Hiobseminar WS 18/19
Index theory

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Introduction

The idea of this seminar is to get an overview over the classical index theory for elliptic differential operators and some more recent aspects involving the K-theory of C^*-algebras and constructions with Lie groupoids. The seminar has three parts:

1. The classical Atiyah-Singer index theorem: Here the main goal is to understand the statement, the deduction of cohomological index formulas, and the application to geometry and topology. We will discuss Dirac operators in detail.

2. C^*-algebra K-theory: Here we want to get a quick overview on the basic construction of K and KK-theory. As appropriate for a HIGHER INVARIANTS OS we will of course construct spectrum-valued functors and use infinity categories and the algebraic K-theory machine. As a basic application we show that the algebra $C(M)$ for a closed manifold is dualizable, calculate its dual, and derive Poincaré duality statements and a first version of the index theorem. We also consider the crossed product construction for G-C^*-algebras and Connes' Thom isomorphism.

3. Here (again in order to pay tribute to the HIGHER aspect of HIOB) we develop the theory of Lie groupoids as presentations of smooth stacks. We then consider the reduced C^*-algebra of a Lie groupoid and (at least morally) its K-theory as a K-theory of the associated smooth stack. Using the sheaf/stack language we can give a simple view on the deformation-to-the-normal-cone construction which allows to define the tangent groupoid of a groupoid and finally the analytic index map for groupoids. The culmination of the seminar is a sketch of Connes' proof of the family index theorem using these Lie groupoid methods.

We have tried to describe the talks in quite a detail in order to keep/force the philosophical red line of the seminar. It is not expected (and not possible in 90 minutes) to give detailed proofs of all results. But the participants should understand the definitions of the objects and get an idea why the statements are true. It is very important to discuss the given examples since they are important to connect the different talks and to glue the theory to an entity. We expect that the speakers have a profound pre-knowledge of the respective topic.

The following is a disclaimer (by Uli). When writing the description of the talks I wrote what I think is true and important to know in order to get the big picture. I have not looked up references in each case. I might have forgotten additional assumptions and probably made sign mistakes in the formulas and other flaws. I would be happy about improvements of any kind. I am also prepared for intense discussions with the prospective speakers before the actual talks.

Part I. The classical index theorem

1. The Atiyah-Singer Index Theorem

The origin of index theory is the desire to calculate the index of an elliptic differential operator D on a closed manifold M explicitly in terms of its principal symbol $\sigma(F)$. Here D is considered as a linear map

$$D : C^\infty(M, E) \to C^\infty(M, F)$$

for complex vector bundles E and F on M, and the index is the integer

$$\mathtt{index}(D) = \dim \mathtt{ker}(D) - \dim \operatorname{coker}(D) \ .$$

This definition should be justified by arguing that the dimensions appearing in this formula are finite.

Remark 1.1. One could argue as follows.

We first introduce the notion of distribution sections, smoothing operators, and a version of Rellich's theorem.

Let $E \to M$ be a vector bundle on some manifold (not necessarily compact). Recall that a locally convex vector space is a topological vector space whose topology is generated by a family of seminorms. For every metric h on E, differential operator P on E, and compact subset K in M we can define a seminorm

$$\|\phi\|_{K,h,P} := \sup_{n \in K} h(P\phi)$$

on $C^\infty(M, E)$ inducing a locally convex structure (actually of a Fréchet space). We define the locally convex vector space of compactly supported sections by taking the colimit in locally convex vector spaces

$$C_c^\infty(M, E) := \mathop{\mathtt{colim}}_{K} C_K^\infty(M, E) \ ,$$

where $C_K^\infty(M, E)$ is the locally convex subspace of $C^\infty(M, E)$ of sections with support in K, and K runs over the compact subsets of M. Let E^* denote the dual bundle of E and $D(M)$ the bundle of densities on M. Then we have a duality pairing

$$C^\infty(M, E) \times C_c^\infty(M, E^* \otimes D(M)) \to \mathbb{C} \ , \quad (\phi, \psi) \mapsto \int_M \langle \phi, \psi \rangle$$

(note that the integrand is a compactly supported density in the natural way and hence can be integrated). One then defines the locally convex vector space of distribution sections of E by

$$C^{-\infty}(M,E) := C_c^\infty(M, E^* \otimes D(M))^* \ .$$

These are objects which in local coordinates and trivializations of E are represented by distributions. The pairing then induces an embedding

$$C^\infty(M,E) \to C^{-\infty}(M,E) \ .$$

Lemma 1.2. *This embedding has a dense range.*

A differential operator $D : D : C^{-\infty}(M,E) \to C^\infty(M,F)$ extends by continuity to an operator $D : C^{-\infty}(M,E) \to C^\infty(M,F)$ (denoted by the same symbol).

We consider a general operator $S : C^\infty(M,E) \to C^\infty(M,F)$.

Definition 1.3. *S is called smoothing if it extends by continuity to an operator (same symbol)*

$$S : C^{-\infty}(M,E) \to C^\infty(M,F) \ .$$

The following is now shown using e.g. pseudo-differential operator techniques.

Lemma 1.4. *The differential operator $D : C^\infty(M,E) \to C^\infty(M,F)$ is elliptic if and only if there exists an operator $Q : C^\infty(M,F) \to C^\infty(M,E)$ (called parametrix) such that $DQ = 1 + S$ and $QD = 1 + T$, where S and T are smoothing.*

For k in $\mathbb{N}$ we can define the subspaces $C^k(M,E)$ of $C^{-\infty}(M,E)$ by closing $C^\infty(M,E)$ with respect to the topology generated by the seminorms $\| - \|_{K,h,P}$ for operators P of degree $\leq k$. We write $C(M,E) := C^0(M,E)$.

Recall that a locally convex vector space is called a Banach space if is isomorphic to a Banach space as a locally convex vector space, i.e., the norm exist but is not part of the data.

Proposition 1.5 (a version of the Rellich theorem)**.** *If M is compact, then $C^k(M,E)$ is a Banach space and the embedding $C^k(M,E) \to C^{-\infty}(M,E)$ is compact.*

Consequently a smoothing operator extends to a compact operator on $C^k(M,E)$

$$C^k(M,E) \stackrel{compact}{\to} C^{-\infty}(M,E) \stackrel{S}{\to} C^\infty(M,E) \to C^k(M,E) \ .$$

Let Q be a parametrix for D and $DQ = 1 + S$ and $QD = 1 + T$ with S and T smoothing. For the moment we write $D^{-\infty}$ and D^∞ for D acting on distribution and smooth sections.

Lemma 1.6. *1. We have an equality*

$$\mathtt{ker}(D^\infty) = \mathtt{ker}(D^{-\infty}) \ .$$

2. *The natural map* $\mathrm{coker}(D^\infty) \to \mathrm{coker}(D^{-\infty})$ *is a linear isomorphism.*

Proof. It is clear that $\mathtt{ker}(D^\infty) \subseteq \mathtt{ker}(D^{-\infty})$. If ϕ is in $\mathtt{ker}(D^{-\infty})$, then we have $\phi = T(\phi)$, hence $\phi \in C^\infty(E, M)$ and so $\phi \in \mathtt{ker}(D^{-\infty})$.

Let $[\phi]$ be in $\mathrm{coker}(D^{-\infty})$. Then we have $[\phi] = [DQ(\phi) - S(\phi)] = -[S(\phi)]$. Therefore $[\phi]$ belongs to the image of $\mathrm{coker}(D^\infty)$. Let now $[\psi]$ be in $\mathrm{coker}(D^\infty)$ which is send to zero in $\mathrm{coker}(D^{-\infty})$. Then we have $\psi = D\phi$ for some $\phi \in C^{-\infty}(M, E)$. This implies $\phi = Q\psi - T\phi$, hence $\phi \in C^\infty(M, E)$ and thus $[\psi] = 0$. □

Proposition 1.7. *If M is compact, then* $\mathtt{ker}(D)$ *and* $\mathrm{coker}(D)$ *are finite-dimensional*

Proof. We equip $\mathtt{ker}(D)$ the topology induced from $C(M, E)$. Then $\mathtt{ker}(D)$ is a Banach space. Moreover $\mathtt{id}_{\mathtt{ker}(D)} = T_{|\mathtt{ker}(D)}$ is compact. Consequently, $\mathtt{ker}(D)$ is finite-dimensional.

We now observe that $\mathtt{im}(D^{-\infty}) \cap C(M, F)$ is closed (this requires some work) and we have an exact sequence

$$0 \to \mathtt{im}(D^{-\infty}) \cap C(M, F) \to C(M, F) \to \mathrm{coker}(D) \to 0 \ .$$

Then we consider on $\mathrm{coker}(D)$ the Banach space topology induced from $C(M, F)$. Since $\mathtt{id}_{\mathrm{coker}(D)}$ coincides with the compact operator induced by S we again conclude that $\mathrm{coker}(D)$ is finite-dimensional. □

□

The Atiyah-Singer index theorem provides a formula for the integer $\mathtt{index}(D)$ in terms of its symbol using homotopy theory. The purpose of the talk is to explain the statement of the Atiyah-Singer index theorem.

It involves the multiplicative generalized cohomology theory KU^* and its version KU^*_S with support in a subset S. We need the following facts:

1. If H^* is a generalized cohomology theory and S is a closed subset of a space X, then one defines
$$H^*_S(X) := H^*_S(X, X \setminus S) \ .$$
Here as assume that the relative version of H^* is defined on pairs (X, U) of spaces with open subsets. If $S \subseteq S'$, then we have the extension-by-zero map $H^*_S(X) \to H^*_{S'}(X)$ induced by the map of pairs $(X, X \setminus S') \to (X, X \setminus S)$. For an arbitrary subset S (or a filtered collection of subsets) we define
$$H^*_S(X) := \mathop{\mathtt{colim}}_{K \subseteq S} H^*_K(X) \ ,$$
where K runs over the closed subsets of X contained in S. From the pair sequence we have the exact sequence
$$\dots \to H^*_S(X) \to H^*(X) \to H^*(X \setminus S) \to H^{*+1}_S(X) \to \dots$$
where $H^*(X \setminus S)$ has to be interpreted appropriately if S is not closed. If $S \subseteq X$ and $X \subseteq Y$ is open, then we have an excision isomorphism $H^*_S(X) \cong H^*_S(Y)$.

2. We have $KU^* \cong \mathbb{Z}[b, b^{-1}]$ with $\deg(b) = 2$. In particular we have fixed an isomorphism $KU^0 \cong \mathbb{Z}$.

3. A vector bundle E on a space M represents a class $[E]$ in $KU^0(M)$. In particular $[\mathbb{C}^n \to *]$ represents the number n in $KU^0(*)$.

4. A map $\sigma : E \to F$ between vector bundles on a space M which is an isomorphism outside of a closed subset S represents a class $[E, F, \sigma]$ in $KU^0_S(M)$ with support S. More generally, a finite complex $\mathcal{E} := (E_0 \to E_1 \to \cdots \to E_k)$ of vector bundles on M represents an element in $K^0_S(M)$, where S is such that $\mathcal{E}$ is exact outside S. For two complexes $\mathcal{E}, \mathcal{F}$ the tensor product represents the product, i.e.,
$$[\mathcal{E}] \cup [\mathcal{F}] = [\mathcal{E} \otimes \mathcal{F}]$$
in $KU^0_{S\cap T}(M)$, where $\mathcal{E}$ is exact outside S and $\mathcal{F}$ is exact outside T.

5. The cohomology theory KU is complex oriented. The complex orientation is fixed by the class
$$\beta^{KU} := b(1 - [L]) \text{ in } KU^2(\mathbb{CP}^\infty) ,$$
where L is the tautological bundle. We then have $KU^*(\mathbb{CP}^\infty) \cong KU^*[[\beta^{KU} b^{-1}]]$.

6. The complex orientation provides a natural Thom class ν^{KU}_H in $KU^2_{0_H}(H)$ for every complex line bundle H on a space M, where 0_H is the zero section, and which is uniquely characterized by
$$\beta^{KU} = 0^*_L \nu^{KU}_L ,$$
where $L \to \mathbb{CP}^\infty$ is the tautological bundle.

7. More generally, by the splitting principle or Remark 1.8, we have a natural Thom class ν^{KU}_V in $KU^{2n}_{0_V}(V)$ for every complex vector bundle V of dimension n such that
$$\nu^{KU}_{V\oplus W} = \mathtt{pr}^*_V \nu^{KU}_V \cup \mathtt{pr}^*_W \nu^{KU}_W .$$

References are [Ati67], [Fri78], [Ada95, II.2] for complex orientations.

Remark 1.8. Here is a direct way to define the Thom class $\nu^{KU}(V)$ of a complex vector bundle.

We start with a complex vector bundle (V, J). Then $V \otimes_{\mathbb{R}} \mathbb{C} \simeq V^{1,0} \oplus V^{0,1}$ is the decomposition into the eigenspaces of $J \otimes 1$ for the eigenvalues i and $-i$. Let $\pi : V \to M$ be the projection. We then define the complex
$$\mathcal{S}^c(V, J) := \pi^*\Lambda^0 V^{1,0} \to \pi^*\Lambda^1 V^{1,0} \to \cdots \to \pi^*\Lambda^{\dim(V)} V^{1,0} ,$$
where the differential at the point v in V is given by exterior multiplication by $v^{1,0}$, where $v \otimes 1 = v^{1,0} + v^{0,1}$ is the canonical decomposition. This complex is exact outside the zero section 0_V. We have
$$\nu^{KU} = b^{\dim(V)/2}[\mathcal{S}^c(V)] .$$

Indeed, one directly checks the compatibility with sums and the normalization on line bundles. $\square$

In order to state the index theorem we furthermore need basic facts about orientations for generalized cohomology theories and the integration in the complex oriented case. Here are more details since I do not know a nice reference. Let H^* be a generalized cohomology theory which is complex oriented.

1. A stable vector bundle on a space X is an equivalence class of vector bundles generated by isomorphism and the operation $V \sim V \oplus \mathbb{R}_X$, where $\mathbb{R}_X$ is the trivial 1-dimensional bundle on X. The set of stable vector bundles is in bijection to the set of homotopy classes $[X, BO]$.

2. If $p : E \to B$ is a map between manifolds, then we have the notion of a stable normal bundle N_p. It is a stable vector bundle over E. If we choose a fibrewise embedding $i : E \to B \times \mathbb{R}^N$, then N_p is represented by the normal bundle $N(i)$ of i.

3. We can choose an open embedding $\tilde{i} : N(i) \to B \times \mathbb{R}^N$ such that $\tilde{i} \circ 0_{N(i)} = i$. As for any open embedding, if K is a closed subset of $N(i)$, then we have the excision isomorphism $\tilde{i}_* : H^*_K(N(i)) \to H^*_K(B \times \mathbb{R}^n)$.

4. After increasing N, if necessary, any two choices of such embeddings i and $\tilde{i}$ are homotopic. In particular the stable normal bundle is well-defined.

5. One can talk about complex structures (as a homotopy theoretic datum) on stable bundles. Such a structure is represented by a complex structure on a bundle $V \oplus \mathbb{R}^n$ for sufficiently large N (such that $\dim(V) + n$ is even) subject to the equivalence relation generated by homotopy and stabilization. Equivalently one can consider homotopy classes of lifts in

$$\begin{tikzcd} & BU \arrow[d] \\ X \arrow[r] \arrow[ur, dotted] & BO \end{tikzcd} .$$

6. We now assume that the stable normal bundle of p has a complex structure. Then any representative $N_p \to E$ has a Thom class $\nu^H_{N_p}$ which is compatible with stabilization. This in particular applies to the representative $\mathtt{pr} : N(i) \to E$.

7. $f : W \to B$, and S is a family of subsets on B, then c/S denotes the collection of subsets K of W such that $f(K) \subseteq T$ for some T in S and $f_{|K}$ is proper. We let $c/B := c/All$ and call $H^*_{c/All}(W)$ the cohomology with proper support over B. In particular, for the projection $B \times \mathbb{R}^n \to B$ we have a suspension isomorphism

$$H^*_{c/S}(B \times \mathbb{R}^n) \simeq H^{*-n}_S(B) .$$

Definition 1.9. *The integration map*

$$p^H_! : H^*_S(E) \to H^{*-\dim(E)+\dim(B)}_S(B)$$

is defined as the composition

$$H^*_{c/S}(E) \stackrel{\mathtt{pr}^*(-)}{\to} H^*_{\mathtt{pr}^{-1}c/S}(N(i)) \stackrel{(-)\cup\nu^H_{N(i)}}{\to} H^{*+\dim(N(i))}_{\mathtt{pr}^{-1}c/S\cap 0_{N(i)}}(N(i)) \stackrel{\tilde{i}_*}{\to} H^{*+\dim(N(i))}_{c/S}(B\times\mathbb{R}^N) \cong H^{*-\dim(E)+\dim(B)}_S(B)$$

Theorem 1.10. *The integration does not depend on the choice of the embedding i.*

But it does in general depend on the choice of the complex structure on the stable normal bundle.

