbb{R}^m \quad \phi^N : \nu(\tilde{g}) \rightarrow L \times \mathbb{R}^n\]

and fibrewise norms on \( \nu(\tilde{f}), \nu(\tilde{g}) \). To see that there is an induced orientation of \( g \circ f \), observe that the normal bundle of

\[
\tilde{g}f := (\tilde{g} \times 1) \circ \tilde{f} : M \rightarrow L \times \mathbb{R}^{n+m}
\]

is

\[
\nu(\tilde{g}f) \cong \nu(\tilde{f}) \oplus \tilde{f}^*\nu(\tilde{g} \times 1) \cong \nu(\tilde{f}) \oplus f^*\nu(\tilde{g}).
\]

We denote the induced orientation by

\[
uv := \Delta^*(u \times_p f^*v) \in E^*((\nu(\tilde{g}f), \nu(\tilde{g}f)'), f^*g^*(q + p r)).
\]

Moreover, the tubular neighbourhoods of \( \tilde{f} \) and \( \tilde{g} \) can be used to construct a tubular neighbourhood of \( \tilde{g}f \) as follows. For any \( w \in \nu(f)_m \), there is a natural path \( \tilde{\gamma}_w \) joining \( f(m) \) and \( \phi^M(w) \), namely

\[
\tilde{\gamma}_w : I \rightarrow N \times \mathbb{R}^m, \ t \mapsto \phi^M(tw).
\]
Choose a connection and thus the notion of parallel transport on $\nu(\tilde{g} \times 1) \cong \nu(\tilde{g}) \times \mathbb{R}^n$ which is compatible with the chosen norm on $\nu(\tilde{g})$. We denote the isomorphism given by parallel transport along $\tilde{\gamma}_w$ by 

$$\gamma_w : \nu(\tilde{g} \times 1)|_{\tilde{f}(m)} \to \nu(\tilde{g} \times 1)|_{\phi^m(w)}.$$ 

A tubular neighbourhood is 

$$\phi := (\phi^N \times 1) \circ \phi^1 : \nu(\tilde{g}f) \to L \times \mathbb{R}^{n+m}$$

with 

$$\phi^1 : \nu(\tilde{g}f) \cong \nu(\tilde{f}) + \tilde{f}^*\nu(\tilde{g} \times 1) \to \nu(\tilde{g}) \times \mathbb{R}^m \cong \nu(\tilde{g} \times 1),$$

$$w_1, w_2 \mapsto \gamma_{w_1}(w_2).$$

**Theorem 56** $(g \circ f)_! = g_! \circ f_!$ for the induced orientation of $gf$.

**Proof:** We use the tubular neighbourhood $\phi$ defined above and the norm on 

$$\nu(\tilde{g}f) = \nu(\tilde{f}) + f^*\nu(\tilde{g})$$

given by 

$$\|w_1 + w_2\| := \max\{\|w_1\|, \|w_2\|\}.$$ 

The advantage of this norm is that the various adjustments of the twists needed to make the tubular neighbourhoods maps over $B$ and the homotopies between the original twists and the deformed ones are compatible in the following sense. The homotopy 

$$h^M : N \times \mathbb{R}^m \times I \to N \times \mathbb{R}^m$$

making $\phi^M$ a map over $B$ is just the restriction of the homotopy $h^1$ which turns $\phi^1$ into a map over $B$. We abuse notation and denote all homotopies by $h$. Since $(g \circ f)_!$ depends only on the orientation, it is sufficient to prove the equality for these choices. We write $D$ for the unit disc bundle of a Riemannian vector bundle and 

$$Y := N \times \mathbb{R}^m, \quad Y^c := N \times \mathbb{R}^m \setminus \phi^M(D), \quad Z := \nu(\tilde{g} \times 1),$$

$$Z^c := \nu(\tilde{g} \times 1) \setminus \phi^1(D), \quad W := L \times \mathbb{R}^{n+m}, \quad W^c := L \times \mathbb{R}^{n+m} \setminus \phi(D),$$

$$X := L \times \mathbb{R}^n, \quad X^c := L \times \mathbb{R}^n \setminus \phi^N(D),$$

$$TF := Th_B(\nu(\tilde{f})), \quad TG := Th_B(\nu(\tilde{g})), \quad TGF := Th_B(\nu(\tilde{gf})).$$

