Lectures on analytic torsion

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Abstract

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1 Analytic torsion - from algebra to analysis - the finite-dimensional case

1.1 Torsion of chain complexes

Let $k$ be a field. If $V$ is a finite-dimensional $k$-vector space and $A : V \to V$ an isomorphism, then we have the **determinant** $\det A \in k^*$. The **torsion of a chain complex** is a generalization of the determinant as we will explain next.

If $V$ is a finite-dimensional $k$-vector space of dimension $n$, then we define the **determinant of $V$** by

$$\det(V) := \Lambda^n V.$$ 

This is a one-dimensional $k$-vector space which functorially depends on $V$. By definition we have

$$\det\{0\} := k.$$ 

Note that $\det$ is a functor from the category of finite-dimensional vector spaces over $k$ and isomorphisms to one-dimensional vector spaces over $k$ and isomorphisms.

If $L$ is a one-dimensional vector space, then we have a canonical isomorphism

$$\text{Aut}(L) \cong k^*.$$ 

Under this identification the isomorphism $\det(A) : \det V \to \det V$ and the functor $\det$ induced by an isomorphism $A : V \to V$ is exactly mapped to the element $\det(A) \in k^*$.

We let $L^{-1} := \text{Hom}_k(L, k)$ denote the dual $k$-vector space.

We now consider a **finite chain complex** over $k$, i.e. a chain complex

$$C : \cdots \to C^{m-1} \to C^m \to C^{m+1} \to \cdots$$

of finite-dimensional $k$-vector spaces which is bounded from below and above.

**Definition 1.1.** We define the **determinant of the chain complex** $C$ to be the one-dimensional $k$-vector space

$$\det C := \bigotimes_n (\det C^n)^{(-1)^n}$$

The determinant $\det(C)$ only depends on the underlying $\mathbb{Z}$-graded vector space of $C$ and not on the differential.

The **cohomology** of the chain complex $C$ can be considered as a chain complex $H(C)$ with trivial differentials. Hence the one-dimensional $k$-vector space $\det H(C)$ is well-defined.
Proposition 1.2. We have a canonical isomorphism
\[ \tau_C : \det C \cong \det H(C) . \]
This isomorphism is called the torsion isomorphism.

Proof. For two finite-dimensional \( k \)-vector spaces \( U, W \) we have a canonical isomorphism
\[ \det(U \oplus W) \cong \det U \otimes \det W . \]
More generally, given a short exact sequence (with \( U \) in degree 0)
\[ \mathcal{V} : 0 \to U \to V \to W \to 0 \]
we can choose a split. It induces a decomposition \( V \cong U \oplus W \) and therefore an isomorphism
\[ \det U \otimes \det W \cong \det V . \]
The main observation is that this isomorphism does not depend on the choice of the split. The torsion of the short exact sequence \( \mathcal{V} \) is the induced isomorphism
\[ \tau_V : \det U \otimes (\det V)^{-1} \otimes \det W \to k . \]
Finally, we can decompose a chain complex \( \mathcal{C} \) into short exact sequences and construct the torsion inductively by the length of the chain complex. Assume that \( \mathcal{C} \) starts at \( n \in \mathbb{Z} \). We consider the two short exact sequences
\[ \mathcal{A} : 0 \to H^n(C) \to C^n \to B^{n+1} \to 0 , \]
\[ \mathcal{B} : 0 \to B^{n+1} \to C^{n+1} \to C^{n+1}/B^{n+1} \to 0 , \]
and the sequence
\[ \mathcal{C}' : 0 \to C^{n+1}/B^{n+1} \to C^{n+2} \to \ldots \to \]
Here \( B^{n+1} := d(C^n) \subset C^{n+1} \) denotes the subspace of boundaries. Note that \( \mathcal{C}' \) starts in degree \( n+1 \). By induction, the isomorphism \( \tau_C \) is defined by
\[
\begin{align*}
\det(\mathcal{C}) & \cong (\det C^n)^{(-1)^n} \otimes (\det C^{n+1})^{(-1)^{n+1}} \otimes \bigotimes_{k=n+2}^{\infty} (\det C^k)^{(-1)^k} \\
\cong & (\det H^n(C))^{(-1)^n} \otimes (\det B^{n+1})^{(-1)^n} \otimes (\det B^{n+1})^{(-1)^{n+1}} \otimes \bigotimes_{k=n+2}^{\infty} (\det C^k)^{(-1)^k} \\
\cong & (\det H^n(C))^{(-1)^n} \otimes (\det(C^{n+1}/B^{n+1}))^{(-1)^n} \otimes \bigotimes_{k=n+2}^{\infty} (\det C^k)^{(-1)^k} \\
\cong & (\det H^n(C))^{(-1)^n} \otimes \det \mathcal{C}' \\
\cong & (\det H^n(C))^{(-1)^n} \otimes \det H(\mathcal{C}') \\
\cong & \det H(\mathcal{C}) .
\end{align*}
\]
Example 1.3. If $A : V \to W$ is an isomorphism of finite-dimensional vector spaces, then we can form the acyclic complex

$$A : V \to W$$

with $W$ in degree 0. Its torsion is an isomorphism

$$\tau_A : (\det V)^{-1} \otimes \det W \to k^*$$

which corresponds to the generalization of the determinant of $A$ as a morphism $\det(A) : \det V \to \det W$. Only for $V = W$ we can interpret this as an element in $k^*$.