For the application to the index theorem we apply integration to the map

$$p : T^*M \to * \ .$$

We need the following facts:

1. A symplectic structure ω on a vector bundle V (i.e., a section ω in $\Lambda^2 V^*$ such that $\Lambda^{\dim(V)/2}\omega$ is nowhere vanishing) determines a complex structure. It is represented by any bundle endomorphism J such that $J^2 = -1$ and $\omega(J-,-)$ is a (positive definite) metric.
2. If V represents a stable bundle, then a complement of V is a stable bundle W together with a (stable) trivialization of $V \oplus W$. In this case there is a bijection $J_V \leftrightarrow J_W$ between the complex structures on V and W fixed by the condition that $J_V \oplus J_W$ is the trivial structure on $V \oplus W$.
3. T^*M has a canonical symplectic structure.
4. The vertical bundle $T^v p \to T^*M$ is given by $T(T^*M)$. It therefore has an induced symplectic structure.
5. $T^v p$ is a complement of N_p. There N_p has an complex structure.

Using all this we get an unabigiously defined integration map

$$p_!^{KU} : KU^0_{c/M}(T^*M) \to KU^{-2\dim(M)}(*)$$

where we assume that M is closed so that $c/M = c/*$.

The principal symbol of D is a bundle morphism

$$\sigma(D) : q^*E \to q^*F \ ,$$

where $q : T^*M \to M$ is the projection.

Remark 1.11. A coordinate independent definition goes as follows. For f in $C^\infty(M)$ we have

$$t^{-\deg(f)} e^{-tf} D e^{tf} = (df)^* \sigma(D) + O(t) \ .$$

Letting f vary this determines the symbol. Note that one must check well-definedness. □

Definition 1.12. *D is called elliptic if $\sigma(D)$ is an isomorphism outside of the zero section.*

Since we assume that D is elliptic we get a class

$$[\sigma(D)] := [q^*E, q^*F, \sigma(D)]$$

in $K^0_{0_{T^*M}}(T^*M)$. Note that $0_{T^*M} \in c/M$.

We can now state the Atiyah-Singer index theorem [AS68]:

Theorem 1.13 (Atiyah-Singer)**.**

$$\mathtt{index}(D) = b^{\dim(M)} p_!^{KU}([\sigma(D)]) \ .$$

2. The family version and the Riemann-Roch theorem

A family of differential operators on a bundle $\pi : M \to B$ of manifolds is a differential operator

$$D : C^\infty(M, E) \to C^\infty(M, F)$$

which differentiates along the fibres of π, i.e., which commutes with multiplication with functions of the form $\pi^* f$ for f in $C^\infty(B)$. Then we can consider $\mathtt{ker}(D)$ and $\mathrm{coker}(D)$ as modules over $C^\infty(B)$. Even better, we can consider $\mathtt{ker}(D)$ and $\mathrm{coker}(D)$ as sheaves of modules over the sheaf of rings C_B^∞ of smooth functions on B.

Let $q : T^v p^* \to M$ be the dual of the vertical bundle $T^v p := \mathtt{ker}(dp)$. The principal symbol of D is now a map

$$\sigma(D) : q^* E \to q^* F \ .$$

The same coordinate-free formula as in the absolute case works (if appropriately interpreted).

Definition 2.1. *D is called fibrewise elliptic if $\sigma(D)$ is an isomorphism outside of the zero section.*

We assume that D is fibrewise elliptic and that p is proper.

We further make the following simplifying assumption:

Assumption 2.2. *The C_B^∞-module $\mathtt{ker}(D)$ is locally free and finitely generated.*

This is equivalent to the condition that the dimension kernel of the fibrewise operators is constant. The sheaf $\mathtt{ker}(D)$ is therefore a vector bundle (better, the sheaf of sections of a vector bundle). Then (since the index of the fibrewise operators is constant by homotopy invariance) $\mathrm{coker}(D)$ is a vector bundle, too.

Definition 2.3. *We define the index of the family of elliptic operators D*

$$\mathtt{index}(D) = [\mathtt{ker}(D)] - [\mathrm{coker}(D)]$$

in $KU^0(B)$.

One could discuss the question how to get rid of the simplifying assumption.

The aim of the families index theorem of Atiyah-Singer is to calculate this index class in terms of homotopy theory from the principal symbol of D.

The vertical bundle of the projection $p : T^v\pi^* \to B$ is q^* of the contangent bundle along the fibres of π and therefore has a symplectic structure. It follows that the stable normal bundle of p (i.e., the complement of the stable bundle represented by the vertical bundle of p) has a complex structure and that we have the integration map

$$p_!^{KU} : KU^*_{c/M}(T^v p^*) \to KU^{*-2\dim(M)+2\dim(B)}(B) \ .$$

We have the symbol class

$$[\sigma(D)] := [\sigma(D), q^*E, q^*F] \in KU^0_{0_{T^v p^*}}(T^v p^*) \ .$$

Theorem 2.4 (Atiyah-Singer for families)**.**

$$\mathtt{index}(D) = b^{\dim(M)-\dim(B)} p_!^{KU}[\sigma(D)]$$

Besides the statement of the families index theorem the main goal of the talk is the statement and proof of the Riemann-Roch theorem. In topology a Riemann-Roch theorem is a statement about the compatibility between the integrations in two generalized cohomology theories connected by a transformation.

In our case we consider the natural transformation of homology theories

$$\mathbf{ch} : KU \to HP$$

given by the Chern character, where HP is ordinary cohomology with coefficients in $\mathbb{Q}[b, b^{-1}]$ with $\deg(b) = 2$. We have an isomorphism

$$H\mathbb{Q}^*(\mathbb{CP}^\infty) \cong \mathbb{Z}[c] \ ,$$

where $c := c_1(L)$. Then $HP^*(\mathbb{CP}^\infty) \cong \mathbb{Q}[b, b^{-1}][[b^{-1}c]]$ and $\beta^{HP} := c$ determines a complex orientation of HP.

The statement of the Riemann-Roch theorem involves the characteristic Todd class $\mathbf{Td}(V)$ in $HP^0(B)$ of complex vector bundles $V \to B$. In view of the splitting principle this class is uniquely determined by the characteristic class property, the relation

$$\mathbf{Td}(V \oplus W) = \mathbf{Td}(V) \cup \mathbf{Td}(W)$$

and

$$\mathbf{Td}(L) = \frac{b^{-1}c}{1 - e^{-b^- c}}$$

in $HP^0(\mathbb{CP}^\infty) \cong \mathbb{Q}[[b^{-1}c]]$ for the tautological bundle L. Observe that the Todd class is invertible and can be defined for stable bundles with complex structure.

We consider a map $p : E \to B$ whose stable normal bundle N has a complex structure. We retain the notation previously introduced.

Theorem 2.5. *The square*

$$\begin{tikzcd}
KU^*_S(E) \arrow[r, "\mathbf{ch}"] \arrow[d, "p_!^{KU}"] & HP^*_S(E) \arrow[d, "p_!^{HP}(-\cup \mathbf{Td}(N_p)^{-1})"] \\
KU^{*-\dim(E)+\dim(B)}(B) \arrow[r, "\mathbf{ch}"] & HP^{*-\dim(E)+\dim(B)}(B)
\end{tikzcd}$$

commutes.

Here are the steps of the proof:

1. By the Thom isomorphism theorem there is a unique (cannibalistic) class c in $HP^0(E)$ such that $\mathtt{pr}^* c(N_p) \cup \nu^{HP}_{N_p} = \mathbf{ch}(\nu^{KU}_{N_p})$
2. We have
$$\mathbf{ch} \circ p_!^{KU} = p_!^{HP}(\mathbf{ch}(-) \cup c(N_p))$$
so that it remains to calculate the cannibalistic class.
3. Using the splitting principle one reduces to the case that N_p is a line bundle.
4. The complex orientations of KU and HP are represented by classes β^{KU} in $KU^2(\mathbb{CP}^\infty)$ and β^{HP} in $HP^2(\mathbb{CP}^\infty)$.
5. We have
$$\beta^{HP} = c\ , \quad \mathbf{ch}(\beta^{KU}) = b(1 - e^{-b^{-1}c})\ .$$
Consequently,
$$c(L) = \frac{1 - e^{-b^{-1}c}}{b^{-1}c} = \mathbf{Td}(L)^{-1}\ .$$

Combining the two theorems we get:

Corollary 2.6 (Cohomological Index Theorem)**.**

$$\mathbf{ch}(\mathtt{index}(D)) = b^{\dim(M)-\dim(B)} p_!^{HP}(\mathbf{ch}([\sigma(D)]) \cup \mathbf{Td}(T^v p^*))\ .$$

3. Dirac operators

Dirac operators form an analytically very well-behaved class of operators which subsumes most operators of geometric interest. The goal of this talk is to explain the definition of Dirac operators and the twisting construction. The basic example is the *Spin*^{c}-Dirac operator $\not{D}$. The talk should explain the derivation of the index formula for $\not{D}$ and the index formulas for the Euler and signature operator from the Cohomological Index Theorem.

Remark 3.1. To an euclidean vector space V we associate the $\mathbb{Z}/2\mathbb{Z}$-graded complex Clifford algebra $\mathrm{Cliff}(V)$ generated by V under the relations $v^2 = -\|v\|^2$ and the convention $\deg(v) = 1$. We have

$$\mathrm{Cliff}(V) \cong \begin{cases} \mathtt{Mat}_{2^{\dim(V)/2}}(\mathbb{C}) & \dim(V) \text{ even} \\ \mathtt{Mat}_{2^{(\dim(V)-1)/2}}(\mathbb{C}) \oplus \mathtt{Mat}_{2^{(\dim(V)-1)/2}}(\mathbb{C}) & \dim(V) \text{ odd} \end{cases}$$

If V is oriented, then we can consider an oriented orthonormal base $(v_1, \dots, v_{\dim(V)})$ and the element $z := i^{\frac{(n-1)n}{2}} v_1 \dots v_{\dim(V)}$. It satisfies $z^2 = 1$ and does not depend on the choice of the basis.

If $\dim(V)$ is even, then z anticommutes with the generators v, and if $\dim(V)$ is odd, it is given by $\mathtt{diag}(1, -1)$ (this fixes the order of the Matrix summands).

If $\dim(V)$ is even, then $\mathrm{Cliff}(V)$ has an irreducible representation $\Delta(V)$ on a $2^{\dim(V)}$-dimensional complex vector space. Every other finite-dimensional representation splits as a direct sum of these.

If $\dim(V)$ is odd, then $\mathrm{Cliff}(V)$ has two irreducible representations $\Delta(V)_\pm$ on $2^{(\dim(V)-1)/2}$-dimensional complex vector spaces distinguished by the action of $\Delta(V)(z) = \pm 1$. Every other finite-dimensional representation splits as a direct sum of these.

These representations are determined uniquely up to isomorphism. Their automorphisms are isomorphic to $\mathbb{C}^*$ by Schur's Lemma.

If $V \to B$ is an euclidean vector bundle, then we can form the Clifford algebra fibrewise and get a bundle of algebras $\mathrm{Cliff}(V)$. If V is oriented, then it has a canonical section z. $\square$

We consider a Riemannian manifold M.

Definition 3.2. *A Dirac bundle* $\mathbf{E}$ *on* M *is given by the following data:*

1. *a complex vector bundle* $E \to M$
2. *a metric on* E *and a hermitean connection* ∇^E
3. *a Clifford multiplication* $c : TM^* \to \mathtt{End}(E)$ *turning* E *into a bundle of* $\mathrm{Cliff}(T^*M)$-*modules.*
4. *a grading*

This data has to satisfy various compatibility relations:

1. $c(v)$ is antihermitean for v in V.
2. c is parallel (where T^*M is equiped with the Levi-Civita connection)
3. the grading is parallel
4. $c(v)$ is odd or even with respect to the grading depending on wether $\dim(M)$ is even or odd.

To a Dirac bundle $\mathbf{E}$ we can associate the Dirac operator

$$D_{\mathbf{E}} : C^\infty(M,E) \stackrel{\nabla^E}{\to} C^\infty(M, T^*M \otimes E) \stackrel{c}{\to} C^\infty(M,E) \ .$$

The grading induces a decomposition $E = E^+ \oplus E^-$. If M is even-dimensional, then we get the restriction

$$D^+_{\mathbf{E}} : C^\infty(M,E^+) \to C^\infty(M,E^-) \ .$$

Lemma 3.3. *$D^+_{\mathbf{E}}$ is elliptic and* $\mathtt{index}(D) = \dim \mathtt{ker}(D^+_{\mathbf{E}}) - \dim \mathtt{ker}(D^-_{\mathbf{E}})$.

Indeed, the symbol of $D_{\mathbf{E}}$ is easy to calculate:

$$\sigma(D_{\mathbf{C}})(\xi) = c(\xi) \ , \quad \xi \in T^*M$$

Since $c(\xi)^2 = -\|\xi\|^2$ this is an isomorphism for $\xi \neq 0$.

Example 3.4. The bundle $\Lambda^* T^*_{\mathbb{C}} M \to M$ with the metric induced by the Riemannian metric, the Levi-Civita connection, and the Clifford multiplication $\omega \mapsto \epsilon_\omega - \iota_\omega$ (where ϵ_ω is exterior multiplication by ω and $\iota_\omega := \epsilon^*_\omega$) is a Dirac bundle.

We can consider the even-odd grading z_{Euler} or the signature grading $z_{sign} := \epsilon(\deg)*$. Here $\epsilon(n)$ is a certain power of i depending on n and $\dim(M)$, and $*$ is the Hodge $*$-operator. The design criterion which fixes this is that $\epsilon(2n) = 1$ if $\dim(M) = 4n$, and compatibility with dimension reduction.

The associated Dirac operator is

$$D_{\mathbf{\Lambda^* T^* M}} = d + d^* \ .$$

We denote $D^+_{\mathbf{\Lambda^* T^* M}}$ by D_{Euler} or D_{sign} depending on the grading.

Lemma 3.5. *If M is even-dimensional and closed, then*

$$\mathtt{index}(D_{Euler}) = \chi(M) \ .$$

If $\dim(M)$ *is divisible by* 4*, then*

$$\mathtt{index}(D_{sign}) = \mathtt{sign}(M) \ .$$

This follows from Hodge theory for the Euler operator, and in addition the definition of the Hodge-$*$-operator for the signature operator.

Remark 3.6. In this remark we propose a way to think about $Spin^c$-structures.

Let $V \to B$ be an oriented euclidean vector bundle. A $Spin^c$-structure on V is represented by a bundle of Cliff(V)-modules $S^c(V)$ which is graded if $\dim(V)$ is even, and fibrewise isomorphic to $\Delta(V_x)$ or $\Delta(V_x)_+$ as (graded) Cliff(V_x)-modules for all x in M depending on the parity of $\dim(V)$.

We get a category of representatives of $Spin^c$-structures.

Definition 3.7. *A $Spin^c$-structure is an isomorphism class of such representatives.*

If V is trivial, then V clearly admits a $Spin^c$-structure. So locally we can always find $Spin^c$-structures. We can consider the stack of local choices of representatives of $Spin^c$-structures on V. It is a gerbe with band the sheaf $C_{\mathbb{C}^*}$ of C^*-valued function. Consequently the obstruction against gobal existence is the triviality of that gerbe, hence its Dixmir-Douady class, usually denoted by $W_3(V)$, in $H^2(B, C^\infty_{\mathbb{C}^*}) \cong H^3(B, \mathbb{Z})$. One can show that $W_3 = \beta(w_2(V))$ and thus $2W_3(V) = 0$, where β is the Bockstein associated to $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ and $w_2(V)$ in $H^2(B, \mathbb{Z}/2\mathbb{Z})$ is the second Stiefel-Whitneay class.

If $L \to B$ is a line bundle, then we can define a new $Spin^c$-structure represented by $S^c(V) \otimes L$.