$\hat{g}f : W_BW^c \xrightarrow{\hat{g}} Z_BZ^c \xrightarrow{f} TGF$ factors over $Z_BZ^c$, using the factorisation of $\phi = (\phi^N \times 1) \circ \phi^1$. In the following commutative diagramm, the left hand vertical morphism is $f_1$, the bottom horizontal one is $g_!$ and the diagonal is
Chapter 3

Examples

3.1 Twisted K-theory

We will now define $BPU$-twisted (complex) $K$-theory. In [11], the authors developed an orthogonal (strict) ring spectrum representing $K$-theory. Their construction uses the picture of $KK$-theory from [4], where unbounded operators are used to represent $KK$-classes. We modify the construction of the $K$-spectrum in the sense that we weaken the product (obtaining an $E_\infty$-spectrum), but make way for a $PU$-action on the spectrum which is compatible with the product structure. First, we recall the results from [11].

Let $A$ be a $C^*$-algebra and $M$ a countably generated $\mathbb{Z}_2$-graded Hilbert $A$-module. We denote by $B(M)$ the $C^*$-algebra of bounded $A$-linear operators that admit an adjoint with respect to the $A$-valued scalar product.

**Definition 57** $F_M := \{ F \in B(M) \mid F^* = F, F^2 - 1$ compact and $||F|| \leq 1 \}$

We equip $F_M$ with the weakest topology such that the maps

$$F_M \to B(M), \quad F \mapsto F^2$$

and

$$F_M \to M, \quad F \mapsto F(\psi)$$

are continuous (with respect to the norm topology on $B(M)$ and $M$) for all $\psi \in M$.

Let $U_M$ be the subspace of unitary elements in $F_M$ and denote by $F_M/U_M$ the quotient (in topological spaces).

**Definition 58** $F_M^+$ is the set $F_M/U_M$ with the topology generated by the open sets in $F_M \setminus U_M$, and for all $\varepsilon > 0$ the sets

$$\{ F \in F_M \mid 0 < ||F^2 - 1|| < \varepsilon \} \cup \ast.$$

**Proposition 59** The conjugation action of the unitary group $U(M) \subset B(M)$ (with the norm topology) on $F_M$ is continuous and extends to a continuous action on $F_M^+$. 
Proof: Let us first show that $U(M)$ acts continuously on $F_M$. Observe that for $U, U_0 \in U(M)$ we have $\|U^{-1} - U_0^{-1}\| = \|U_0(U_0 - U)U_0^{-1}\| = \|U - U_0\|$. Now, for $\psi \in M$ and $F, F_0 \in F_M$ we have

$$\|UFU^{-1}\psi - U_0F_0U_0^{-1}\psi\| \leq \|UFU^{-1}\psi - UF_0U^{-1}\psi\| + \|UF_0U_0^{-1}\psi - U_0F_0U_0^{-1}\psi\| \leq 2\|U - U_0\|\|\psi\| + \|(F - F_0)U_0\psi\|.$$ 

Moreover,

$$\|(UFU^{-1})^2 - (U_0F_0U_0^{-1})^2\| \leq \|UF^2U^{-1} - UF^2U_0^{-1}\| + \|UF_0^2U_0^{-1} - UF_0^2U_0^{-1}\| \leq 2\|U - U_0\| + \|F^2 - F_0^2\|.$$ 

so the action of $U(M)$ on $F_M$ is indeed continuous. Since $U_M$ is invariant under this action and $\|F^2 - 1\| = \|UF^2U^{-1} - 1\|$, it extends to an action on $F^+_M$. $\square$

Recall that an $A$-linear operator $D$ on $M$ is called regular if it is closed densely defined, $\text{dom} \ D^*$ is dense and $(1 + D^*D)$ has dense image. The unbounded picture of $KK$-theory as developed in [4] has the advantage that it allows for an easier description of the product.

