Seminar: Morsetheory

Ulrich Bunke/N.N

The goal of this seminar is to present the basic constructions of Morse theory and first applications. Interesting outcomes are (the Morse-) inequalities relating the number of critical points of a function on a manifold with the Betti numbers of the manifold, a proof of the fact that a smooth manifold has the homotopy type of a CW-complex, applications to the homotopy type of path- and loop spaces, and a way to calculate the homology of a manifold using gradient flows. Prerequisites for this seminar are the basic theory of smooth manifolds, some elements of Riemannian geometry, and some basic notions of algebraic topology. The seminar will follow the books [Mil63] by J. Milnor and [AD14] by M. Audin and M. Damian. Additional material will be taken from Hatcher's book [Hat02] and [Nic07] by L. Nicolaescu.

1 Morse functions

Introduce the notion of a non-degenerate critical point of a real-valued function on a manifold and the notion of a Morse function. Show that Morse functions always exist and that every function can be approximated by a Morse function arbitrary well. Follow [Mil63, Sec. 6] and [AD14, Sec. 1.1 and 1.2]. Sard's theorem about the measure of critical values and Whitney's embedding theorem should be stated precisely, but used as a black-box.

2 CW-complexes

Introduce the notion of a CW-complex. In particular make precise the meaning of "attaching a cell". Construct the cellular chain complex of a CW-complex and show that it calculates the (singular) homology. Provide examples of CW-complexes and their homologies, e.g. $\mathbb{CP}^n$ or S^n. The material can be found in the book [Hat02].

3 Critical points and flows

Show the Morse Lemma giving the structure of a smooth function near a non-degenerate critical point. Show that compactly supported vector field generates a flow of diffeomorphisms. Provide pictures where the Morse function is interpreted as a height function. Follow [Mil63, Sec. 1 & 2] and [AD14, Sec. 1.3].

4 The layers betwen levels

Show [Mil63, Thm. 3.1] that the layer between two level sets without critical points is a cylinder over the lower boundary. Show [Mil63, Thm. 3.1] which provides the structure of the layer between two levels containing precisely one non-degenerate critical point. Show [Mil63, Thm. 3.5] describing the layer above a level with only non-degenerate critical points as a relative CW-complex with respect to its boundary. See also [AD14, Sec. 2].

5 First applications

Show Reeb's [Mil63, Thm. 4.1] stating that a compact manifold with a function with precisely two critical points is diffeomorphic to a sphere. Show that manifolds have the homotopy type of a CW-complex. Finally obtain the Morse Inequalities [Mil63, Thm. 5.2], see also [Nic07, Sec. 2.3].

6 The path space

Explain the space of piecewise smooth paths on a manifold and how analysis is understood on this space. Define the Energy functional and calculate its first variation. Recall the relation between energy and length and that geodesics are critial poits of the energy functionional. Follow [Mil63, Sec. 11 & 12].

7 The second variation of the energy

Calculate the second variation of the energy functional. Recall the notion of a Jacobi field and its relation with the zero space of the Hessian of the energy functional. Show that

Jacobi fields are precisely the tangent vectors to variations of geodesics. Follow [Mil63, Sec. 13 & 14].

8 The Morse index theorem

Recall the notion of conjugate points on a geodesic. Show the Morse Index Theorem [Mil63, Thm. 15.1] and its consequences. Follow [Mil63, Sec. 15].

9 Finite CW-approximations to finite energy path spaces

Show the Theorems 16.2 and 16.3 in [Mil63] which state that the energy level sets of the path space between to points in a complete Riemannian manifold are finite CW-complexes.

10 The full path space

Show that the full path space of a complete Riemannian manifold is a countable CW-complex [Mil63, Thm 17.3]. Deduce a corresponding statement for the based loop space. Draw conclusions about the homotopy type of the loop space of S^n [Mil63, Cor. 17.4 & Cor. 17.5].

11 The Smale condition

Introduce the stable and unstable manifold of a critical point of a gradient-like vector field associated to a Morse function. Explain the Smale condition. Show Smale's Theorem [AD14, Thm. 2.2.5] stating that every gradient like vector field can be C^1-approximated by a one satisfying the Smale condition.

12 The Morse complex

Introduce the Morse complex associated to a Morse function. In particular explain in detail how the boundary operator is defined and why its square vanishes. Follow [AD14, Sec. 3.1 & Sec. 3.2]

13 Invariance of Morse homology

Show [AD14, Thm. 3.4.2] stating the the homology of the Morse complex does not depend on the choice of the Morse function and the gradient-like vector field. Follow [AD14, Sec. 3.4].

14 Open topic - outview

References

[AD14] Michèle Audin and Mihai Damian. *Morse Theory and Floer Homology.* Springer London, 2014.

[Hat02] A. Hatcher. *Algebraic Topology.* Cambridge University Press, 2002.

[Mil63] John Milnor. *Morse Theory. (AM-51).* Princeton University Press, dec 1963.

[Nic07] L. Nicolaescu. *An Invitation to Morse Theory.* Springer New York, 2007.