Lemma 3.8. *The group of isomorphism classes of line bundles (i.e., $H^2(B, \mathbb{Z})$) acts simply transitively on the set of $Spin^c$-structures of V.*

□

Remark 3.9. A complex structure J on an euclidean vector bundle $V \to B$ with $J^* = -J$ induces naturally a $Spin^c$-structure on V. We construct the representative $S^c(V)$.

To this end decompose $V \otimes_{\mathbb{R}} \mathbb{C} = V^{1,0} \oplus V^{0,1}$ into the ± 1-eigenspaces of J. Then we form $S^c(V) := \Lambda^*(V^{0,1})$. For v in V let $\bar{v}$ be the projection of $v \otimes 1$ to $V^{0,1}$. We then define the Clifford multiplication by $c(v) := \epsilon_{\bar{v}} - \epsilon^*_{\bar{v}}$.

Example 3.10. If T^*M has a $Spin^c$-structure, then the underlying bundle $S^c(T^*M)$ can be refined to a Dirac bundle $\mathbf{S}^c(T^*M)$ (this involves choices of connections) given by the complex spinor bundle. The Dirac operator associated to $\mathbf{S}^c(T^*M)$ will be denoted by $\not{D}$.

If M has an almost complex structure, and $\mathbf{S}^c(T^*M)$ comes from the $Spin^c$-structure given by the complex structure, then

$$\not{D} = \bar{\partial} + \bar{\partial}^* + \text{zero order terms} .$$

If M is Kähler, then the zero order terms vanish. □

If $\mathbf{E}$ is a Dirac bundle and $\mathbf{V}$ is a (graded) complex vector bundle with metric and hermitean connection we can construct a new Dirac bundle $\mathbf{E} \otimes \mathbf{V}$ with underlying bundle $E \otimes V$, product connection and Clifford multiplication $c \otimes \mathtt{id}_V$. We call the associated operator

$$D_{\mathbf{E}}(\mathbf{V}) := D_{\mathbf{E} \otimes \mathbf{V}}$$

the twist of $D_{\mathbf{E}}$.

Proposition 3.11. *Assume for simplicity that* $\dim(M)$ *is even. Then every Dirac bundle on M is isomorphic to*

$$\mathbf{S}^c(T^*M) \otimes \mathbf{V}$$

for a uniquely determined twist $\mathbf{V}$.

There is an analog in the odd-dimensional case with a more complicated formulation.

Given $\mathbf{E}$ one gets the twist by

$$\mathbf{V} := \mathtt{Hom}_{\mathrm{Cliff}(T^*M)}(\mathbf{S}^c(T^*M), \mathbf{E})$$

with the naturally induced metric, connection, and grading.

We have the characteristic class $\hat{\mathbf{A}}^c$ for $Spin^c$-bundles V. Before we provide the general definition we consider a special case.

Remark 3.12. If the $Spin^c$-structure comes from a complex structure, then we define

$$\hat{\mathbf{A}}^c(V) := \mathbf{Td}(V)$$

(using the complex structure on V in order to define the Todd class).

We consider the case $V = L$ on $\mathbb{CP}^\infty$. From the formula for $\mathbf{Td}(L) = \frac{c/b}{1-e^{-c/b}}$ we get

$$\hat{\mathbf{A}}^c(L) = e^{\frac{c}{2b}} \frac{\frac{c}{2b}}{\sinh(\frac{c}{2b})} \ .$$

We note that the power series

$$\frac{\frac{x}{2}}{\sinh(\frac{x}{2})}$$

is even. It corresponds to the characteristic class $\hat{\mathbf{A}}$ of the underlying real vector bundle. In general this calculation implies the relation

$$\hat{\mathbf{A}}^c(V) := e^{\frac{c_1(\det_{\mathbb{C}} V)}{2b}} \hat{\mathbf{A}}(V) \ .$$

$\square$

In the following a $Spin^c$-structure on a real oriented euclidean vector bundle bundle $V \to M$ is a Spinor module S^c.

To the $Spic^c$-structure we associate the complex line bundle

$$L(S^c) := \mathtt{Hom}_{\mathrm{Cliff}(V)}(S^{c,*}, S^c) \ .$$

Example 3.13. Assume that V has a complex structure and $S^c = S^c(V)$ associated to the complex structure. We have a non-degenerate pairing

$$\langle -, - \rangle : S^c(V) \otimes S^c(V) \to \Lambda^{max} V^{0,1}$$

given by the wedge product and taking the component of highest degree. One checks that $\langle c(v)\alpha, \omega \rangle = (-1)^{deg(\alpha)} \langle \alpha, c(v)\omega \rangle$. Hence the pairing induces an isomorphism if $\mathrm{Cliff}(V)$-modules

$$S^c(V)^* \cong S^c(V) \otimes (\Lambda^{max} V^{0,1})^{-1} \ .$$

It follows

$$L(S^c(V)) \cong \Lambda^{max} V^{0,1} \ .$$

Consequently

$$\hat{\mathbf{A}}^c(V) = e^{-\frac{c_1(L(S^c(V)))}{2b}} \hat{\mathbf{A}}(V) \ .$$

□

Definition 3.14. *We define the characteristic class $\hat{\mathbf{A}}^c$ of vector bundles V with $Spin^c$-structure S^c by*

$$(V, S^c) \mapsto e^{-c(L(S^c))/2b} \hat{\mathbf{A}}(V) \ .$$

If $V \to M$ has a $Spin^c$-structure represented by S^c and $H \to M$ is a complex line bundle, then we can form a new $Spin^c$-structure represented by $S^c \otimes H$. In this case we have the relations

$$L(S^c \otimes H) \cong L(S^c) \otimes H^2$$

and

$$\hat{\mathbf{A}}^c(V, S^c \otimes H) = \mathbf{ch}(H) \cup \hat{\mathbf{A}}^c(V, S^c) \ . \tag{3.1}$$

We furthermore have

$$\hat{\mathbf{A}}^c(V, S^{c,*}) \cong e^{c(L(S^c))/2b} \cup \hat{\mathbf{A}}(V) \ .$$

We now specialize the cohomological index theorem for families to families of Dirac operators. We consider a proper submersion $\pi : M \to B$ and assume that $T^v\pi^*$ has a $Spin^c$-structure S^c. We set $d := \dim(M) - \dim(B)$ and assume that d is even.

We let $\mathbf{E}$ be a fibrewise Dirac bundle and write $\mathbf{E} \cong S^c \otimes \mathbf{V}$ for the twisting bundle $\mathbf{V}$.

Theorem 3.15 (Family index Theorem for twisted Dirac Operators)**.** *We have*

$$\mathbf{ch}(\mathtt{index}(\not{D}(\mathbf{V})^+)) = b^d \pi_!^{HP}(\mathbf{ch}(\mathbf{V}) \cup \hat{\mathbf{A}}^c(T^v\pi^*, S^c)) \ .$$

Here are the main steps.

1. For simplicity assume that $\mathbf{V} = \mathbf{V}^+$.
2. We can decompose
$$p : T^v\pi^* \xrightarrow{q} M \xrightarrow{\pi} * \ .$$
3. The Vvertical tangent bundle of p is given by $q^*T^v\pi^* \oplus q^*T^v\pi$.
4. The bundle $T^v\pi^* \oplus T^v\pi$ has the symplectic structure $(v^*, v) \otimes (w^*, w) \mapsto v^*(w) - w^*(v)$. The symplectic structure fixes a class of complex structures which is used to define $\mathbf{Td}(T^v\pi^* \oplus T^v\pi)$ below.
5. The $Spin^c$-structure induces an (ordinary) orientation of $T^v\pi^*$ and hence orientations of π, q and p for HP. With these orienations we have
$$p_!^{HP} = \pi_!^{HP} \circ q_!^{HP} \ .$$

6. The symbol of $\not{D}(\mathbf{V})^+$ is given by $\sigma(\not{D})^+\otimes q^*V$. Hence $[\sigma(\not{D}(\mathbf{V})^+)] = [\sigma(\not{D}^+)]\cup q^*[V]$ in $K^0_{c/M}(T^*\pi)$.
7. From the cohomological families index theorem we get
$$\mathbf{ch}(\mathtt{index}(\not{D}(\mathbf{V})^+)) = b^d p_!^{HP}(\mathbf{ch}([\sigma(\not{D}^+)])\cup q^*\mathbf{ch}(V)\cup q^*\mathbf{Td}(T^v\pi^*\oplus T^v\pi)\ .$$
8. We use the projection formula in order to get
$$\mathbf{ch}(\mathtt{index}(\not{D}(\mathbf{V})^+)) = b^d \pi_!^{HP}(q_!^{HP}\mathbf{ch}([\sigma(\not{D}^+)])\cup \mathbf{Td}(T^v\pi^*\oplus T^v\pi)\cup \mathbf{ch}(V))$$
9. We define characteristic class for $Spin^c$ bundles (V,S^c) by
$$\Phi(V,S^c) := q_!^{HP}\mathbf{ch}([\sigma(\not{D}^+_{S^c})])\cup \mathbf{Td}(V^*\oplus V)\ .$$
Here we use that the symbol class $[\sigma(\not{D}^+_{S^c})]$ in $K^0_{c/M}(V)$ can be defined for every even-dimensional vector bundle with a $Spin^c$-structure S^c. One must verify naturality. In the application $V=T^v\pi^*$ with the fixed $Spin^c$-structure S^c.

   It remains to show:
$$\Phi(V,S^c) = \hat{\mathbf{A}}^c(V,S^{c,*}) \tag{3.2}$$
10. We check that both sides of (3.2) are additive:
$$\Phi(V\oplus V', S^c\otimes S^{c,\prime}) = \Phi(V,S^c)\cup\Phi(V',S^{c,\prime})$$
$$\hat{\mathbf{A}}^c(V\oplus V', S^c\otimes S^{c,\prime}) = \hat{\mathbf{A}}^c(V,S^{c,*})\cup\hat{\mathbf{A}}^c(V',S^{c,\prime,*})$$
11. Both sides are compatibe with the shift of $Spin^c$-structure, i.e.
$$\Phi(V,S^c\otimes H) = \Phi(V,S^c)\cup\mathbf{ch}(H)$$
and (3.1).
12. We observe that $BU\to BSpin^c$ induces injection in rational cohomology. Hence it suffices to verify the equality in the case where V is complex and $S^c=S^c(V)$. We use the splitting sprinciple in order to reduce to line bundles.
13. It suffices to show the equality for the tautological bundle $L\to\mathbb{CP}^\infty$. Have seen above:
$$\hat{\mathbf{A}}^c(L) = \mathbf{Td}(L) = \frac{c/b}{1-e^{-c/b}}\ .$$
We observe that $L\oplus L^*$ as complex bundle (coming from symplectic structure) is isomorphic to $L\oplus L^*$ with its canonical complex structure. Hence
$$\mathbf{Td}(L\oplus L^*) = \frac{c/b}{1-e^{-c/b}}\cdot\frac{-c/b}{1-e^{c/b}}\ .$$

14. We show that
$$q_!^{HP}[\mathbf{ch}(\sigma(\not{D}^+_{S^c(L)}))] = \frac{1-e^{-c/b}}{c/b} \ .$$
To this end we use that $\sigma(\not{D})^+_{S^c(L)} : \mathbb{C} \to \bar{L}$ is the morphism which over l in L is given by $\lambda \mapsto \overline{\lambda l}$. We embed L into the total space of bundle $q' : \mathbb{P}(L \oplus \mathbb{C}) \to M$ with fibre $\mathbb{CP}^1$.

- glue $\sigma(\not{D})^+_{S^c(L)} : \mathbb{C} \to \bar{L}$ with
$$\mathtt{id} : (\mathbb{P}(L \oplus \mathbb{C}) \setminus \{0\}) \times \mathbb{C} \to (\mathbb{P}(L \oplus \mathbb{C}) \setminus \{0\}) \times \mathbb{C}$$
over $L \setminus \{0\}$ by
$$\begin{array}{ccc} (L \setminus \{0\}) \times \mathbb{C} & \xrightarrow{\lambda \mapsto \overline{\lambda l}} & \bar{L}_{|L\setminus\{0\}} \\ \cong \downarrow \mathtt{id} & & \cong \downarrow k \mapsto \overline{k/l} \\ (L \setminus \{0\}) \times \mathbb{C} & \xrightarrow{\mathtt{id}} & (L \setminus \{0\}) \times \mathbb{C} \end{array} \ .$$
The target is a bundle $H \to \mathbb{P}(L \oplus \mathbb{C})$ Using the locality of the push-forward we see that
$$q_!^{HP}(\mathbf{ch}([\sigma(\not{D})^+])) = q_!^{\prime,HP}(1 - \mathbf{ch}(H)) \ .$$
We now observe that the bundle H has a linear transition function. It has a global section. This implies
$$q'_!(c_1(H)) = -1 \ .$$
We have
$$c_1(H) + c_1(H^\perp) = q'^{,*}c_1(L \oplus \mathbb{C}) = q'^{,*}c_1(L) \ .$$
Furthermore
$$(c_1(H) + c_1(H^\perp))^2 = c_1(H)^2 + c_1(H^\perp)^2 + c_1(H)c_1(H^\perp) = q'^{,*}c_1(L)^2 \ .$$
Using $c_1(H)c_1(H^\perp) = q'^{,*}c_2(L \oplus \mathbb{C}) = 0$ and inserting $c_1(H^\perp) = q'^{,*}c_1(L) - c_1(H)$ we get
$$2c_1(H)^2 - 2q'^{,*}c_1(L)c_1(H) = 0 \ .$$
Hence $c_1(H)^n = c_1(H)q'^{,*}c_1(L)^{n-1}$.

15. This gives (by projection formula)
$$q_!^{\prime,HP}(1 - \mathbf{ch}(H)) = \frac{e^{c_1(L)/b} - 1}{c_1(L)/b} \ .$$

Putting everything together we get

$$\Phi(L) = \frac{c/b}{1-e^{-c/b}} \frac{-c/b}{1-e^{c/b}} \frac{e^{c/b}-1}{c/b} = \frac{c/b}{1-e^{-c/b}} = e^{c/2b} \frac{c/2b}{\sinh(c/2b)} = \hat{\mathbf{A}}^c(V, S^c(L))$$

This finishes the verification of (3.2).

We let $\mathbf{V}_{Euler}$ and $\mathbf{V}_{sign}$ be the twisting bundles yielding the Euler and signature grading on $\mathbf{\Lambda^* T^* M}$.

Theorem 3.16 (Hirzebruch Signature Formula). *If* $\dim(M)$ *is divisible by* 4*, then we have* $\mathbf{V}_{sign} = S(M)^*$ *(with grading forgotten) and*

$$\mathtt{sign}(M) = \mathtt{index}(D_{sign}) = \langle \mathrm{L}(TM), [M] \rangle \ .$$

1. The identification of $\mathbf{V}_{\mathtt{sign}}$ is a calculation in the representation ring of $Spin^c(4n)$.
2. We use Lemma 3.5.
3. The index formula follows with the definition

$$\mathrm{L}(TM) := \mathbf{ch}(\mathbf{S}^*_{ungr}(M)) \cup \hat{\mathbf{A}}^c(T^*M) \ .$$

4. In the talk one should calculate $\mathrm{L}(TM)$ in terms of Pontrjagin classes of M, or discuss the corresponding formal power series $\mathrm{L}(L)$ in $HP^0(\mathbb{CP}^\infty)$.

Theorem 3.17 (Chern-Gauss-Bonnet). *If* $\dim(M)$ *is even, then we have* $\mathbf{V}_{Euler} = \mathbf{S}(M)^*$ *(with grading) and*

$$\chi(M) = \mathtt{index}(D_{Euler}) = \langle e(TM), [M] \rangle \ .$$

1. The identification of $\mathbf{V}_{Euler}$ is a calculation in the representation ring of $Spin^c(2n)$.
2. We use Lemma 3.5.
3. The index formula follows from

$$e(TM) := \mathbf{ch}(\mathbf{S}^*) \cup \hat{\mathbf{A}}^c(T^*M) \ .$$

4. On should say something about the characteristic class $\chi(V)$, e.g. discuss the relation $\chi(V)^2 = p_{\dim(V)}$.

A reference for the derivation of these special cases is [BGV04, Sec. 2+4].

4. Geometric applications of the index theorem for Dirac operators

This is a free talk which could address the following topics:

1. The application of the index theory of Dirac operators in order to construct obstructions against the existence of metrics with positive scalar curvature.
2. The application of the index theory of Dirac operators in order to prove the homotopy invariance of higher signatures.