**Definition 60** [11]

$$R_M := \{D \text{ regular on } M \mid D^* = D, \ D \text{ odd}, (1 + D^2)^{-1} \text{ compact}\}.$$ 

Let $J : R_M \rightarrow F_M$ be the map $D \mapsto D(1 + D^2)^{-\frac{1}{2}}$. $J$ maps $R_M$ indeed to $F_M$, is injective, and $\text{im}(J) \cap U_M = \emptyset$, hence we may regard $R_M$ as a subset of $F^+_M$ [4, 11].

**Definition 61** Let $R^+_M$ be the pointed set $R_M \cup *$, equipped with the subspace topology induced by $R^+_M \subset F^+_M$.

Since $R^+_M \subset F^+_M$ is invariant under the conjugation action of the even part $U^0(M)$ of $U(M)$, the $U(M)$-action on $F_M$ induces a continuous action of $U^0(M)$ on $R^+_M$.

If $M$ and $M'$ are Hilbert modules over $A$ and $A'$, we define

$$\mu : R_M \times R_{M'} \rightarrow R_{M \otimes M'}, \ (D, D') \mapsto D \otimes 1 + 1 \otimes D'.$$
3.1 Twisted K-theory

Proposition 62 \([11, 3.6]\) \(\mu : R_M \times R_{M'} \to R_{M \oplus M'}\) is continuous and has a continuous pointed extension \(\mu^+ : R^+_M \wedge R^+_M \to R^+_{M \oplus M'}\).

For a finite dimensional real inner product space \((V, g)\), we denote the Clifford algebra of \((V \otimes \mathbb{C}, g \otimes \mathbb{C})\) by \(\text{Cl}(V)\). Note that \(\text{Cl}(V)\) is functorial in \((V, g)\) and that there are natural isomorphisms \(\text{Cl}(V) \otimes \text{Cl}(W) \cong \text{Cl}(V \oplus W)\). Consider the \(\mathbb{Z}_2\)-graded vector space \(C^\infty_0(V, \text{Cl}(V))\) and define \(E_V := L^2(V, \text{Cl}(V))\) to be its \(L^2\)-completion.

Definition 63 \([11]\)

- The Dirac operator \(\partial_V \in R_{E_V}\) is the closure of the operator on \(C^\infty_0(V, \text{Cl}(V))\) given on the functions of pure degree by

\[
\partial_V \sigma(x) := (-1)^{|\sigma|} \sum_{i \in I} \left( \frac{\partial \sigma}{\partial v_i}(x) \right) v_i, \quad x \in V
\]

where \((v_i)_{i \in I}\) is any orthonormal basis of \(V\).

- The Clifford operator \(L_V \in R_{E_V}\) is defined by

\[
L_V(\sigma)(v) = v \sigma(v), \quad v \in V, \quad \sigma \in C^\infty_0(V, \text{Cl}(V))
\]

- The Bott-Dirac operator \(D_V \in R_{E_V}\) is \(D_V := \partial_V + L_V\).

Proposition 64 \([11, 3.9]\) There are canonical isomorphisms \(E_V \otimes E_W \cong E_{V \oplus W}\). Using these isomorphisms, we have identities

\[
\partial_{V \oplus W} = \mu(\partial_V, \partial_W) \quad \text{and} \quad D_{V \oplus W} = \mu(D_V, D_W).
\]

Recall that an orthogonal spectrum is defined as a module over the orthogonal sphere spectrum, therefore the data for an orthogonal spectrum \(K\) is given by

- a pointed space \(K(V)\) with a left \(O(V)\)-action
- \(O(V) \times O(W)\)-equivariant maps \(\sigma_{V,W} : K(V) \wedge S^W \to K(V \oplus W)\)

for all finite dimensional real inner product spaces \(V, W\). Moreover, the appropriate associativity and unit diagrams have to commute.

