Seminar: Lie algebras and Lie groups

Ulrich Bunke/Christoph Winges

The main object of study in this seminar are *Lie algebras*. These are vector spaces equipped with an additional operation, called the Lie bracket, which behaves like a commutator: one basic example of a Lie algebra is the vector space of linear endomorphism $\mathrm{End}(V)$ of some vector space V, equipped with the operation $[f,g] = fg - gf$. The first talks (1–4) will introduce this concept and develop some structure theory for Lie algebras, including the concepts of solvability, nilpotence and semisimplicity. Semisimple Lie algebras will occupy most of our attention in the sequel. Starting with talk 5, we will study Lie algebras through their representations on vector spaces. In the last of couple of talks, we will take a cursory look at Lie groups (roughly speaking, manifolds equipped with a group structure). These provide an important source of examples of Lie algebras by considering the left-invariant vector fields with the Lie bracket given by commutator of vector fields.

1 Lie algebras - basics (2021-10-18)

Define the notion of a Lie algebra [Hum80, p. 1] and explain briefly the obvious notions that build on this definition: homomorphisms of Lie algebras, monomorphisms, epimorphisms, isomorphisms, (inner) automorphisms [Hum80, Sec. 2.2 & 2.3], Lie subalgebras [Hum80, p. 1]. Introduce and discuss the important (families of) examples of Lie algebras: $\mathfrak{gl}(V)$, triangular matrices, diagonal matrices, and the families A_ℓ, B_ℓ, C_ℓ, D_ℓ [Hum80, Sec. 1.2].

Define the notion of an ideal in a Lie algebra; mention examples like the center, the derived subalgebra and the kernel of a homomorphism [Hum80, p. 6]. Explain what it means to be simple [Hum80, p. 6f]. Prove the homomorphism theorem [Hum80, p. 7].

Define the notion of a representation of a Lie algebra [Hum80, p. 8] and introduce the adjoint representation [Hum80, Sec. 1.3].

References: [Hum80, Sec. 1 & 2]

2 Solvable and nilpotent (2021-10-25)

Define the notion of solvability [Hum80, p. 10] and discuss constructions which preserve solvability [Hum80, p. 11]. Explain what it means to be semisimple [Hum80, p. 11].

Define the notion of nilpotence [Hum80, p.] and how it relates to solvability. Discuss constructions which preserve nilpotence [Hum80, p. 12].

Prove Engel's theorem [Hum80, Sec. 3.3] and the sufficient criterion for nilpotence derived from it [Hum80, p. 12].

References: [Hum80, Sec. 3]

3 Theorems of Cartan and Lie (2021-11-08)

Prove Lie's theorem [Hum80, Sec. 4.1]. Discuss the Jordan–Chevalley decomposition [Hum80, Sec. 4.2] and prove Cartan's criterion for solvability [Hum80, Sec. 4.3].

References: [Hum80, Sec. 4]

4 The Killing form (2021-11-15)

Define the Killing form [Hum80, p. 21] and prove the characterization of semisimplicity in terms of the Killing form [Hum80, p. 22].

Show that every semisimple Lie algebra decomposes into a sum of simple ideals [Hum80, Sec. 5.2].

Prove that all derivations on a semisimple Lie algebra are inner [Hum80, Sec. 5.3].

References: [Hum80, Sec. 5.1–5.3]

5 Reducibility of representations (2021-11-23)

Introduce the notion of a module over a Lie algebra [Hum80, p. 25]. Briefly explain notions that derive from this concept, e.g. homomorphism, isomorphism and submodule [Hum80,

p. 25] as well as constructions like tensor products and duals [Hum80, p. 26f].

Explain the notion of irreducibility and prove Schur's lemma [Hum80, p. 26].

Define the Casimir element of a faithful representation [Hum80, p. 27] and prove Weyl's theorem [Hum80, Sec. 6.3].

If time permits, show that the abstract Jordan decomposition [Hum80, Sec. 5.4] is compatible with the linear representations of a semisimple Lie algebra [Hum80, Sec. 6.4]

References: [Hum80, Sec. 6.1–6.3, Sec. 5.4]

6 Representations of $\mathfrak{sl}(2, F)$ (2021-11-30)

If this has not been covered in the previous talk, prove the results of [Hum80, Sec. 6.4] on the abstract Jordan decomposition [Hum80, Sec. 5.4].

Then discuss the representation theory of $\mathfrak{sl}(2, F)$. Introduce the notions of weights and weight spaces [Hum80, p. 31] and prove the classification of irreducible modules over $\mathfrak{sl}(2, F)$. Present explicit constructions of the various irreducible modules [Hum80, Exercise 4 of Sec. 7].

References: [Hum80, Sec. 7]

7 Cartan subalgebras (2021-12-06)

Introduce the root space decomposition of a semisimple Lie algebra relative to a given Cartan algebra (maximal toral subalgebra) [Hum80, p. 35]. Show that a maximal toral subalgebra is its own centralizer [Hum80, Sec. 8.2].

Discuss the behaviour of the Killing form [Hum80, Prop. on p. 37, but also p. 36 and Cor. on p. 37].

References: [Hum80, Sec. 8.1–8.3]

8 More geometry of roots (2021-12-13)

Show that the multiplicity of a root is 1, and no non-trivial multiple of a root can be a root [Hum80, Sec. 8.4]. State and verify the proposition on [Hum80, p. 39]. State and verify the Theorem on [Hum80, p. 40]. Solve [Hum80, Ex. 1 & 2 on p. 40] and present the solution.

References: [Hum80, Sec. 8.4]

9 Root systems (2021-12-20)

Explain the notion of a reflection at a hyperplane [Hum80, Sec. 9.1]. State the axioms of a root system [Hum80, Sec. 9.1] and introduce the Weyl group. Present the material from [Hum80, Sec. 9.3] and [Hum80, Sec. 9.4].

References: [Hum80, Sec. 9]

10 Weyl chamber (2022-01-10)

Introduce the notion of a Weyl chamber and its relation with a choice of basis [Hum80, Sec. 10.1]. Introduce the notion of a simple root and present the material from [Hum80, Sec. 10.2]. Show the theorem from [Hum80, Sec. 10.3] and present the material from [Hum80, Sec. 10.4].

References: [Hum80, Sec. 10]

11 Classification of Root systems (2022-01-17)

Explain and prove the classification theorem of root systems [Hum80, Sec. 11.4].

References: [Hum80, Sec. 11]

12 Lie groups (2022-01-24)

Present the definition of a Lie group and give various examples. Define the associated Lie algebra of left-invariant vector fields and construct the functor which associates to a Lie group its associated Lie algebra. Give as many proofs as time permits.

References: [BtD95, Sec. 1 & 2]

13 Compact Lie groups and representations - overview (2022-01-31)

This is an overview talk about the representation theory of compact Lie group. Relate compactness of a Lie group with negative definiteness of the Killing form of its associated Lie algebra. Introduce the notion of a maximal torus and construct the weight decomposition of finite-dimensional complex representations of a compact Lie group. Explain the classification of irreducible representations of a compact Lie group in terms of highest weights. Do the examples S^1, $SO(3)$ and $SU(2)$ on the way.

A basic reference for this talk is [BtD95].

References

[BtD95] Th. Bröcker and T. tom Dieck. *Representations of compact Lie groups.* Graduate Texts in Mathematics. 98. New York, NY: Springer, 1995.

[Hum80] James E. Humphreys. *Introduction to Lie algebras and representation theory. 3rd printing, rev*, volume 9. Springer, New York, NY, 1980.