3. The derivation of the Riemann-Roch formula in complex geometry as a special case of the index theory of Dirac operators applied to the Dolbeault complex.

Part II. K-theory of C^*-algebras

5. K-theory of C^*-algebras

The goal of this talk is to introduce topological K-theory for C^*-algebras as a functor

$$K^{top} : C^*\mathbf{Alg}^{nu} \to \mathbf{Sp}$$

from the category $C^*\mathbf{Alg}^{nu}$ of (non-unital) C^*-algebras to the ∞-category of spectra. Furthermore it should explain the basic properties. In particular it should discuss the Example 5.5 in detail.

Remark 5.1. The definition of the K-theory groups $K_i^{top}(A)$ of a C^*-algebra is immediate.

1. In the unital case $K_0^{top}(A)$ is the Grothendieck group of the monoid of isomorphism classes of finitely generated projective A-modules (this does not involve the topology of A). In general one sets $K_0^{top}(A) := \mathtt{ker}(K_0^{top}(A^+) \to K_0^{top}(\mathbb{C}))$, where A^+ is the unitalization of A.
2. $K_1^{top}(A)$ is the group of components of $GL_1(A \otimes \mathbf{K})$ (invertibles in $(A \otimes \mathbf{K})^+$), where $\mathbf{K}$ is the C^*-algebra of compact operators on a Hilbert space. This involves the topology of A.
3. The groups are two-periodic.

These classical definitions serve as the design criterion for the construction of the functor K^{top}. The main point is to define a spectrum valued functor $A \mapsto K^{top}(A)$ with these homotopy groups $K_i^{top}(A) \cong \pi_i K^{top}(A)$.

Note that classically the development of the datum of C^*-algebra K-theory consist of the definition of the groups (easy task as above) and the ad-hoc construction of various exact sequences and the verification of Bott periodicity (complicated task). Different references have different formulas and their comparison and the naturality properties are in general not easy to see.

The modern philosophy to be promoted in this talk is first to define a spectrum-valued (lax symmetric monoidal) functor by general machinery. It provides the datum of the sequences and the candidate for Bott periodicity map automatically. Technically one must verify that certain excision maps are equivalences, i.e., one must verify properties of the construction, which is then the complicated task. □

Here are options:

1. One could consider the lax-symmetric monoidal (non-connective) algebraic K-theory functor
$$K^{alg} : \mathbf{Rings} \to \mathbf{Sp}$$
as being defined (very easy using modern ∞-category technology). For every, say spectrum valued, functor F on unital C^*-algebras one can define the homotopification
$$FH(A) := \operatorname*{\mathtt{colim}}_{\Delta^{op}} F(C(\Delta^-) \otimes A) \ .$$
One furthermore defines the functor on not necessarily unital algebras by
$$F^+(A) := \mathrm{Fib}(F(A^+) \to F(\mathbb{C})) \ .$$
Then one defines $K^{top} : C^*\mathbf{Alg}^{nu} \to \mathbf{Sp}$ by
$$K^{top}(A) := (K^{alg}_{|C^*\mathbf{Alg}} H)^+(A \otimes \mathbf{K}) \ .$$
Here is the justification. By construction, this functor is stable and homotopy invariant, lax symmetric monoidal (this uses a good version of $\mathbf{K} \otimes \mathbf{K} \cong \mathbf{K}$). Using excision for algebraic K-theory (Suslin-Wodzicki) one checks that it sends exact sequences of C^*-algebras to fibre sequences. Using the $+$-construction model of algebraic K-theory one can check that the positive homotopy groups are correct. Using the $\cup$-product and an argument of Karoubi one deduces Bott periodicity in negative degrees and hence that our construction gives the correct homotopy groups in all degrees. Alternatively one can use the gument of Cuntz to deduce Bott periodicity from stability, excision, and homotopy invariance, see [Con94, Lemma 15, App. B]. Then one must only check that π_0 and π_1 are correct.

   A good starting reference for all this is [Ros05].

2. One can also define
$$K^{top}(A) := (K^{alg}_{|C^*\mathbf{Alg}})^+(A \otimes \mathbf{K})$$
(i.e., without H).

   Here is again the justification. This functor is stable, lax symmetric monoidal, and sends exact sequences of C^*-algebras to fibre sequences, again by (Suslin-Wodzicki) excision for algebraic K-theory. Consequently it is homotopy invariant by a theorem of Higson.

   Furthermore, one checks directly that it has the correct π_0 and π_1. Using the $\cup$-product and the argument of Karoubi or of Cuntz one deduces Bott periodicity in negative degrees.

   Consequently the natural map
$$(K^{alg}_{|C^*\mathbf{Alg}})^+(A \otimes \mathbf{K}) \to (K^{alg}_{|C^*\mathbf{Alg}} H)^+(A \otimes \mathbf{K})$$

is an equivalence. This statement is also called the Karoubi conjecture (a theorem by now [SW90]).

Philosophically this definition is the simplest and most easy to work with, but the justification is quite involved.

3. One can follow [Joa03], where a functor to symmetric spectra is constructed. Its more concrete, but much more complicated.

The following theorem states the basic properties of K^{top} which we will use later in the seminar.

Theorem 5.2. *1. (homotopy invariance) The functor K^{top} is homotopy invariant.*

2. *(continuity) The functor K^{top} preserves finite sums and filtered colimits.*
3. *(symmetric monoidal) K^{top} is lax symmetric monoidal (this is an additional structure, e.g. inherited from the corresponding structure on algebraic K-theory, on C^***Alg** we consider the maximal tensor product).*
4. *(Bott periodicity) The product with the Bott class b in $\pi_2(K^{top}(C_0(\mathbb{R}^2)))$*

$$K^{top}(\mathbf{A}) \to \Omega^2 K^{top}(C_0(\mathbb{R}^2) \otimes A)$$

   is an equivalence.
5. *(stability) The inclusion $A \to A \otimes \mathbf{K}$ induces an isomorphism $K^{top}(A) \to K^{top}(A \otimes \mathbf{K})$.*
6. *The functor K^{top} sends exact sequences*

$$0 \to I \to A \to A/I \to 0$$

   of C^*-algebras to fibre sequences

$$K^{top}(I) \to K^{top}(A) \to K^{top}(A/I) \xrightarrow{\partial} \Sigma K^{top}(I) \ .$$

The class b in $\pi_2(K^{top}(C_0(\mathbb{R}^2)))$ should be given explicitly.

Example 5.3. The following discussion provides the bridge between topological K-theory and C^*-algebra K-theory.

Theorem 5.4 (Higher Swan Theorem)**.** *For X in CW^{fin} we have a natural equivalence*

$$\mathtt{map}(\Sigma^+_\infty X, KU) \simeq K^{top}(C(X)) \ .$$

For the proof:

1. One shows that $X \mapsto K^{top}(C(X))$ a cohomology theory on CW^{fin}.
2. One calculates $K^{top}(\mathbb{C}) \simeq KU$. The choice here determines the equivalence above.

$\square$

Example 5.5. In this example we provide the K-theory interpretation of the index of Fredholm operators. Let $\mathbf{B}$ be the C^*-algebra of the bounded operators on a separable Hilbert space.

Lemma 5.6. *We have*

$$K^{top}(\mathbf{B}) \simeq 0 \ .$$

Here we use an Eilenberg-Swindle argument.

We consider the exact sequence

$$0 \to \mathbf{K} \to \mathbf{B} \xrightarrow{\pi} \mathbf{Q} \to 0 \ .$$

The quotient $\mathbf{Q}$ is called the Calkin algebra.

Corollary 5.7. *We have equivalences*

$$K^{top}(\mathbf{Q}) \simeq \Sigma K^{top}(\mathbf{K}) \simeq \Sigma K^{top}(\mathbb{C}) \simeq \Sigma KU \ .$$

Definition 5.8. *An operator F in $\mathbf{B}$ is called a Fredholm operator if $\pi(F)$ is invertible.*

If F is Fredholm, then $\pi(F)$ in $GL_1(\mathbf{Q})$ can be considered as a point in $GL_1(\mathbf{Q} \otimes \mathbf{K})$. It represents a class $[\pi(F)]$ in $\pi_1 K^{top}(\mathbf{Q})$.

The sequence above gives a boundary map

$$\partial : K^{top}(\mathbf{Q}) \to \Sigma K^{top}(\mathbf{K}) \ .$$

In particular we have the class $\partial[\pi(F)]$ in

$$\pi_1 \Sigma K^{top}(\mathbf{K}) \cong \pi_0 K^{top}(\mathbf{K}) \cong \pi_0 KU \cong \mathbb{Z} \ .$$

Lemma 5.9. *Under these identifications*

$$\partial[\pi(F)] = \mathtt{index}(F) \ .$$

Here

$$\mathtt{index}(F) = \dim(\mathtt{ker}(F)) - \dim(\mathrm{coker}(F)) \ .$$

Here we should see an example of a calculation of a boundary operator in K-theory. Note that ∂ for a sequence $0 \to A \to B \to C \to 0$ of C^*-algebras comes from excision for algebraic K-theory whose essential part is the identification of the fibre of $K^{alg}(B \otimes \mathbf{K}) \to K^{alg}(C \otimes \mathbf{K})$ with $K^{alg}(A \otimes \mathbf{K})$. So the calculation of ∂ is not at all obvious. But the formulas from the book [Mil71] will help.

One could explain that the index is exactly the obstruction against deforming F by a compact operator to an invertible operator in $\mathbf{B}$.

6. Kasparov KK-theory

The aim of this talk is to give an overview about the construction of the Kasparov K-theory functor

$$KK : C^*\mathbf{Alg}^{op}_{gr,sep} \times C^*\mathbf{Alg}_{gr,sep} \to \mathbf{Ab} ,$$

the Kasparov product

$$KK(A,B) \otimes KK(B,C) \to KK(A,C)$$

and basic examples of classes.

Given a C^*-algebra A one can consider the notion of a (right) A-Hilbert module, the obvious generalization of the notion of a Hilbert space replacing the ground field $\mathbb{C}$ by A. On such a Hilbert A-module one has the C^*-algebra $B(H)$ of bounded adjointable (existence of adjoints this is an additional condition) and the subalgebra of compact operators $K(H)$ generated by the finite rank operators $\theta_{v,w}$, $x \mapsto \langle w, x\rangle$ for v, w in H.

For graded C^*-algebras A, B one then introduces a notion of a Kasparov (A,B) module (H, F, ρ):

1. H is a graded B Hilbert module
2. $\rho : A \to B(H)$ is a graded representation
3. $F \in B(H)^{odd}$ is an operator satisfying
   a) $\rho(a)(F^* - F) \in K(H)$ for all a in A
   b) $\rho(a)(F^2 - 1) \in K(H)$ for all a in A
   c) $[F, \rho(a)]$ in $K(H)$ for all a, b in A

Let $\mathcal{E}(A,B)$ be the set of isomorphism classes of Kasparov (A,B)-modules. For homomorphisms $f : A' \to A$ or $g : B \to B'$ we get morphisms

$$f^* : \mathcal{E}(A,B) \to \mathcal{E}(A',B) , \quad (H,\rho,F) \mapsto (H, \rho \circ f, F)$$

and

$$f_* : \mathcal{E}(A,B) \to \mathcal{E}(A,B') , \quad (H,\rho,F) \mapsto (H \otimes_B B', \rho \otimes 1, F \otimes 1) .$$

1. There is a natural notion of a sum of Kasparov (A,B) modules turning $\mathcal{E}$ into a functor
$$C^*\mathbf{Alg}^{op} \times C^*\mathbf{Alg} \to \mathbf{Monoid} .$$
2. A homotopy between Kasparov (A,B) modules M_0, M_1 is a Kasparov module M in $\mathcal{E}(A, C([0,1]) \otimes B)$ which evaluates to M_0 and M_1.
3. (H,ρ,F) is called degenerate if $F^* = F$, $F^2 = 1$ and $[F,\rho(a)] = 0$ for all a in A.

Definition 6.1. *$KK(A,B)$ is the Grothendieck group of the monoid of homotopy classes in $\mathcal{E}(A,B)$. For n in $\mathbb{Z}$ we set*

$$KK_n(A,B) := KK(A, B \otimes \mathrm{Cliff}_n)$$

if $n \geq 0$ and

$$KK_n(A,B) := KK(A, B \otimes C_0(\mathbb{R}^{-n}))$$

if $n < 0$.

Lemma 6.2. *Degenerate modules represent the zero element.*

Proposition 6.3. *We have an isomorphism*

$$K_0(B) \cong KK(\mathbb{C}, B) \ .$$

Lemma 6.4. *We have canonical isomorphisms*

$$KK(A,B) \cong KK(M_n(A), M_m(B))$$

and

$$KK(A,B) \cong KK(A \otimes \mathbf{K}, B) \cong KK(A, B \otimes \mathbf{K}) \cong KK(A \otimes \mathbf{K}, B \otimes \mathbf{K})$$

all induced by the left-upper corner inclusion.

Example 6.5. The isomorphism $\mathrm{Cliff}_{2n} \cong \mathtt{Mat}_{2^n}(\mathbb{C})$ induces the formal Bott periodicity

$$KK_{2n}(A,B) \cong KK(A,B) \ .$$

Example 6.6. A homomorphism $f : A \to B$ gives a class

$$[f] := [B, f, 0] \text{ in } KK(A,B) \ .$$

Example 6.7. We have a Thom class

$$\nu_n \ \text{ in } \ KK_n(\mathbb{C}, C_0(\mathbb{R}^n))$$

represented by $(C_0(\mathbb{R}^n) \otimes \mathrm{Cliff}_n, \rho, F)$, where

$$F(x) = \frac{x}{\sqrt{1+x^2}}$$

(left multiplication).

Example 6.8. Let $D : C^\infty(M,E) \to C^\infty(E)$ be a symmetric Cliff_n-equivariant odd elliptic differential operator on M (where E is a bundle of graded Cliff_n-modules). Then we form the Cliff_n-Hilbert module.

$$H := L^2(M,E) \ .$$

We have a representation $\rho : C_0(M) \to B(H)$. We consider D as an unbounded essentially selfadjoint operator and set

$$F := \frac{D}{\sqrt{1+D^2}} \ \text{ in } \ B(H)^{odd}$$

Then

$$(H, \rho, D) \in \mathcal{E}(C_0(M), \mathrm{Cliff}_n) \ .$$

It represents the class $[D]$ in $KK_n(C_0(M), \mathbb{C})$.

Example 6.9. We consider the odd Cliff_n-equivariant operator $D_n := \sum_{i=1}^n e_i\partial_i$ on the bundle $\mathbb{R}^n \times \mathrm{Cliff}_n \to \mathbb{R}^n$. It provides a class

$$[D_n] \in KK_n(C_0(\mathbb{R}^n), \mathbb{C}) .$$

□

Example 6.10. We consider a hermitean $\mathbb{Z}^2$-graded bundle V of Cliff_n-modules on M. Then $C_0(M,V)$ is a $C_0(V) \otimes \mathrm{Cliff}_n$-Hilbert module in the natural way. We get a class

$$[V] = [C_0(M,V), \rho, 0] \in KK_n(C_0(M), C_0(M)) .$$

□

Let (H,ρ,F) be in $\mathcal{E}(A,B)$ and (H',ρ',F') be in $(\mathcal{E}(B,C)$. Then

1. We can form the Hilbert C-module $H'' := H \otimes_B H'$
2. $\rho'' := \rho \otimes 1$
3. one can choose a F'-connection F'' in $B(H'')^{odd}$ characterized up to operator homotopy by
   a) $T_x \circ F' - F'' \circ T_x \in K(H', H'')$ for all x in H (where $T_x = x \otimes -$).
   b) $F' \circ T_x^* - T_x^* \circ F'' \in K(H'', H')$ for all x in H.

Definition 6.11. *The Kasparov product*

$$KK(A,B) \otimes KK(B,C) \to KK(A,C)$$

is defined such that

$$([H,\rho,F]\ [H',\rho',F']) \mapsto (H'', \rho'', F \otimes \mathtt{id} + F'') .$$

Here is the main theorem about the Kasparov product.

Theorem 6.12. *1. The Kasparov product is well-defined.*

2. It is associative and defines a category (enriched in abelian groups) KK with objects the separable C^*-algebras and morphisms $KK(A,B)$.

3. $f \mapsto [f]$ defines a functor $C^*\mathbf{Alg}_{sep} \to KK$.