To obtain the spaces of the spectrum representing complex \(K\)-theory, we regard \(H'(V) := \text{Cl}(V) \otimes E_V\) as a right \(\text{Cl}(V)\)-module, where \(\text{Cl}(V)\) acts by right multiplication on \(\text{Cl}(V)\) and trivially on \(E_V\), and define \(K'(V) := R^+_{H'(V)}\).

To construct the \(O(V)\)-action, let \(\text{Pin}^c(V) \subset \text{Cl}(V)^*\) be the subgroup generated by elements of \(V\) of norm one. The \(\text{Pin}^c\)-representation

\[
\rho_V : \text{Pin}^c(V) \to O(V)
\]
is defined as follows. The subspace $V \subset \text{Cl}(V)$ is invariant under the twisted conjugation action

$$Pin^c(V) \times \text{Cl}(V) \to \text{Cl}(V), \quad (g, v) \mapsto (-1)^{|g|} gv g^{-1}$$

and $\rho_V$ is simply the restriction of this action to $V$. $Pin^c(V)$ is a central $S^1$-extension of $O(V)$, i.e.

$$1 \to S^1 \to Pin^c(V) \xrightarrow{\rho_V} O(V) \to 1.$$  

We define an action of $Pin^c(V)$ on $E_V$ by

$$g\sigma(v) := \rho_V(g)\sigma(\rho_V(g)^{-1}v)$$

for $g \in Pin^c(V), \sigma \in C^\infty_0(V, \text{Cl}(V))$ and $v \in V$. Combining this with the $Pin^c(V)$-action on $\text{Cl}(V)$ given by left multiplication, we obtain a unitary representation $\tilde{\mu}_V : Pin^c(V) \to U \bigl(\mathbb{H}'(V)\bigr)$. The twisted conjugation action on $\mathbb{H}'(V)$ given by $p : F \mapsto (-1)^{\deg(p)} g'_V(p) F g'_V(p)^{-1}$ factors through an $O(V)$-action. Moreover, this $O(V)$-action induces one on $\mathbb{H}'_+(V)$ [11]. The subspace $\mathbb{R}'_+(V)$ is $O(V)$-invariant, thus we get an action on $K'_+(V)$.

For $v \in V$, denote the operator on $\text{Cl}(V)$ given by left Clifford multiplication by $l_v$ and observe that the map

$$\tilde{\eta}_V : V \to \mathbb{R}'_+(V), \quad v \mapsto \mu(l_v, D_V)$$

has a continuous pointed extension $\eta_V : S^V \to \mathbb{R}'_+(V) = K'_+(V)$ since

$$\|J(\tilde{\eta}_V(v))^2 - 1\| = \left\| \frac{(l_v \otimes 1 + 1 \otimes D_V)^2}{1 + (l_v \otimes 1 + 1 \otimes D_V)^2} - 1 \right\| \overset{v \to \infty}{\to} 0.$$

We define the structure maps of $K'$ by

$$\begin{array}{c}
K'(V) \wedge S^W \xrightarrow{1 \wedge \eta_W} K'(V) \wedge K'(W) \xrightarrow{\mu} R^+_H(V \oplus W) \cong R^+_H(V \oplus W) = K'_+(V \oplus W).
\end{array}$$

These maps are $O(V) \times O(W)$-equivariant, associative and unital, hence $K'$ is indeed an orthogonal spectrum.


Moreover,

$$\mu : K' \wedge K' \to K' \text{ given by } K'(V) \wedge K'(W) \xrightarrow{\mu} R^+_H(V \oplus W) = K'(V \oplus W)$$

induces the product in $K$-theory and $\mu, \eta$ make $K'$ a ring spectrum.
With this definition of the $K$-spectrum at hand, we will now describe the modification that enables us to define $BPU$-twisted $K$-theory. The spaces of the spectrum $K'$ are given as spaces of operators on Hilbert modules indexed on $I$. The idea is to tensor all these modules by a fixed Hilbert space in such a way that the projective unitary group of this space acts on the spectrum, so let us fix the separable $\mathbb{Z}_2$-graded Hilbert space $H := L^2(\mathbb{R}, \mathcal{Cl}(\mathbb{R}))$ and define

$$PU := U^0(H)/S^1 \quad H(V) := H'(V) \otimes H \quad K(V) := R_{H(V)}^+.$$ 

To get the $O(V)$-action, we extend the unitary $\mathcal{Cl}(V)$-representation $\varrho' : Pin^c(V) \rightarrow U_{H'(V)}$ trivially, i.e.