Example 1.4. Let $C$ be a finite chain complex and $W$ be a finite-dimensional $k$-vector space. Then we form the chain complex

$$W : W^{\text{id}_W} \to W$$

starting at 0 and let $n \in \mathbb{Z}$. We say that the chain complex $C' := C \oplus W[n]$ is obtained from $C$ by a simple expansion.

If $C'$ is obtained from $C$ by a simple expansion, then we have canonical isomorphisms $\det C \cong \det C'$ and $H(C) \cong H(C')$. Under these isomorphisms we have the equality of torsion isomorphisms $\tau_C = \tau_{C'}$.

1.2 Torsion and Laplace operators

We now assume that $k = \mathbb{R}$ or $k = \mathbb{C}$.

A metric $h^V$ on $V$ is a (hermitean in the case $k = \mathbb{C}$) scalar product on $V$. It induces a metric $h^{\det V}$ on $\det V$. This metric is fixed by the following property. Let $(v_i)_{i=1,\ldots,n}$ be an orthonormal basis of $V$ with respect to $h^V$, then $v_1 \wedge \cdots \wedge v_n$ is a normalized basis vector of $\det V$ with respect to $h^{\det V}$.

Example 1.5. Let $(V, h^V)$ and $(W, h^W)$ be finite-dimensional $k$-vector spaces with metrics and $A : V \to W$ be an isomorphism of $k$-vector spaces. Then we can choose an isometry $U : W \to V$. We have the number $\det(UA) \in k^*$ which depends on the choice of $U$. We now observe that

$$| \det A | := | \det(UA) | \in \mathbb{R}^+$$

(1)

does not depend on the choice of $U$. The analytic torsion generalizes this idea to chain complexes.

A metric $h^C$ on a chain complex is a collection of metrics $(h^C_n)_n$. Such a metric induces a metric on $\det C$ and therefore, by push-forward, a metric $\tau_{C,h^C}$ on $\det H(C)$. 


**Definition 1.6.** Let \( C \) be a finite chain complex and \( h^C \) and \( h^{H(C)} \) metrics on \( C \) and its cohomology \( H(C) \). Then the **analytic torsion** \( T(C, h^C, h^{H(C)}) \in \mathbb{R}^+ \) is defined by the relation
\[
h^{\det H(C)} = T(C, h^C, h^{H(C)}) \tau_{C,*} h^{\det C}.
\]

**Example 1.7.** In general the analytic torsion depends non-trivially on the choice of metrics. For example, if \( t, s \in \mathbb{R}^+ \), then we have the relation
\[
T(C, sh^C, th^{H(C)}) = (\frac{t}{s})^{\chi(C)},
\]
where \( \chi(C) \in \mathbb{Z} \) denotes the Euler characteristic of \( C \). But observe that if \( C \) is acyclic, then \( \chi(C) = 0 \) and \( T(C, h^C, h^{H(C)}) \) does not depend on the scale of the metrics.

**Example 1.8.** In this example we discuss the dependence of the analytic torsion on the choice of \( h^{H(C)} \). Assume that \( h_i^{H(C)} \), \( i = 0, 1 \) are two choices. Then we define numbers \( v_k(h_0^{H(C)}, h_1^{H(C)}) \in \mathbb{R}^+ \) uniquely such that
\[
v_k(h_0^{H(C)}, h_1^{H(C)}) h_1^{\det H(C)} = h_0^{\det H(C)}.
\]
We further set
\[
v(h_0^{H(C)}, h_1^{H(C)}) := \prod_k v_k(h_0^{H(C)}, h_1^{H(C)})(-1)^k.
\]
Then we have
\[
T(C, h^C, h_i^{H(C)}) = v(h_0^{H(C)}, h_1^{H(C)}) T(C, h^C, h_i^{H(C)}).
\]

We consider the \( \mathbb{Z} \)-graded vector space
\[
C := \bigoplus_{n \in \mathbb{Z}} C^n
\]
and the differential \( d : C \to C \) as a linear map of degree one. The metric \( h^C \) induces a metric \( h_i^C \) such that the graded components are orthogonal. Using \( h^C \) we can define the adjoint \( d^* : C \to C \) which has degree \(-1\). We define the **Laplace operator**
\[
\Delta := (d + d^*)^2.
\]
Since \( d^2 = 0 \) and \((d^*)^2 = 0\) we have \( \Delta = dd^* + d^*d \). Hence the Laplace operator preserves degree and therefore decomposes as
\[
\Delta = \bigoplus_{n \in \mathbb{Z}} \Delta_n.
\]
As in Hodge theory we have an orthogonal decomposition

\[ C \cong \text{im}(d) \oplus \ker \Delta \oplus \text{im}(d^\ast) , \quad \ker(d) = \text{im}(d) \oplus \ker \Delta . \]

In particular, we get an isomorphism of graded vector spaces

\[ H(C) \cong \ker(\Delta) . \]

This isomorphism induces the Hodge metric \( h^H_{Hodge} \) on \( H(C) \).