On can generalize the product to

$$KK(A, B \otimes D) \otimes KK(B, C \otimes E) \to KK(A, C \otimes D \otimes E)$$

Example 6.13. If D is a Dirac operator and V is a vector bundle, then we have

$$[V] \otimes_{C(M)} [D] = [D_V]$$

in $KK(C_0(M), \mathbb{C})$. □

Example 6.14. The class

$$\nu_n \otimes_{C_0(\mathbb{R}^n)} [D] \in KK_{2n}(\mathbb{C}, \mathbb{C})$$

corresponds to $[\mathtt{id}_{\mathbb{C}}]$ under formal Bott periodicity.

We have an obviously defined exterior product

$$KK(A, B) \otimes KK(C, D) \to KK(A \otimes C, B \otimes D) \ .$$

Example 6.15. The exterior product with $[\mathtt{id}_{\mathrm{Cliff}_1}]$ is an isomorphism. This follows from matrix stability and $\mathrm{Cliff}_2 \cong \mathtt{Mat}_2(\mathbb{C})$.

Proposition 6.16. *The map*

$$KK(A, B) \stackrel{\times[\mathtt{id}\dots]}{\to} KK_2(A \otimes C_0(\mathbb{R}^2) \otimes \mathrm{Cliff}_2, B \otimes C_0(\mathbb{R}^2)) \stackrel{\nu_2 \otimes -}{\to} KK_2(A, B \otimes C_0(\mathbb{R}^2))$$

is an isomorphism with inverse induced by $[D_2]$ *in the obvious way.*

This is the Bott-periodicity isomorphism. For $A = \mathbb{C}$ and we get the Bott periodicity isomorphism

$$K_0^{top}(B) \cong K_0^{top}(C_0(\mathbb{R}^2) \otimes B) \ .$$

Corollary 6.17. *We have*

$$K_n^{top}(B) \cong KK_q(\mathbb{C}, B \otimes C_0(\mathbb{R}^p))$$

as long as $n = q - p$ *with* q, p *in* $\mathbb{N}$.

Corollary 6.18. *We have for all* n *in* $\mathbb{Z}$

$$KK_n(A, B) \cong KK_q(A, B \otimes C_0(\mathbb{R}^p))$$

as long as $n = q - p$ *with* q, p *in* $\mathbb{N}$.

A reference for this talk is the book [Bla98].

7. The KK-category

An element a in a C^*-algebra A is called positive if it can be written in the form $a = b^*b$ for some element b in A.

Let $f : A \to B$ be a *linear* map between C^*-algebras and $\mathtt{Mat}_n(f) : \mathtt{Mat}_n(A) \to \mathtt{Mat}_n(B)$ its extension to matrizes.

Definition 7.1. *f is called (completely) positive if f (or $\mathtt{Mat}_n(f)$ for all $n \in \mathbb{N}$) sends positive elements to positive elements.*

The main feature of completely positive contraction maps is that they appear as $e\rho e : A \to B$, where $\rho : A \to \mathtt{Mat}_n(B)$ is a C^*-algebra homomorphism and e projects onto the first coordinate. This is Stinespring's theorem.

Definition 7.2. *A exact sequence of C^*-algebras is called cps (completely positive split) if*

$$0 \to I \to A \xrightarrow{\pi} A/I \to 0$$

if π admits a completely positive norm-decreasing left inverse.

Note that this split-exact implies cps-exact.

At this point one should discuss a major aspect of KK left out in the previous talk, namely the exact sequences:

A cps exact sequence

$$0 \to I \to A \to A/I \to 0$$

gives rise to an element

$$\delta \in KK(A/I, C_0(R) \otimes I) \cong KK_{-1}(A(I, I) \ .$$

This element should be described in terms of a Kasparov cycle [Bla98, 17.6.4].

Theorem 7.3. *For every separable C^*-Algebra B we have long exact sequences*

$$\dots \to KK_n(B, I) \to KK_n(B, A) \to KK(B, A/I) \stackrel{-\otimes_{A/I}\delta}{\to} KK_{n-1}(B, I) \to \dots$$

and

$$\dots \to KK_n(A/I, B) \to KK_n(A, B) \to KK_n(I, B) \stackrel{\delta\otimes_I -}{\to} KK_{n-1}(A/I, B) \ .$$

A reference is [Bla98].

At this point one could comment that KK can be refined to a triangulated category.

Definition 7.4. *A functor F defined on C^***Alg** to some ∞-category is called:*

1. *stable: if $F(A) \to F(A \otimes \mathbf{K})$ (induced by the left upper corner inclusion) is an equivalence*
2. *homotopy invariant: if the evaluation maps at 0 and 1 induce equivalent morphisms $F(C([0,1]) \otimes A) \to F(A)$*
3. *cps exact: if it sends cps exact sequences of C^*-algebras to fibre sequences.*

Example 7.5. The functor $C^*\mathbf{Alg}_{sep} \to KK$ is stable and homotopy invariant.

C^***Alg** is symmetric monoidal with respect to the maximal tensor product.

The following is the main theorem of this talk.

Theorem 7.6. *1. There exists a universal functor*

$$\iota : C^*\mathbf{Alg}_{sep} \to \mathbf{KK}$$

to a stable ∞-category which is stable and cps-exact.

2. *This functor be obtained as a localization*

$$C^*\mathbf{Alg}_{sep} \to C^*\mathbf{Alg}_{sep}[W_{KK}^{-1}]$$

inverting so-called KK-equivalences.

3. *This functor is homotopy invariant.*

4. *The functor ι has a lax symmetric monoidal refinement.*

The argument is based on the thesis of Higson [Hig87] and given in [LN18, Sec. 3].

We write

$$\mathbf{KK}(A, B) := \mathtt{map}_{\mathbf{KK}}(\iota(A), \iota(B))$$

for the mapping spectrum between the images of A and B in $\mathbf{KK}$. Then the proof of the relation between the homotopy groups of this spectrum with the classical Kasparov groups

$$\pi_i \mathbf{KK}(A, B) \cong KK_i(A, B) \ .$$

The topological K-theory of separable C^*-algebras is representable in $\mathbf{KK}$:

Theorem 7.7. *For A in $C^*\mathbf{Alg}_{sep}$ we have an equivalence of spectra*

$$K^{top}(A) \simeq \mathbf{KK}(\mathbb{C}, A) \ .$$

Remark 7.8. One could in fact define the K-theory functor for C^*-algebras by

$$A \mapsto KK(\mathbb{C}, A) \ .$$

One uses left Kan-extensions to extend from separable to all C^*-algebras. This is an alternative to the definition in the previous talk. The advantage is that in this picture it is much simpler to understand the boundary map for the exact sequences. □

In particular, by the Higher Swan Theorem, for X in CW^{fin} we have the equivalence

$$KU(X) \simeq \mathbf{KK}(\mathbb{C}, C(X)) \ .$$

Theorem 7.9 (Homological Higher Swan Theorem)**.** *For X in CW^{fin} we have a natural equivalence of spectra*

$$KU \wedge \Sigma^\infty_+ X \simeq \mathbf{KK}(C(X), \mathbb{C}) \ .$$

Again one shows that the right-hand side is a homology theory and compares coefficients.

8. Duality

We have the symmetric monoidal universal functor

$$\iota : C^*\mathbf{Alg}_{sep} \to \mathbf{KK} \ .$$

In particula can talk about duality between separable C^*-algebras A, B in the ∞-category **KK**.

Definition 8.1. *A duality between A, B is the datum of a unit and a counit*

$$u \ in \ KK_*(\mathbb{C}, A \otimes B) \ , \quad v \ in \ KK_{-*}(A \otimes B, \mathbb{C})$$

satisfying the triangle identities. If A admits a dual, then it is called dualizable.

1. The dual of A is determined uniquely up to KK-equivalence.
2. A duality between A and B induce a natural equivalence of bifunctors

$$\mathbf{KK}(- \otimes A, -) \simeq \mathbf{KK}(-, B \otimes -) \ .$$

We consider a smooth closed manifold M. If we choose a Riemannian metric, then we can define the bundle of complex Clifford algebras $\mathrm{Cliff}(T^*M) \to M$ associated to the cotangent bundle $T^*M \to M$. We set

$$\mathrm{Cliff}(M) := C_c(M, \mathrm{Cliff}(T^*M)) \ .$$

Up to isomorphism this algebra does not depend on the metric.

A reference for the following result is [Kas88, Sec. 4].

Theorem 8.2. *If M is a smooth closed manifold, then the C^*-algebra $C(M)$ is dualizable in KK with dual represented by* $\mathrm{Cliff}(M)$.

The proof has the following steps.

1. One writes down the Kasparov modules representing the unit and the counit.
2. One verifies the triangle identities.

Here are my candidates (this should indicate that the unit and the counit of the duality can be understood explicitly):

1. The Dirac operator D of the Dirac bundle $\mathrm{Cliff}(T^*M)$ provides a class $[D]$ in $KK(\mathrm{Cliff}(M) \otimes C(M), \mathbb{C})$. It is represented by

$$(L^2(M, \mathrm{Cliff}(T^*M), \rho, \frac{D}{\sqrt{1+D^2}}) \ ,$$

   where D uses the left multiplication, and ρ the right-multiplication.

2. We have an isomorphism of algebras

$$\mathrm{Cliff}_0(M) \otimes C(M) \cong C(M \times M, \mathtt{pr}_1^*\mathrm{Cliff}(T^*M))$$

sending $X \otimes f$ to the section $(x, y) \mapsto X(x)f(y)$. We get an element

$$[C(M, \mathtt{pr}_1^*\mathrm{Cliff}(T^*M)), scal, F] \ ,$$

where F near the diagonal of $M \times M$ is the multiplication by $d\sigma$, where $\sigma(x, y) := \mathrm{dist}(x, y)^2$, and otherwise continued appropriately.

Let $n := \dim(M)$. If M has a $Spin^c$-structure, then the sections of the Cliff_n-equivariant spinor bundle $\mathbf{S}(M) := C_0(M, S(M))$ can be considered as a $\mathrm{Cliff}(M)$-$C(M) \otimes \mathrm{Cliff}_n$-bimodule.

Proposition 8.3. *The bimodule* $\mathbf{S}(M)$ *induces a Morita equivalence between* $\mathrm{Cliff}_0(M)$ *and* $C(M) \otimes \mathrm{Cliff}_n$*.*

This follows from the fact that the fibre $S(M)_m$ of the spinor bundle induces a Morita equivalence between $\mathrm{Cliff}(T^*_mM)$ and Cliff_n.

Corollary 8.4. *For separable* $\mathbb{C}$*-algebras* A *and* B *we get the Poincar'e duality equivalence*

$$PD^{an} : \Sigma^n\mathbf{KK}(A \otimes C(M), B) \simeq \mathbf{KK}(A, B \otimes C(M)) \ .$$

Applying this to $A = B = \mathbb{C}$ and using the higher Swan theorems for homology and cohomology we get the following homotopy theoretic statement:

Corollary 8.5 (Poincaré duality)**.** *A* $Spin^c$*-structure on* M *induces an equivalence*

$$PD^{an} : \Sigma^nKU \wedge M_+ \simeq \mathtt{map}(\Sigma^\infty_+M, KU) \ .$$

We write

$$PD^{an} : KU^*(M) \to KU_{*-n}(M)$$

for the corresponding isomorphism from KU-cohomology to KU homology.

The next goal is to compare this with a topological version of Poincaré duality.

Theorem 8.6 (Atiyah-Bott-Shapiro)**.** $Spin^c$*-vector bundles are naturally and additively* K*-oriented.*

([ABS64] is a reference for the real version).

We thus have a natural Thom class ${}^{Spin^c}\nu^{KU}_V$ in $KU^{\dim(V)}_{0_V}$ for a $Spin^c$-vector bundle V and the relation

$${}^{Spin^c}\nu^{KU}_{V\oplus W} = \mathtt{pr}_V^* \, {}^{Spin^c}\nu^{KU}_V \cup \mathtt{pr}_W^* \, {}^{Spin^c}\nu^{KU}_W$$

This allows to define the integration

$$p^{KU}_! : KU^*_S(E) \to KU^{*-\dim(E)+\dim(B)}(B)$$

for proper maps $p : E \to B$ whose stable normal bundle has a $Spin^c$-structure.

Remark 8.7. If $q : V \to M$ has an almost complex structure, then it has an induced $Spin^c$-structure and we have

$$ {}^{Spin^c}\nu_V^{KU} = \nu_V^{KU} \ . $$

In this case a $Spin$-structure is fixed by the choice of a root L of $\Lambda^{max} V^*$, and the Thom class associated to the $Spin$-orientation is given by $\nu_V^{KU} \cup q^*[L] = {}^{Spin}\nu_V^{KU}$.

Proposition 8.8. *The following square commutes*

$$\begin{array}{ccc} KU^*(M) & \xrightarrow{p_!} & KU^{*-n}(*) \\ \Big\downarrow{\scriptstyle PD^{an}} & & \cong\Big\downarrow{\scriptstyle PD^{an}} \\ KU_{*-n}(M) & \xrightarrow{p_*} & KU_{*-n}(*) \end{array} \ .$$

1. Using the Thom classes for $Spin^c$ bundles and the $Spin^c$-structure on TM one can construct a KU-theory fundamental class on M.
2. This fundamental class naturally induces a Poincaré duality PD^{top}.
3. The square commutes if one replaces PD^{an} by PD^{top}.
4. One verifies the "index theorem" stating $PD^{an} = PD^{top}$. This is essentially the assertion that the image of $[M]_{KU}$ in $KK_n(C(M), \mathbb{C})$ is given by class

   $$[L^2(M, \mathbf{S}(M)), \rho, \frac{\not{D}}{\sqrt{\not{D}^2 + 1}}]$$

   and a comparison with the explicit representative of the counit of the duality given above.

A Dirac operator D on M induces a class

$$[L^2(M, E), \rho, \frac{\not{D}}{\sqrt{\not{D}^2 + 1}}]$$

in $KK(C(M), \mathbb{C}) = K_0(M)$. Let $p : M \to *$.

Lemma 8.9. *If* $\dim(M)$ *is even, then we have*

$$\mathtt{index}(D) = p_*[D] \in KU_0(*) \cong \mathbb{Z} \ .$$

If M has dimension $2n$ and a $spin^c$-structure and $\mathbf{V}$ is a geometric vector bundle on M, then we can form the twisted Dirac operator $\not{D}_{\mathbf{V}}$ and the class $[\mathbf{V}] \in KU^0(M)$.

Proposition 8.10. *We have the equality* $[\not{D}_{\mathbf{V}}] = PD^{an}([\mathbf{V}])$ *in* $KU_{-2n}(M)$.

Here we use that

$$[\not{D}_{\mathbf{V}}] = [V] \otimes_{C(M)} [D] \ .$$

This proposition and $PD^{an} = PD^{top}$ immediately implies a proof of the Atiyah-Singer index theorem for twisted Dirac operators.

9. Crossed products and Connes' Thom isomorphism

Let A be a C^*-algebra.

Then we can consider the C^*-categories $\mathrm{Rep}_C(A)$ of representations on Hilbert C-moduln $A \to B(H)$, and unitary equivalences, for auxiliary C^*-algebras C. We get a functor

$$Rep_C : C^*\mathbf{Alg}^{op} \to C^*\mathbf{Cat}\ , \quad A \mapsto \mathrm{Rep}_C(A)\ .$$

Let G be a locally compact group. A G-C^*-algebra is a C^*-algebra A with an action $\alpha = (\alpha_g)_{g\in G}$ of G on A considered just as a group and a $*$-algebra such that for every a in A the map $g \mapsto \alpha_g(a)$ is continuous.

Remark 9.1. If G is discrete we could just say that a G-C^*-algebra is an object of $\mathbf{Fun}(BG, C^*\mathbf{Alg})$. For locally compact groups we have an additional continuity condition. We could topologically enrich $C^*\mathbf{Alg}$ using the strong (not the norm!) topology and consider enriched functors. □

A covariant representation of (G, A) is a triple (H, π, ρ) of a strongly continuous unitary representation (H, π) of G on some Hilbert C^*-module over some auxiliary C^*-algebra and a representation $\rho : A \to B(H)$ of C^*-algebras such that $\pi(g)\rho(a)\pi(g^{-1}) = \rho(\alpha_g(a))$ for all a in A and g in G. We let $\mathrm{CovRep}_C(A, \alpha)$ be the C^*-category of covariant representations of (A, α) on Hilbert C-modules. We have a functor

$$\mathrm{CovRep}_C : GC^*\mathbf{Alg}^{op} \to C^*\mathbf{Cat}\ , \quad (A, \alpha) \mapsto \mathrm{CovRep}_C(A, \alpha)$$

We let $\widehat{\mathrm{Rep}_C}$ and $\widehat{\mathrm{CovRep}_C}$ denote the Grothendieck constructions of these functors.