$$\varrho(V) := \varrho'(V) \otimes 1 : Pin^c(V) \rightarrow U_{H(V)} = U_{H'(V) \otimes H}.$$ 

As before, the twisted conjugation action factors through an $O(V)$-action, which yields an $O(V)$-action on $K(V)$. Similarly, we define the structure maps by

$$K(V) \otimes S^W \xrightarrow{\varrho W} K(V) \otimes K'(W) \xrightarrow{\mu^+} R_{H(V) \otimes H'(W)} \cong R_{H(V \oplus W)} = K(V \oplus W).$$ 

The $O(V) \times O(W)$-equivariance as well as the unitality and associativity follow as before, so we have an orthogonal spectrum, again. The $PU$-action defined by

$$PU \times K(V) \quad (u, D) \mapsto (1 \otimes u)D(1 \otimes u)^{-1}$$

is $O(V)$-equivariant as well as compatible with the structure maps and thus yields a $PU$-action on $K$.

Moreover,

$$\phi_V := \mu(\cdot, D_R) : K'(V) = R_{H'(V)}^+ \rightarrow R_{H(V)}^+ = K(V)$$

is a weak equivalence for all $V \neq \{0\}$ [11, 3.12]. Since these maps are $O(V)$-equivariant and compatible with the structure maps, they assemble to a stable equivalence $\phi : K' \rightarrow K$.

**Definition 66** Let $\mathcal{P}$ be the $E_\infty$-operad given by $\mathcal{P}(j) := \text{Isom}^0(H^\otimes j, H)$ with the obvious $\Sigma_j$-action and structure maps, where $\text{Isom}^0$ denotes the space of even isometric isomorphisms with the norm topology.

Recall that $\mathcal{P}[\mathcal{U}]$ is the set of algebras over $\mathcal{P}$, i.e. spaces $X \in \mathcal{U}$ with morphisms $\theta_j : \mathcal{P}(j) \times X^j \rightarrow X$ which are compatible with the operadic structure of $\mathcal{P}$.

**Lemma 67** $PU$ is a group object in $\mathcal{P}[\mathcal{U}]$. 


Proof: Define the action of $\mathcal{P}$ by

$$\theta : \mathcal{P}(j) \times PU^j \to PU, \quad (p, [u_1], ..., [u_j]) \mapsto [p(u_1 \otimes ... \otimes u_j)p^{-1}]$$

and check that it commutes with the product $\mu : PU \times PU \to PU$ and the unit $e : * \to PU$. □

Since the category of spectra $\mathcal{S}$ is enriched, tensored and cotensored over $\mathcal{U}$, the category $\mathcal{P}[\mathcal{S}]$ of $\mathcal{P}$-algebras in $\mathcal{S}$ is defined.

**Lemma 68** $K$ is in $\mathcal{P}[\mathcal{S}]$.

**Proof:** Each $p \in \mathcal{P}(j)$ defines a map $R_{M \otimes H ... \otimes H} \to R_{M \otimes H}$ for all $M$, given by $D \mapsto (1 \otimes p)D(1 \otimes p)^{-1}$. Using this, the action of $\mathcal{P}$ is defined by the maps

$$\phi_{V_1, ..., V_j} : \mathcal{P}(j)_+ \wedge K(V_1) \wedge ... \wedge K(V_j) \xrightarrow{1 \times \mu^+} \mathcal{P}(j) \times R_{H'(V_1 \oplus ... \oplus V_j) \otimes H ... \otimes H} \to R_{H'(V_1 \oplus ... \oplus V_j) \otimes H} = K(V_1 \oplus ... \oplus V_j).$$

□

**Lemma 69** The $PU$-action on $K$ is compatible with the $\mathcal{P}$-actions on $PU$ and $K$, i.e. $m : PU_+ \wedge K \to K$ is a map in $\mathcal{P}[\mathcal{S}]$.

**Proof:** We have to show commutativity of

$$\begin{array}{ccc}
\mathcal{P}(j)_+ \wedge PU_+ \wedge K(V_1) \wedge ... \wedge PU_+ \wedge K(V_j) & \xrightarrow{m} & \mathcal{P}(j)_+ \wedge K(V_1) \wedge ... \wedge K(V_j) \\
PU_+ \wedge K(V_1 \oplus ... \oplus V_j) & \xrightarrow{m} & K(V_1 \oplus ... \oplus V_j). \\
\end{array}$$

On inspection of the definitions, this reduces to the observation that for all $D_i \in K(V_i)$ and $[u_i] \in PU$

$$\mu^+((1 \otimes u_1)D_1(1 \otimes u_1)^{-1}, ..., (1 \otimes u_j)D_j(1 \otimes u_j)^{-1}) = (1 \otimes u_1 ... \otimes u_j)\mu^+(D_1, .., D_j)(1 \otimes u_1 ... \otimes u_j)^{-1}. \Box$$

**Proposition 70** Any $p \in \mathcal{P}(2)$ endows $Ho\mathcal{S}_{BPU}$ with the structure of a symmetric monoidal category.

**Proof:** First, recall that $BPU$ is a $K(\mathbb{Z}, 3)$-space. We will make repeated use of this fact throughout the proof.

In Proposition 32 we have shown how to construct a symmetric monoidal structure on $Ho\mathcal{S}_B$ from an action of a pointed $E_\infty$-operad on $B$. $\mathcal{P}$ is an $E_\infty$-operad and it acts on $BPU$, but it is not pointed and therefore we can’t
apply Proposition 32 directly. However, the only parts of the proof where the pointedness of the operad were used were the construction of the unit isomorphism and the coherence diagram involving it, so we will just reproof that.

We define the bifunctor associated to $p \in P$ to be

$$p := L(\theta(p), \circ \triangledown) : \text{Ho} \mathcal{S}_{BPU} \times \text{Ho} \mathcal{S}_{BPU} \to \text{Ho} \mathcal{S}_{BPU}$$

and the unit to be $1_{BPU} := i_* S$.

To get the unit isomorphism, we need to find a homotopy from $p_e : BPU \rightarrow BPU \times BPU \theta(p) \rightarrow BPU$ to the identity. Note that

$$\theta(p) : BPU \times BPU \to BPU$$

induces the sum on $H^3$, i.e.

$$\theta(p)_* = + : H^3(X, \mathbb{Z}) \times H^3(X, \mathbb{Z}) \to H^3(X, \mathbb{Z}) \ \forall X \in \mathcal{U}$$

and thus $p_e$ is indeed homotopic to the identity. Choosing such a homotopy defines by Proposition 26 a natural transformation $\phi : Lp_e ! \to Lid_1$. This yields the required natural isomorphism

$$p(1_{BPU}, X) \cong Lp_e X \cong QX \cong X.$$ 

Moreover, the identity component of $\mathcal{U}(BPU, BPU)$ is simply connected (since $H^2(BPU, \mathbb{Z}) = 0$), so all homotopies connecting $p_e$ and $id_{BPU}$ are homotopic and the isomorphism does not depend on the choice of the homotopy. It remains to check commutativity of