Proposition 9.2. *There exists a functor*

$$\rtimes : GC^*\mathbf{Alg} \to C^*\mathbf{Alg}\ , \quad (A, \alpha) \mapsto A \rtimes_{m,\alpha} G$$

such that for every C^*-algebra C there is a (1-categorical) pullback square of categories

$$\begin{tikzcd} \widehat{\mathrm{CovRep}_C} \arrow[r, "\hat{\rtimes}_C"] \arrow[d] & \widehat{\mathrm{Rep}_C} \arrow[d] \\ GC^*\mathbf{Alg} \arrow[r, "\rtimes_m"] & C^*\mathbf{Cat} \end{tikzcd}\ .$$

Definition 9.3. *The C^*-algebra $A \rtimes_{m,\alpha} G$ is called the (maximal) crossed product of A with G.*

1. $A \rtimes_{m,\alpha} G$ is constructed as a completion of the convolution algebra $C_c(G, A)$ in the norm $\|-\|_m$ described as follows. Given a covariant representation (H, π, ρ) of (A, α) (for any C) we define a representation of $C_c(G, A)$ on H by $f \mapsto \int_G \pi(g)f(g)dg$. It induces a norm $\|f\|_{(H,\pi,\rho)}$ on $C_c(G, A)$. We define the norm by

$$\|f\|_m := \sup_{(H,\pi,\rho)} \|f\|_{(H,\pi,\rho)}\ .$$

2. The representation of $C_c(G,A)$ on H extends to a representation of $A \rtimes_{m,\alpha} G$. This defines the functor $\hat{\rtimes}_C$ on the fibre over (A,α).
3. In order to show the cartesian property we must in particular reconstruct the triple (H,π,ρ) from a representation σ of $A \rtimes_{m,\alpha} G$ on H. Let (u_i) be an approximate unit (in general a net, if A is not σ-unital) of A and (δ_i) be a delta sequence in $C_c(G)$. Then we get π and ρ by $\pi(g) := \mathtt{lim}_i\, \sigma(u_i \otimes R_g\delta_i)$ and $\rho(a) := \mathtt{lim}_i\, \sigma(a \otimes \delta_i)$.

Using the universal property one can easily check:

Lemma 9.4. *The functor*

$$- \rtimes_m G : GC^*\mathbf{Alg} \to C^*\mathbf{Alg}$$

preserves exact sequences of C^*-algebras.

There is a particular covariant representation of (A,α) on the Hilbert A-module $L^2(G)\otimes A$. It induces a norm $\| - \|_r$.

Definition 9.5. *The reduced crossed product $A \rtimes_{m,\alpha} G$ of A with G is defined as the completion of $C_c(G,A)$ with respect to $\| - \|_r$.*

By construction we have a homomorphism:

$$A \rtimes_{m,\alpha} G \to A \rtimes_{r,\alpha} G \ .$$

Lemma 9.6. *If G is amenable, then this homomorphism is an isomorphism.*

Example 9.7. The crossed products $\mathbb{C} \rtimes_{m,triv} G$ and $\mathbb{C} \rtimes_{r,triv} G$ are the maximal and reduced group C^*-algebras usually denoted by $C^*_m(G)$ and $C^*_r(G)$. □

Example 9.8. Unital C^*-algebras embed naturally in C^*-categories which have a natural homotopy theory with weak equivalences the unitary equivalences.

For the following we assume that G is discrete. A unital G-C^*-algebra can be considered as a C^*-category with G-action with one object.

Lemma 9.9. *We have a equivalence*

$$A \rtimes_{m,\alpha} G \simeq \mathtt{hocolim}_{BG} A$$

□

We now assume that G is abelian. Then it is amenable and we can just talk about crossed products.

The dual group $\hat{G}$ acts on the set of covariant representations by

$$(\chi,(H,\pi,\rho)) \mapsto (H,\chi\pi,\rho) \ , \quad \chi \in \hat{G} \ .$$

This is compatible with the action $\hat{\alpha}$ of $\hat{G}$ on $A \rtimes_\alpha G$ induced by

$$\chi(a \otimes f) \mapsto a \otimes \chi f \ .$$

Theorem 9.10 (Takai-duality)**.** *There is an isomorphism of G-C^*-algebras*

$$(A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G} \cong A \otimes \mathbf{K}$$

with respect to the actions $\hat{\hat{\alpha}}$ and $\alpha \otimes \mathtt{id}_{\mathbf{K}}$.

This is a nice exercise with the Fourier transformation for locally compact groups.

We now specialize to the group $G = \mathbb{R}^n$.

Remark 9.11. Later in the seminar when we discuss smooth stacks we will see that for a manifold M with an action of $\mathbb{R}^n$ the crossed product $C(M) \rtimes \mathbb{R}^n$ and the groupoid C^*-algebra of the action groupoid $C^*_r(\mathbb{R}^n \curvearrowright M)$ are isomorphic. Moreover $K^{top}(C^*_r(\mathbb{R}^n \curvearrowright M))$ is morally the K-theory of the quotient stack $M//\mathbb{R}^n$. Now $M \to M//\mathbb{R}^n$ is a $\mathbb{R}^n$-principal bundle and therefore trivial. In particular we expect a Thom isomorphism

$$KU(M) \simeq \Sigma^n KU_c(M//\mathbb{R}^n) \ ,$$

in other words, an equivalence

$$K^{top}(C(M)) \simeq \Sigma^n K^{top}(C(M) \rtimes \mathbb{R}^n) \ .$$

Such an isomorphism indeed exists, not only in the commutative case. $\square$

Let (A, α) be an $\mathbb{R}^n$-algebra.

Theorem 9.12 (Connes' Thom isomorphism)**.** *There is an equivalence*

$$K^{top}(A) \simeq \Sigma^n K^{top}(A \rtimes_\alpha \mathbb{R}^n) \ .$$

This map should be constructed as a natural transformation from $\mathbb{R}^n$-algebras to spectra.

Here is the idea: Let (A, α) be a G-C^*-algebra.

1. We reduce to the case $n = 1$ by working stages.
2. We consider the $\mathbb{R}$-space $\mathbb{R}_\infty \cup \{\infty\}$ with ∞ a fixed point.
3. We form the $\mathbb{R}$-algebras $C(A) := C_0(\mathbb{R}_\infty) \otimes A$ and $S(A) := C_0(\mathbb{R}) \otimes A$ with the diagonal action.
4. We have an exact sequence of $\mathbb{R}$-C^*-algebras

   $$0 \to S(A) \to C(A) \to A \to 0$$

   given by evaluation at ∞.
5. By exactness of $\rtimes_m \mathbb{R}$ we get the exact sequence

   $$0 \to S(A) \rtimes \mathbb{R} \to C(A) \rtimes \mathbb{R} \to A \rtimes_\alpha \mathbb{R} \to 0 \ .$$

6. Using Takai duality one shows that $S(A) \rtimes \mathbb{R} \cong A \otimes \mathbf{K}$ (see e.g. [Bla98, Lemma 10.9.1])
7. The Thom equivalence is then induced by the boundary map ∂ as follows:

$$\Sigma K^{top}(A \rtimes_\alpha \mathbb{R}) \stackrel{\partial}{\to} \Sigma^2 K^{top}(S(A) \rtimes \mathbb{R}) \simeq \Sigma^2 K^{top}(A) \stackrel{Bott}{\simeq} K^{top}(A) \ .$$

8. In order show that this is an equivalence we show that $K^{top}(C(A) \rtimes \mathbb{R}) \simeq 0$. The (quite complicated) argument is given in [Bla98, Sec. 10.9]. The C^*-algebra $C(A)$ is contractible after forgetting the $\mathbb{R}$-action. So if the can interpret the crossed product as a sort of homotopy quotient, then this result is plausible.

Example 9.13. Consider A with the trivial action of $\mathbb{R}$. If we compose the Thom isomorphisms for the action of $\mathbb{R}$, then of the dual group $\hat{\mathbb{R}}$, and then use the Takai-duality, we get the Bott periodicity isomorphism

$$K^{top}(A) \simeq \Sigma K^{top}(A \rtimes_{triv} \mathbb{R}) \simeq \Sigma^2 K^{top}((A \rtimes_{triv} \mathbb{R}) \rtimes_{\hat{triv}} \hat{\mathbb{R}}) \simeq \Sigma^2 K^{top}(A \otimes \mathbf{K}) \simeq \Sigma^2 K^{top}(A) \ .$$

Note that $\hat{triv}$ is not trivial.

A starting reference for most of this is [Con94, Appendix C]. Complete proofs are given in [Bla98, Sec. 10].

Part III. Lie groupoids

10. Lie groupoids

We consider the site **Mf** of manifolds with the usual topology generated by open coverings. On this site we can consider sheaves of sets and groupoids, i.e. stacks. A reference for stacks on **Mf** is [Hei05]. We identify manifolds with sheaves using the Yoneda embedding which we do note write explicitly. We also consider sheaves as stacks in the canonical way.

Since **Groupoids** is 2-categorically complete, the 2-category of stacks also has all (2-categorical) limits.

Definition 10.1. *A morphism between stacks $X \to Y$ is called representable if for any map $M \to X$, where M is (the Yoneda image of) a manifold, the stack $M \times_Y X$ is also a manifold.*

Example 10.2. A submersion between manifolds is representable. I do not know whether a representable map between manifolds must be a submersion. □

The properties *proper, surjective, submersion, admit local sections, local diffeomorphism* of maps between manifolds are stable under pull-back. Such a property $\mathbf{P}$ extends to representable morphisms between stacks:

Definition 10.3. *A representable morphism $X \to Y$ between stacks has the property P if the induced morphism $M \times_Y X \to M$ has the property $\mathbf{P}$ for every map $M \to X$ from a manifold.*

Example 10.4. For a Lie group G acting on a manifold N we have the quotient stack $G\backslash\backslash N$ which associates to a manifold M the groupoid $(G\backslash\backslash N)(M)$ of pairs (P,ρ) of G-principal bundles $P \to M$ (where we G acts from the left) and G-equivariant maps $\rho : P \to N$, and isomorphisms of such pairs.

The canonical map $N \to G\backslash\backslash N$ corresponds to the object $(G \times N \to N, \mathtt{pr}_N)$ of the groupoid $(G\backslash\backslash N)(N)$ consisting of the trivial bundle and the projection to the first factor.

For a manifold M with a map $M \to G\backslash\backslash N$ represented by (P,ρ) we have $M \times_{G\backslash\backslash N} N \cong P$. $\square$

We consider a stack X.

Definition 10.5. *An atlas of X is a representable surjective submersion $M \to X$ from a manifold. A stack which admits an atlas is called smooth.*

Example 10.6. The quotient stack $G\backslash\backslash N$ from above is smooth. The map $N \to G\backslash\backslash N$ is an atlas. $\square$

Example 10.7. Orbifolds are special smooth stacks.

Definition 10.8. *A smooth stack which admits and atlas which is addition a proper local diffeomorphism is called an orbifold.*

If G is finite, then $G\backslash\backslash N$ is an orbifold. $\square$

Smooth stacks form a full sub 2-category of all stacks which can be studied using Lie groupoids. As we shall see a Lie groupoid is essentially the same datum as a smooth stack with a choice of an atlas.

Definition 10.9. *A Lie groupoid is a groupoid in* $\mathbf{Mf}$ *whose source and range maps are submersions.*

Lie groupoids from a 2-category in the natural way.

Our notation for a Lie groupoid is

$$\text{mfd of morphisms} \rightrightarrows \text{mfd of objects}$$

where the two arrows indicate the source and the range map.

Example 10.10. Here are examples of Lie groupoids:

1. A manifold M is a Lie groupoid $M \rightrightarrows M$.
2. A Lie group is a Lie groupoid $G \rightrightarrows *$.
3. If a Lie group G acts on M, then we can form the action groupoid

$$G \curvearrowright M := (G \times M \rightrightarrows M) \ .$$

   Here source, range and composition are given by $s(g, m) := m$, $r(g, m) := gm$ and $(g, hm) \circ (h, m) = (gh, m)$.

$\square$

Definition 10.11. *An (left)-action of a Lie groupoid $G = (G^1 \rightrightarrows G^0)$ on a manifold M is given by a map $M \to G^0$ and an action map $a : G^1 \times_{s,G^0} M \to M$ such that*

$$\begin{tikzcd} G^1 \times_{s,G^0} M \arrow[rr] \arrow[dr, "r"] & & M \arrow[dl] \\ & G^0 & \end{tikzcd}$$

commutes and the obvious unit and associativity relations are satisfied.

Example 10.12. $G^1 \rightrightarrows G^0$ acts on G^0 in the obvious way. $\square$

Assume that G acts on manifolds M and N.

Definition 10.13. *An equivariant map is a map $f : N \to M$ such that*

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr] & & N \arrow[dl] \\ & G^0 & \end{tikzcd} \quad , \quad \begin{tikzcd} G^1 \times_{G^0} M \arrow[r] \arrow[d, "(\mathtt{id}_{G^1}, f)"] & M \arrow[d, "f"] \\ G^1 \times_{G^0} N \arrow[r] & N \end{tikzcd}$$

commute.

If a Lie groupoid G acts freely and properly on a manifold M, then one can also form the classical quotient manifold $G \backslash M$ (by the equivalence relation generated by the action).

Definition 10.14. *A G-principal bundle is given by a submersion $p : P \to M$ together with a free and proper fibrewise action of G on P (fibrewise means that the map $P \to G^0$ factors over p and the action map preserves p) such that $G \backslash P \cong M$.*

Example 10.15. Generalizing the case of Lie group actions, if a Lie groupoid G acts on a manifold N, then we can form the quotient stack $G \backslash\backslash N$. Its value on a manifold M is the groupoid of pairs (P, ρ) of G-principal bundles $P \to M$ together with an equivariant map $P \to N$.

On N we have the G-principal bundle $Q := G^1 \times_{s,G^0} N \to N$ with the obvious action. The action map of G on M is then an equivariant map $Q \to N$. In this way we get a canonical map

$$N \to G\backslash\backslash N .$$

One checks that for any manifold M with a map $M \to G\backslash\backslash N$ represented by a G-principal bundle $P \to M$ we have $M \times_{G\backslash\backslash N} N \cong P$. In particular, $G\backslash\backslash N$ is a smooth stack with atlas $N \to G\backslash\backslash N$. □

Example 10.16. If a Lie groupoid G acts freely and properly on a manifold M, then we have an equivalence of stacks $G\backslash M \simeq G\backslash\backslash M$.

A Lie groupoid $G := (G^1 \rightrightarrows G^0)$ represents a stack $Y(G)$, the sheafification of the presheaf of groupoids represented by G.

Lemma 10.17. *The stack $Y(G)$ is equivalent to $G\backslash\backslash G^0$.*

Corollary 10.18. *The canonical map $G^0 \to Y(G)$ is an atlas of $Y(G)$ and $Y(G)$ is therefore a smooth stack.*

We have seen the way from Lie groupoids to smooth stacks. We now consider the other direction. Let $M \to X$ is an atlas of a smooth stack.

Lemma 10.19. *Then*

$$G(M \to X) := (M \times_X M \rightrightarrows M)$$

is a Lie groupoid and there is a canonical equivalence of stacks $Y(G(M \to X)) \simeq X$.

We now can check:

Proposition 10.20. *The constructions above produce an equivalence of ordinary categories*

$$\{\textit{Lie groupoids}\} \simeq \{\textit{smooth stacks with an atlas}\} .$$

In particular every smooth stack is the quotient of a groupoid action on a manifold. But it is in many ways corresponding to the choices of an atlas.

Example 10.21. Let a groupoid G act on a manifold M. Then we can form the action groupoid $G \curvearrowright M$ extending the definition of an action groupoid in the case of a Lie group action.

We can find the explicit construction from general principles. We consider the atlas $M \to G\backslash\backslash M$. Then the action groupoid is given by $M \times_{G\backslash\backslash M} M \rightrightarrows M$. Explicitly, it is the groupoid

$$G^1 \times_{s,G^0} M \rightrightarrows M$$

with obvious structure maps. We have of course an equivalence

$$Y(G \curvearrowright M) \simeq G\backslash\backslash M .$$

□

Example 10.22. Let G be a Lie groupoid and $P \to M$ be a G-principal bundle. Then we have an equivalence of stacks $G\backslash\backslash P \simeq M$. $\square$

Example 10.23. We can consider the pair groupoid $M \times M \rightrightarrows M$. One checks that the canonical map $(M \times M \rightrightarrows M) \to (* \rightrightarrows *)$ induces an equivalence of stacks. The choice of a base point in M determines an inverse up to isomorphism $(* \rightrightarrows *) \to (M \times M \rightrightarrows M)$ on the groupoid level.

Let $P \to M$ be a non-trivial G-principal bundle for a Lie group G. Then we have an equivalence $G\backslash\backslash P \simeq M$ but there is no inverse equivalence of the map between the action groupoids

$$(G \times P \rightrightarrows P) \to (M \rightrightarrows M)$$

since P has no section. We see that the canonical functor

$$\{\text{Lie groupoids}\}[\{\text{equivalences}\}^{-1}] \to \{\text{smooth stacks}\}$$

is not an equivalence of 2-categories. $\square$

The following discussion fixes this problem. Let G and H be two Lie groupoids.

Definition 10.24. *A (G, H)-bibundle is a H-principal bundle $P \to G^0$ with a commuting right-action of G. It is called a Morita bi-bundle if it is also a G-principal bundle.*

If P is Morita, then we use the notation $\hat{P}$ for P considered in the other direction.

Note that a bibundle (G, H)-bibundle induces a G-equivariant map $G^0 \to Y(H)$ and therefore descends to a map $Y(G) \to Y(H)$. If it is Morita, then the induced map is an equivalence.

Remark 10.25. Here is a more precise description of this map. Let M be a manifold and (Q, ρ) be an object in $Y(G)(M)$, where $Q \to M$ is a G-principal bundle and $\rho : Q \to G^0$ is equivariant. Then the map sends this object to the object in $Y(H)$ consisting of the associated H-principal bundle $P \times_G Q \to M$ and the H-equivariant map $P \times_G Q \to P \to H^0$, where the second map is the base projection of the H-principal bundle P. Note that we use ρ in order to define the associated bundle $P \times_G Q$.

One can compose bi-bundles. If Q is a (H, L)-bibundle, then we can form the (G, L)-bi-bundle $P \times_H Q \to L^0$. This is compatible with composition of the associated morphisms between the stacks.

Definition 10.26. *We define the 2-category $\{Lie\ groupoids\}_{Morita}$ of Lie groupoids, bi-bundles, and isomorphisms of bi-bundles.*

The construction above then induces a functor

$$\{Lie\ groupoids\}_{Morita} \to \{smooth\ stacks\}\ .$$

A morphism of Lie groupoids $f : G \to H$ induces a (G, H)-bi-bundle

$$G^0 \times_{H^0,s} H^1 \to G^0$$

with G-action naturally induced by f. This gives a functor

$$\{\textit{Lie groupoids}\} \to \{\textit{Lie groupoids}\}_{Morita} \ .$$

We let W_{Morita} be the morphisms in $\{\textit{Lie groupoids}\}$ which induce equivalences of stacks.

Proposition 10.27. *We have equivalences 2-categories*

$$\{\textit{Lie groupoids}\}[W^{-1}_{Morita}] \simeq \{\textit{Lie groupoids}\}_{Morita} \simeq \{\textit{smooth stacks}\} \ .$$

Example 10.28. If $p : E \to B$ is a fibre bundle, then we can consider the fibrewise pair groupoid $P(p)$. It is Morita equivalent to $B \rightrightarrows B$ and therefore a presentation of the stack B (note that $E \to B$ is an atlas).

The Morita equivalence $P(p) \sim (B \rightrightarrows B)$ is given by $E \to B$ is considered as a bi-bundle. Indeed, $E \to B$ has a $P(p)$-right action and the (trivial) $(B \rightrightarrows B)$-action. It is a $P(p)$-principal bundle over B and a $(B \rightrightarrows B)$-principal bundle over E, hence a Morita bi-bundle. □

Example 10.29. A vector bundle $\pi : E \to M$ gives rise to a Lie groupoid with $s = r = \pi$ and the groupoid structure is induced by the fibrewise addition. □

Let G be a Lie groupoid.

Definition 10.30. *The underlying vector bundle of the Lie algebroid of G is defined by*

$$A(G) := e^* T^v r \ ,$$

where $e : G^0 \to G^1$ is the identity section and $T^v r$ is the vertical tangent bundle along r.

The Lie algebroid has further structures:

1. the anchor map $ds : A(G) \to TG^0$.
2. The sections $C^\infty(G^0, A(G))$ have a natural Lie algebra structure which satisfies a Leibnitz rule
$$[X, fY] = f[X, Y] + ds(X)(f)Y \ .$$
The bracket is induced from the bracket on $C^\infty(G^1, TG^1)$ in the natural way.

This construction generalizes the construction of the Lie algebra of a group.

On could address the question whether a Lie algebroid A on a manifold G^0 comes from a Lie groupoid with objects G^0 and how unique this Lie groupoid is.

Example 10.31. Let $p : E \to B$ be a fibre bundle and $P(p)$ denote the fibrewise pair groupoid. Then its Lie algebroid is given by the vertical bundle $P^v p$.

11. C^*-algebras associated to Lie groupoids

The goal of this talk is to understand the association of a C^*-algebra to a Lie groupoid, its functoriality and its behaviour under Morita equivalences. Furthermore the talk should provide the basic examples. A reference for the material is [Con94].

Let G be a Lie groupoid. On G^1 we have the vector bundles of half densities $\Omega^{1/2}T^v r$ and $\Omega^{1/2}T^v s$ along the fibres of the range and source maps, and we set

$$\Omega^{1/2} := \Omega^{1/2}T^v r \otimes \Omega^{1/2}T^v s \ .$$

On $C_c(G^1, \Omega^{1/2})$ we can define a convolution product

$$(f \circ g)(\gamma) = \int_{G^{r(\gamma)}} f(\sigma) g(\sigma^{-1}\gamma)$$

(note that the integrand is a density along the fibres of r) and the $*$-operation

$$f^*(\gamma) := \bar{f}(\gamma^{-1}) \ .$$

We get a $*$-representation of $C_c(G, \Omega^{1/2})$ on $L^2(G^x, \Omega^{1/2}T^v r)$ for every x in G^0 by convolution. The family of these representations induces a the reduced C^*-norm $\| - \|_r$ on $C_c(G, \Omega^{1/2})$.

Definition 11.1. *The reduced C^*-algebra $C^*_r(G)$ of the Lie groupoid G is defined as the closure of $C_c(G, \Omega^{1/2})$ with respect to $\| - \|_r$.*

The association of the C^*-algebra to a Lie groupoid has the follwing functorialities. If $H \to G$ is an open and $L \to G$ is a closed embedding of Lie groupoids (i.e. induced from the corresponding embeddings $H^0 \to G^0$ and $L^0 \to G^0$ of saturated morphism spaces), then we have morphisms

$$C^*_r(H) \to C^*(G) \ , \quad C^*_r(G) \to C^*_r(L)$$

induced by extension by zero and restriction.

We must understand that bi-bundles induce (strong) Morita equivalences between C^*-algebras

Let A, B be C^*-algebras.

Definition 11.2. *A (A, B)-bi-module is a Hilbert B-module E with a representation $\rho : A \to B(E)$. If the representation ρ takes values in $K(E)$, then we call E small.*[1]

Example 11.3. We consider A as a small (A, A)-bi-module in the obvious way.

If $f : A \to B$ is a homomorphism, then we can consider B as a small (A, B)-bi-module in the natural way. □

[1] This does not seem to be a standard notion.

Given (A, B) and (B, C) bi-modules E and F we can define the (A, C)-bimodule $E \otimes_B F$ as a closure of the algebraic tensor product with respect to the norm induced by the C-valued scalar product

$$\langle e \otimes f, e' \otimes f' \rangle := \langle f, \langle e, e' \rangle f' \rangle \ .$$

If E and F are small, then so is $E \otimes_B C$.

Example 11.4. If E is a small (A, B)-bi-module with representation $\rho : A \to K(E)$, then we get an (A, B) Kasparov module $[E, \rho, 0]$. We let $[E]$ in $KK(A, B)$ denote the corresponding element and $[E]_* : K^{top}(A) \to K^{top}(B)$ be the induced morphism of K-theory spectra. If F is a second small (B, C)-bi-module, then we have the relations

$$[E \otimes_B C] = [E] \otimes_B [C]$$

and

$$[E \otimes_B C]_* = [C]_* \circ [E]_* \ . \tag{11.1}$$

Definition 11.5. *A (strong) Morita equivalence between A and B is induced by (A, B) and (B, A)-bi-modules E and F such that $E \otimes_B F \cong A$ and $F \times_A E \cong B$.*

Remark 11.6. If E and F induce a Morita equivalence, then E and F are automatically small. A (strong) Morita equivalence between A and B induces equivalences

$$[E]_* : K^{top}(A) \stackrel{\simeq}{\to} K^{top}(B)$$

with inverse $[F_*]$ and also a KK-equivalence if A and B are separable. The latter follows from the fact that the strong Morita equivalence implies $A \otimes \mathbf{K} \cong B \otimes \mathbf{K}$ [Con94, Thm. A.8], [BGR77]. The first assertion can be deduced from (11.1). $\square$

We consider a (G, H)-bi-bundle $\pi : P \to G^0$ and associate to it a $(C_r^*(G), C_r^*(H))$-bi-module. On $C_0(P, \Omega^{1/2} T^v \pi)$, using the right-action of H, we define the $C_r^*(H)$-valued scalar product

$$\langle f, f' \rangle(h) := \int_{P^{r(h)}} f(p) f'(ph) \ .$$

Here one must check that the integral takes values in the fibre of $\Omega_h^{1/2}$. After taking closures we get a Hilbert $C_r^*(H)$-module $E(P)$. The action of G on P defines an action of $C_r^*(G) \to B(E(P))$ by convolution

$$(\phi f)(p) := \int_{G^{\pi(p)}} \phi(g) f(g^{-1} p) \ .$$

Again one must observe that the integrand is canonically a density on $G^{\pi(p)}$. One can check that this bi-module is small.

If P is Morita, then we also have a $(C_r^*(H), C_r^*(G))$-bi-module $E(\hat{P})$.

Proposition 11.7. *If the (G, H) bi-bundle P is Morita, then $E(P)$ and $E(\hat{P})$ induce a strong Morita equivalence between between C^*-algebrs $C^*(G)$ and $C^*(H)$.*

This is a calculation of tensor products.

Remark 11.8. This might be an interesting small research project: We can consider the category

$$C^*\mathbf{Alg}_{Morita}$$

of C^*-algebras, bi-modules and isomorphisms of bi-modules. There are 2-functors

$$E : \{\textit{Lie groupoids}\}_{Morita} \to C^*\mathbf{Alg}_{Morita} \ , \quad \tilde{K}^{top} : C^*\mathbf{Alg}_{Morita} \to \mathbf{Sp} \ .$$

We then get a topological K-theory functor for smooth stacks by

$$\{\textit{smooth stacks}\} \simeq \{\textit{Lie groupoids}\}_{Morita} \stackrel{\tilde{K}^{top}}{\to} \mathbf{Sp} \ .$$

□

In the following we relate constructions with Lie groupoids with constructions in C^*-algebras.

Let H be a Lie group acting on a manifold M. Then H acts on $C_0(M)$.

Lemma 11.9. *We have an isomorphism* $C_r^*(H \curvearrowright M) \cong C_0(M) \rtimes H$*.*

In particular $C_r^*(G) \cong C_r^*(G \curvearrowright *)$.

Let $E \to B$ be a vector bundle and G be the associated groupoid.

Lemma 11.10. *We have an isomorphism of C^*-algebras*

$$C_r^*(G) \cong C_0(E^*) \ .$$

This is shown using the fibrewise Fourier transformation.

Let $\lambda : G \to \mathbb{R}^N$ be a homomorphism of Lie groupoids. A homomorphism induces an action on $G^0 \times \mathbb{R}^N$ in the natural way. Then the dual group $\hat{\mathbb{R}}^N$ of $\mathbb{R}^N$ acts on $C_r^*(G)$ by

$$(\chi f)(\gamma) := \chi(\lambda(\gamma))f(\gamma) \ .$$

In this situation we have:

Proposition 11.11. *We have a Morita equivalence*

$$C_r^*(G) \rtimes \hat{\mathbb{R}}^N \simeq C_r^*(G \curvearrowright (G^0 \times \mathbb{R}^N)) \ .$$

Remark 11.12. The following should explain why this proposition is true. We have an equivalence of stacks $\mathbb{R}^N//\mathbb{R}^N \simeq *$ and therefore

$$G\backslash\backslash G^0 \simeq G\backslash\backslash(G^0 \times \mathbb{R}^N//\mathbb{R}^N) \simeq (G\backslash\backslash(G^0 \times \mathbb{R}^N))//\mathbb{R}^N \ .$$

Hence we have Morita equivalence

$$C_r^*(G) \sim C_r^*(G \curvearrowright (G^0 \times \mathbb{R}^N)) \rtimes \mathbb{R}^N \ .$$

This explains the action of $\hat{\mathbb{R}}^N$ on $C_r^*(G)$. Finally by Takai duality an get

$$C_r^*(G) \rtimes \hat{\mathbb{R}}^N \simeq C_r^*(G \curvearrowright (G^0 \times \mathbb{R}^N)) \otimes \mathbf{K} \ .$$

□

Using Proposition 11.11 and Connes' Thom isomorphism for the $\hat{\mathbb{R}}^N$-action we get

Corollary 11.13 (Generalized Thom Isomorphism)**.** *We have an equivalence*

$$K^{top}(C_r^*(G \curvearrowright (G^0 \times \mathbb{R}^N))) \simeq \Omega^N K^{top}(C_r^*(G)) \ .$$

This transformation should be constructed as a natural transformation between (contravariant for closed and covariant for open embeddings) spectrum-valued functors on Lie-groupoids with action on $\mathbb{R}^N$.

Example 11.14. We the groupoid $B \rightrightarrows B$ with the unique homomorphism to $\mathbb{R}^N$. Then $(B \rightrightarrows B) \curvearrowright (B \times \mathbb{R}^N) \cong (B \times \mathbb{R}^N \rightrightarrows B \times \mathbb{R}^N)$. Consequently, the generalized Thom isomorphism provides an equivalence

$$KU_c(B \times \mathbb{R}^N) \simeq K^{top}(C_0(B \times \mathbb{R}^N)) \simeq \Omega^N K^{top}(C_0(B)) \simeq \Omega^N KU_c(B) \ .$$

This is equivalence is the suspension map known from topology.

If G acts freely and properly on $G^0 \times \mathbb{R}^N$, then the classical quotient $G\backslash(G^0 \times \mathbb{R}^N)$ exists. In this case we have an equivalence

$$K^{top}(C_0(G\backslash(G^0 \times \mathbb{R}^N))) \simeq \Omega^N K^{top}(C_r^*(G)) \ ,$$

i.e. we can express the K-theory of the non-commutative C^*-algebra $C_r^*(G)$ in terms of the K-theory of a commutative C^*-algebra. This is the basic idea in Connes' approach to index theorems.

12. Deformation to the normal cone

The goal of the talk is to define a groupoid version of an index map using the deformation to the normal cone construction (DNC) and to relate it with the classical analytic index map of Atiyah-Singer.

We consider a closed submanifold L in M and let $\mathtt{pr} : N_L^M \to L$ be the normal bundle of L in M. We consider $[0, \infty]$ as a manifold diffeomorphic by $t \mapsto e^{-t}$ with $[0, 1]$[2]. The multiplicative group $\mathbb{R}_+^*$ of positive reals acts by multiplication with fixed points 0 and ∞.

Definition 12.1. *On* **Mf** *we define the sheaf $DNC(M, L)$ (deformation to the normal cone) as follows: for every manifold N the set $DNC(M, L)(N)$ consists of*

1. *a map $\lambda : N \to [0, \infty]$,*
2. *a map $f : N \to M$ a map*
3. *$\xi : N_{\lambda=0} \to N_L^M$*

such that

1. *λ is smooth,*
2. *f is smooth,*
3. *$\mathtt{pr} \circ \xi = f_{|N_{\lambda=0}}$*
4. *for every smooth function $g \in C^\infty(M)$ with $g_{|L} = 0$ the function*

$$n \mapsto \begin{cases} g(f(n))\lambda(n)^{-1} & \lambda(n) \neq 0 \\ dg(f(n))(\xi(n)) & \lambda(n) = 0 \end{cases}$$

is smooth.