$$\tilde{p}(1_{BPU}, p(X, Y)) \tilde{p}(X, Y).$$

Recall that the associativity transformation was defined using a path in the operad connecting $\gamma(p; 1, p)$ and $\gamma(p; p, 1)$. We denote the natural isomorphisms induced by the various homotopies by $\phi_i$. The commutativity of the
The above diagram is the same as the commutativity of
\[
\theta(p):(i_* \times id)\theta(p):(QX \wedge QY) \overset{\sim}{\longrightarrow} p_\ast \theta(p):(QX \wedge QY)
\]
\[
\theta(p):(1 \times \theta(p)):(i_* \times id):(QX \wedge QY) \overset{\phi_0}{\longrightarrow}
\]
\[
\theta(p):(\theta(p) \times 1):(i_* \times id):(QX \wedge QY) \overset{\phi_1}{\longrightarrow}
\]
\[
\theta(p):(p_e \times id):(QX \wedge QY) \overset{\phi_2}{\longrightarrow}
\]
for all \(X,Y \in \mathcal{S}_{BPU}\). Note that \(H^2(BPU \times BPU, \mathbb{Z}) = 0\) and hence all homotopies between maps \(f, g : BPU \times BPU \to BPU\) are homotopic, so the application of Propositions \(26, 27\) finishes the proof. \(\Box\)

**Proposition 71** The gauge group \(G(EPU)\) of bundle automorphisms of the principal \(PU\)-bundle \(EPU \to BPU\) is connected.

**Proof:** A gauge transformation \(g \in G(EPU)\) defines a principal \(PU\)-bundle \(P_g : BPU \times S^1\) by identifying \(EPU \times \{0\}\) and \(EPU \times \{1\}\) via \(g\) in \(EPU \times I\). Now, \(H^3(BPU \times S^1, \mathbb{Z}) \cong H^3(BPU, \mathbb{Z})\) and therefore \(P_g\) is isomorphic to \(\pi^*EPU\), where \(\pi : BPU \times S^1 \to BPU\)
is the projection. This implies that one can extend \(P_g\) to a \(PU\)-bundle on \(BPU \times D^2\). The choice of such an extension yields a path connecting \(g\) and \(id_{EPU}\) in \(G(EPU)\). \(\Box\)

**Lemma 72** \(E := BPUK := EPU \times_{PU} K\) is a monoid in \((Ho\mathcal{S}_{BPU}, p)\) for all \(p \in \mathcal{P}(2)\).

**Proof:** The non-pointedness of \(\mathcal{P}\) obstructs the direct application of Lemma \(35\) thus we will redo those parts of the proof where the basepoints of the operad were used, namely the construction of the unit \(\eta : 1_{BPU} \to E\). Note that the map \(\tilde{\eta} : S \xrightarrow{\eta} K' \xrightarrow{\phi} K\) is a homotopy unit for each \(p \in \mathcal{P}(2)\) since
\[
\begin{align*}
K'(V) \wedge K'(W) &\xrightarrow{\mu^+} K'(V \oplus W) \\
\downarrow_{\phi \wedge \phi} &\quad \downarrow_{\phi} \\
K(V) \wedge K(W) &\xrightarrow{\mu^+} R^+_{H\,(V \oplus W) \otimes H \otimes H} K(V \oplus W)
\end{align*}
\]
is homotopy commutative. Define $\eta : 1_{BPU} = i_\ast S \to E$ to be the adjoint of $\tilde{\eta}$. The last thing to check is the commutativity of

$$p(1_{BPU}, E) \longrightarrow p(E, E) \longrightarrow E.$$ 

By the definition of the unit isomorphism, all we have to find are maps $\psi_1, \psi_2$ that make the following diagram homotopy commutative. In the proof of Lemma 35, these were constructed from the action of $\mathcal{P}$. Since $p_e$ need not be of the form $\theta(q)$ for some $q \in \mathcal{P}(1)$, we cannot copy that construction.

To obtain $\psi_1$, let us first choose an isomorphism $\tilde{\psi}_1' : EPU \to p_e^* EPU$ and use this to define $\psi_1 : E \to p_e^* E$.

Letting $\psi_1 : p_e E \to E$ be the adjoint, the quadrangle $\text{I}$ in the diagram commutes in $HoS_{BPU}$ since $\tilde{\eta}$ is a homotopy unit. Moreover, since the gauge group
of $EPU$ is connected, one can extend the isomorphism $\tilde{\psi}^\prime_1$ to $\tilde{\psi}^\prime_2 : \pi^*EPU \to h^*EPU$ in such a way that $i_0^*(\tilde{\psi}^\prime_2) = \tilde{\psi}^\prime_1$ and $i_1^*(\tilde{\psi}^\prime_2) = id_{EPU}$. Now, $\tilde{\psi}^\prime_2$ yields $\tilde{\psi}_2 : \pi^*E \to h^*E$. Defining $\psi_2$ to be the adjoint of $\tilde{\psi}_2$ completes the argument. 

3.2 Twisted $Spin^c$-cobordism

To define twisted $Spin^c$-cobordism, we build upon the construction of $MSpin^c$ in [23]. Recall from the previous section the representation

$$\varrho(V) : Pin^c(V) \to U_{H(V)}.$$  

Restricting it to the subgroup of even elements $Spin^c(V) \subset Pin^c(V)$, we obtain a representation $Spin^c(V) \to U_{H(V)}^0$ which we call again $\varrho(V)$. Since $U_{H(V)}^0$ is contractible, $U_{H(V)}^0 \times_{Spin^c(V)} V$ is a model for the universal $Spin^c(V)$-bundle. We define the spaces of the spectrum as

$$MSpin^c(V) := Th(U_{H(V)}^0 \times_{Spin^c(V)} V) = U_{H(V)}^0 \wedge_{Spin^c(V)} S^V.$$  

The $O(V)$-action is obtained by combining the $O(V)$-action induced by the conjugation action of $Spin^c(V)$ on $U_{H(V)}^0$ and the standard action on $S^V$. To define the structure maps

$$\sigma_{V,W} : MSpin^c(V) \wedge S^W \to MSpin^c(V \oplus W),$$  

we use the canonical isomorphism $H(V \oplus W) \cong H(V) \otimes H'(W)$ and the homeomorphism $f : S^V \wedge S^W \to S^{V \oplus W}$ and define

$$\sigma_{V,W}[U,v,w] := [U \otimes 1, f(v,w)].$$  

Note that $PU = PU^0(H)$ acts on $MSpin^c$ via

$$PU \times MSpin^c(V) \to MSpin^c(V), \quad ([u], [U,v]) \mapsto [(1 \otimes u)U(1 \otimes u)^{-1}, v].$$  

**Lemma 73** $MSpin^c$ is in $\mathcal{P}(S)$ and the $PU$-action on $MSpin^c$ is compatible with the $\mathcal{P}$-algebra structures on $PU$ and $MSpin^c$.

Finally, there is an obvious map

$$\alpha : MSpin^c \to K, \quad \alpha(U,v) := U\tilde{\eta}(v)U^{-1} \text{ for } U \in U_{H(V)}^0, v \in V$$

which is a $PU$-equivariant map of $\mathcal{P}$-spectra.
3.2 Twisted $\text{Spin}^c$-cobordism

**Proposition 74** [23] $M\text{Spin}^c$ represents $\text{Spin}^c$-cobordism and $\alpha$ induces the classical orientation homomorphism $M\text{Spin}^c \to K^*$.

Analogous to the $K$-spectrum over $BPU$, we define an ex-spectrum over $BPU$ with fibre $M\text{Spin}^c$ by $\mathcal{M}_{\text{Spin}^c}(V) := EPU \times_{PU} M\text{Spin}^c$. A similar proof as for the parametrised $K$-spectrum shows that $\mathcal{M}_{\text{Spin}^c}$ is a parametrised ring spectrum over $BPU$.

**Definition 75** The parametrised $M\text{Spin}^c$-orientation is

$$\beta := 1 \times \alpha : EPU \times_{PU} M\text{Spin}^c \to EPU \times_{PU} K.$$
Bibliography


[24] G.M. Kelly, On MacLane’s Conditions for Coherence of Natural Associativities, Commutativities, etc, Journal of Algebra 1 (1964)