Proposition 12.2. *$DNC(M, L)$ is a smooth manifold.*

We have smooth maps and cartesian squares

$$\begin{tikzcd}
 & {[0,\infty] \times M} & \\
(0,\infty] \times M \arrow[r] \arrow[d] & DNC(M,L) \arrow[u] \arrow[d, "\pi"] & N_L^M \arrow[l] \arrow[d] \\
(0,\infty] \arrow[r] & {[0,\infty]} & \{0\} \arrow[l]
\end{tikzcd}$$

We have an action of $\mathbb{R}_+^*$ on $DNC(M, L)$ given on the value $DNC(M, L)(N)$ by

$$(t, \lambda) \mapsto t\lambda \ , \quad (t, f) \mapsto f \ , \quad (t, \xi) \mapsto t^{-1}\xi \ .$$

[2]this avoids exponential functions in the formulas below

Remark 12.3. If L has codimension ≥ 1, then the group $\mathbb{R}_+^*$ acts freely and properly on $DNC(M,L)_{[0,\infty)}$ and the quotient $DNC(M,L)_{[0,\infty)}/\mathbb{R}_+^*$ is called the blow-up of M along L. □

Example 12.4. We have $DNC(M,M) \cong [0,\infty] \times M$.

If $V \to B$ is a vector bundle, then

$$DNC(V, 0_V)_{[0,\infty)} \cong [0,\infty) \times V$$

by the map

$$(\lambda, v, \xi) \mapsto \begin{cases} (\lambda, \lambda^{-1}v) & \lambda \neq 0 \\ (0, \xi) & \lambda = 0 \end{cases}$$

In contrast, the canonical map $DNC(V, 0_V) \to [0,\infty] \times V$ is given by

$$(\lambda, v, \xi) \mapsto (\lambda, v) \ .$$

Given a map of pairs

$$h : (M, L) \to (M', L')$$

we get a functorially induced $\mathbb{R}_+^*$-equivariant map

$$DNC(h) : DNC(M,L) \to DNC(M',L') \ , \quad \lambda \mapsto \lambda \ , \quad f \mapsto h \circ f, \quad \xi \mapsto \bar{d}h_{|L} \circ \xi \ .$$

If h and $h_{|L}$ are surjective submersions, then so is $DNC(h)$.

For a pull-back square with all intersections transverse

$$\begin{tikzcd} (M_1 \times_M M_2, L_1 \times_L L_2) \arrow[r] \arrow[d] & (M_2, L_2) \arrow[d] \\ (M_1, L_1) \arrow[r] & (M, L) \end{tikzcd}$$

we get the pull-back

$$\begin{tikzcd} DNC(M_1 \times_M M_2, L_1 \times_L L_2) \arrow[r] \arrow[d] & DNC(M_2, L_2) \arrow[d] \\ DNC(M_1, L_1) \arrow[r] & DNC(M, L) \end{tikzcd}$$

Here we use a corresponding pull-back diagram for the normal bundles.

Consequently we can apply the DNC construction to pairs (G, H) of Lie groupoids and an embedded subgroupoid.

Remark 12.5. There is a well-defined notion of a locally closed embedded substack Y of a smooth stack X. The DNC-construction can be applied to the pair (X, Y) and produces again a smooth stack. It is compatible with the construction for Lie groupoids.

Definition 12.6. *For a Lie groupoid G let*

$$\mathcal{T}(G) := DNC(G, G^0 \rightrightarrows G^0) \ .$$

be called the tangent groupoid of G.

A reference for these constructions is [DS, Sec. 4.1], suitably interpreted in order to read off Proposition 12.2. For what follows a starting point is [Con94, Ch. 2.5], but most of this is also in [DS]. See also [His10].

We have an open and a closed embdding

$$\mathcal{T}(G)_{(0,\infty]} \to \mathcal{T}(G) \leftarrow \mathcal{T}(G)_{\{0\}} \ .$$

We have further an embedding $j : G \to \mathcal{T}(G)_{\{1\}}$.

We obtain an exact sequence of C^*-algebras

$$0 \to C^*_r(\mathcal{T}(G)_{(0,\infty]}) \to C^*_r(\mathcal{T}(G)) \to C^*_r(\mathcal{T}(G)_{\{0\}}) \to 0 \ .$$

Proposition 12.7. *1. $C^*_r(\mathcal{T}(G)_{(0,\infty]})$ is contractible and hence the restriction map*

$$K(C^*_r(\mathcal{T}(G))) \to K(C^*_r(\mathcal{T}(G)_{\{0\}})$$

is an equivalence.

2. $\mathcal{T}(G)_{\{0\}}$ is isomorphic to the Lie groupoid associated to the vector bundle $A(G) \to G^0$ and consequently

$$C^*_r(\mathcal{T}(G)_{\{0\}}) \cong C_0(A(G)^*) \ .$$

Definition 12.8. *The composition*

$$\mathtt{index}_G : K^{top}(C_0(A(G)^*)) \simeq K^{top}(C^*_r(\mathcal{T}(G)_{\{0\}})) \simeq K^{top}(C^*_r(\mathcal{T}(G))) \stackrel{j^*}{\to} K^{top}(C^*_r(G))$$

is called index map of G.

Let $p : E \to B$ be a proper submersion and $P(p)$ be the fibrewise pair groupoid.

Proposition 12.9. *1. We have a canonical Morita equivalence $C^*_r(P(p)) \sim C_0(B)$.*

2. We have an isomorphism of vector bundles $A(P(p)) \cong T^vp$ over E.

3. The composition

$$\mathtt{index}^{grpd}_{an} : K^{top}(C_0(T^vp)) \simeq K^{top}(C_0(A(G)^*)) \stackrel{\mathtt{index}_{P(p)}}{\to} K^{top}(C^*_r(P(p))) \simeq K(C_0(B))$$

is equivalent to the analytic index map for families $\mathtt{index}^{AS}_{an}$ of Atiyah-Singer.

Here we implicitly also use the equivalences

$$K^{top}(C_0(A(G)^*)) \simeq KU_c(A(G)^*) \ , \quad K^{top}(C_c(B)) \simeq KU_c(B)$$

induced from the Cohomological higher Swan Theorem (which must be extended to locally compact spaces and a version with supports).

Remark 12.10.

1. We have a Morita equivalence of Lie groupoids $P(p) \sim (B \rightrightarrows B)$ see earlier.
2. We have calculated that $A(P(p)) \cong T^v p$ earlier.
3. Here is a sketch for the case of a single manifold. The family case is similar.

   We use pseudodifferential operators. We fix an operator convention Op for M. Given a complete symbol σ in $C_c^\infty(T^*M)$ we let $\sigma_\lambda(\xi) := \sigma(\lambda^{-1}\xi)$. It gives rise to the family of smooth distribution kernels $\mathrm{Op}(\sigma_\lambda)$ on $M \times M$. One can check that this family extends by continity to a family of kernels on $\mathcal{T}(G)$. In this way we get an element

   $$(\mathrm{Op}_\lambda(\sigma_\lambda))_{\lambda \in [0,\infty]} \in C_r^*(\mathcal{T}(G)) \ .$$

   In order to see this extension we use the coordinates $(x+y, x-y)$. Then

   $$\mathrm{Op}(\sigma_\lambda)(x+y, \lambda^{-1}(x-y))$$

   extends to $\lambda = 0$. The basic calculation in the euclidean case is that

   $$\lambda^{-\dim(M)} \int_{\mathbb{R}^{\dim(m)}} e^{i\langle \eta, \lambda^{-1}(x-y)\rangle} \sigma(\lambda^{-1}\eta, x+y) d\eta$$

   (the pre-factor comes from the fact that the kernels take values in half-densities) does not depend on λ at all.

   Consequently, $\mathtt{index}_{an}^{grpd}([\sigma])$ is the index of the pseudodifferential operator $\mathrm{Op}(\sigma)$. In order to produce non-trivial results this final step requires to extend all constructions to vector bundles over M.

13. Proof of the families index theorem (a la Connes)

The goal of this talk is to provide a proof of the families index theorem using the groupoid approach. This is due to Connes and sketched in the book [Con94] at least in the case of a single manifold. Another reference with a different proof for Dirac operators is [His10].

We consider a proper submersion $p : E \to B$ with compact B. We fix an embedding $i : E \to \mathbb{R}^N$.

Lemma 13.1. *The embedding naturally induces a homomorphism of groupoids $j : \mathcal{T}(P(p)) \to \mathbb{R}^N$.*

We describe this homomorphism in detail in language of points $\mathcal{T}(P(p))(M)$. It is clear on objects so we concentrate on morphisms. A point in $\mathcal{T}(E)^1(M)$ is tuple (λ, f, ξ), where

1. $\lambda : M \to [0, \infty)$ is smooth.
2. $f : M \to E \times_B E$ is smooth.
3. $\xi : M_{\lambda=0} \to T^v p$

satisfying

1. $\mathtt{pr} \circ \xi = f_{|M_{\lambda=0}}$ (note that this means that $f_{|M_{\lambda=0}}$ takes values in the diagonal submanifold of $E \times_B E$.
2. The map
$$M \ni m \mapsto \begin{cases} g(f(m))\lambda(m)^{-1} & \lambda(m) \neq 0 \\ dg(f(m))(\xi(n)) & \lambda(m) = 0 \end{cases} \in \mathbb{R}$$
is smooth for every smooth function g on $E \times_B E$ vanishing on the diagonal.

In order to define j we associate to such a pair naturally a map $M \to \mathbb{R}^N$. Note that $g_j := j \circ r - j \circ s$ vanishes on the diagonal. We take

$$j(\lambda, f, \xi)(m) := \begin{cases} g_f(f(m))\lambda(m)^{-1} & \lambda(m) \neq 0 \\ dg_f(f(m))(\xi(n)) & \lambda(m) = 0 \end{cases}$$

More explicitly,

$$j(\lambda, f, \xi)(m) := \begin{cases} (j(r(f(m))) - j(s(f(m))))\lambda(m)^{-1} & \lambda(m) \neq 0 \\ dj(s(f(m)))(\xi(n)) & \lambda(m) = 0 \end{cases}$$

We now check that this defines a homomorphism. To this end we consider an element in the evaluation on M of

$$\mathcal{T}(P(p))^1 \times_{\mathcal{T}(P(p))^0} \mathcal{T}(P(p))^1 \cong DNC(P(p)^1 \times_{P(p)^0} P(p)^1, \mathtt{diag})$$

Such an element is given by
$$(\lambda, (f_1, f_0), (\xi_1, \xi_0)) \ ,$$
where we use that the normal bundle of the diagonal is $T^v p \oplus T^v p$ naturally. Note that $s \circ f_1 = r \circ f_0$. This implies $p \circ r \circ f_1 = p \circ s \circ f_0$ so that the pair $(r(f_1), s(f_0))$ below is well-defined. The groupoid operation in $\mathcal{T}(P(p))$ sends this element to

$$(\lambda, (r(f_1), s(f_0)), \xi_1 + \xi_0)$$

On the other hand, obviously

$$j(\lambda, f_1, \xi_1) + j(\lambda, f_0, \xi_0) = j(\lambda, (r(f_1), s(f_0)), \xi_1 + \xi_0) \ .$$

The homomorphism gives an action of $\mathcal{T}(P(p))$ on $[0, \infty] \times E \times \mathbb{R}^N$ (note that $[0, \infty] \times E \cong \mathcal{T}(P(p))^0$).

We get complementary open and closed subgroupoids of the action groupoid

$$G := \mathcal{T}(P(p)) \curvearrowright ([0,\infty] \times E \times \mathbb{R}^N) \ .$$

$$G_{(0,\infty]} := \mathcal{T}(P(p))_{(0,\infty]} \curvearrowright ((0,\infty] \times E \times \mathbb{R}^N) \ , \quad G_{\{0\}} := \mathcal{T}(P(p))_{\{0\}} \curvearrowright (E \times \mathbb{R}^N)$$

and a corresponding exact sequence of C^*-algebras.

$$0 \to C_r^*(G_{(0,\infty]}) \to C_r^*(G) \to C_r^*(G_{\{0\}}) \to 0 \ .$$

Restricting to $\lambda = 1$ we get a groupoid $G_{\{1\}} \cong P(p) \curvearrowright (E \times \mathbb{R}^N)$.

Proposition 13.2. *1. $C_r^*(G_{(0,\infty]})$ is contractible.*

2. We have Morita equivalences

a) $P(p) \sim (B \rightrightarrows B)$,

b) $G_{\{1\}} \sim (B \times \mathbb{R}^N \rightrightarrows B \times \mathbb{R}^N)$

c) $G_{\{0\}} \sim N_E^{B\times\mathbb{R}^N}$

For 2.b we use that G_1 acts freely and properly on $E \times \mathbb{R}^N$. Two points (e,v) and (e',v') are equivalent if $p(e) = p(e')$ and $v - v' = j(e) - j(e')$. Hence equivalence classes are parametrized by $B \times \mathbb{R}^N$.

For 2.c we use that two points (e,v) und (e',v') in $E \times \mathbb{R}^N$ are equivalent if $e = e'$ and there exists ξ in $T^v p_e$ such that $dj(e)(\xi) = v' - v$. The equivalence classes are therefore the points the normal bundle of the embedding $(p,j) : E \to B \times \mathbb{R}^N$.

The proof of the index theorem is now based on the following diagram:

$$\begin{tikzcd}
K^{top}(C_0(T^v p)^*) \arrow[r, "\simeq"] \arrow[ddr, bend right, "topThom"'] & K^{top}(C_r^*(\mathcal{T}(P(p))_{\{0\}})) \arrow[d, "gen.Thom"', "\simeq"] & K^{top}(C_r^*(\mathcal{T}(P(p)))) \arrow[l, hook', "\simeq"'] \arrow[r, "j^*"] \arrow[d, "gen.Thom"', "\simeq"] & K^{top}(C_r^*(P(p))) \arrow[d, "genThom"', "\simeq"] \arrow[ddd, bend left=60, "\simeq"', "Morita"] \\
& \Sigma^N K^{top}(C_r^*(G_{\{0\}})) \arrow[d, "Morita"', "\simeq"] & \Sigma^N K^{top}(C_r^*(G)) \arrow[l, "\simeq"'] \arrow[r, "j^*"] & \Sigma^N K^{top}(C_r^*(G_{\{1\}})) \arrow[d, "Morita"', "\simeq"] \\
& \Sigma^N K^{top}(C_0(N_E^{B\times\mathbb{R}^N})) \arrow[rr] & & \Sigma^N K^{top}(C_0(B \times \mathbb{R}^N)) \\
& & & K^{top}(C_0(B)) \arrow[u, "gen.Thom"', "\simeq"]
\end{tikzcd}$$

The upper right-down composition is $\mathtt{index}_{an}^{gprd}$. The down-right composition is the topological index map where we use that the last gen. Thom isomorphism is the same as the suspension equivalence (we have seen this earlier). The proof of the Atiyah-Singer index theorem is accomplished by the observation that all cells commute.

The squares in the middle commute by the naturality of the generalied Thom isomorphism for closed and open embeddings.

The right square expresses the compatibility of Morita equivalence with the generalized Thom isomorphism.

Finally, the left square is the deepest and the reference [Con94] does not really give the argument. First we need to produce a filler.

In order to show that it is an equivalence we use homotopy invariance and excision. We can first reduce (by decomposing into cells) to the case that B is a point. Here we use naturality of the morphisms between spectrum-valued functors. We then can further decompose E into cells. So finally we must check the diagram for $\mathbb{R}^n \to *$ and the identity embedding.

$$\begin{tikzcd}
K^{top}(C_0(T^*\mathbb{R}^N)) \arrow[r, "fibrewise fourier"] \arrow[drr, "top.Thom"'] & K^{top}(C_r(T\mathbb{R}^N)) \arrow[r, "genThom"] & \Sigma^N K^{top}(C_r^*(T^*\mathbb{R}^n \curvearrowright \mathbb{R}^N)) \arrow[d, "Morita"] \\
& & \Sigma^N K^{top}(C_0(*))
\end{tikzcd}$$

14. The Baum-Connes conjecture

The aim of this free talk is to present the statement of the Baum-Connes conjecture and indicate various applications.

The talk should indicate the status of the conjecture and some interesting consequences (e.g. Novikov conjecture and Gromov-Lawson conjecture). One should try to revisit as much as possible the topics discussed in previous talks of the seminar.

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