We let \( \Delta'_n \) be the restriction of \( \Delta_n \) to the orthogonal complement of \( \ker(\Delta_n) \).

**Lemma 1.9.** We have the equality

\[ T(C, h^C, h^H_{Hodge}) = \sqrt{\prod_{n \in \mathbb{Z}} \det(\Delta'_n)^{(-1)^n} n} . \]

**Proof.** The differential \( d \) induces an isomorphism of vector space

\[ d_k : C^k \supseteq \ker(d|_{C^k})^\perp \cong \text{im}(d|_{C^k}) \subseteq C^{k+1} . \]

First show inductively that

\[ \prod_k |\det d_k|^{(-1)^{k+1}} \tau_{\mathcal{C}} : \det \mathcal{C} \to \det H(C) \]

is an isometry (see [II] for notation), hence

\[ T(C, h^C, h^H_{Hodge}) = \left[ \prod_k |\det d_k|^{(-1)^{k+1}} \right] . \quad (2) \]

We repeat the construction of the torsion isomorphism. But in addition we introduce factors to turn each step into an isometry. Note that

\[ |\det(d_k)|^{-1} \det(d_k) : \det(\ker(d|_{C^k})^\perp) \to \det(\text{im}(d|_{C^k})) \]

is an isometry. This accounts for the first correction factor in the following chain of
isometries.

\[
\det(C) \cong (\det C^n)^{-n} \otimes (\det C^{n+1})^{-n+1} \otimes (\det C^k)^{-1}
\]

\[
\prod_{k=n+2}^\infty (\det C^k)^{-1}
\]

We now observe that

\[
|\det d_n|^{-n+1} \tau_A \cong (\det H^n(C))^{-n} \otimes (\det B^{n+1})^{-n} \otimes (\det C^{n+1})^{-n+1} \otimes (\det C^k)^{-1}
\]

\[
\prod_{k=n+2}^\infty (\det C^k)^{-1}
\]

\[
\tau_B \cong (\det H^n(C))^{-n} \otimes (\det(C^{n+1}/B^{n+1}))^{-n+1} \otimes (\det C^k)^{-1}
\]

\[
\prod_{k=n+2}^\infty (\det C^k)^{-1}
\]

\[
\tau_C' \cong (\det H^n(C))^{-n} \otimes \det(C')
\]

\[
\prod_{k=n+1}^\infty \det H(C) .
\]

1.3 Torsion and Whitehead torsion

Let now \(G\) be a group and \(\mathcal{X}\) be an acyclic based complex of free \(\mathbb{Z}[G]\)-modules. Its \textbf{Whitehead torsion} is an element

\[
\tau(\mathcal{X}) \in Wh(G) .
\]

Let \(\rho : G \rightarrow SL(N, \mathbb{C})\) be a finite-dimensional representation of \(G\). It induces a ring homomorphism

\[
\rho : \mathbb{Z}[G] \rightarrow \text{End}(\mathbb{C}^N) .
\]

Then we can form the complex

\[
\mathcal{C} := \mathcal{X} \otimes_{\mathbb{Z}[G]} \mathbb{C}^N .
\]
Lemma 1.10. This complex is acyclic.

Proof. The acyclic complex of free (or more generally, of projective) $\mathbb{Z}[G]$-modules $\mathcal{X}$ admits a chain contraction. We get an induced chain contraction of $\mathcal{C}$.

The basis of $\mathcal{X}$ together with the standard orthonormal basis of $\mathbb{C}^N$ induces a basis of $\mathcal{C}$ which we declare to be orthonormal, thus defining a metric $h^C$. Since $H(\mathcal{C}) = 0$ we have a canonical metric $h^{H(\mathcal{C})}$ on $H(\mathcal{C}) = \mathbb{C}$ and the analytic torsion $T(\mathcal{C}, h^C) := T(\mathcal{C}, h^C, h^{H(\mathcal{C})})$ is defined.

The representation $\rho$ induces a homomorphism

$$K_1(\mathbb{Z}[G]) \to K_1(\text{End}(\mathbb{C}^N)) \cong K_1(\mathcal{C}) \cong \mathbb{C}^* \to \mathbb{R}^+.$$

Under this homomorphism

$$K_1(\mathbb{Z}[G]) \ni [\pm g] \mapsto |\det(\pm \rho(g))| = 1 \in \mathbb{R}^+.$$

Therefore, by passing through the quotient, we get a well-defined homomorphism

$$\chi_\rho : Wh(\mathcal{G}) = K_1(\mathbb{Z}[G])/(\pm [g]) \to \mathbb{R}^+. \tag{4}$$

Proposition 1.11. The Whitehead torsion and the analytic torsion are related by

$$T(\mathcal{C}, h^C) = \chi_\rho(\tau(\mathcal{X})).$$

Proof. We can define the Whitehead torsion of based complexes $\mathcal{X}$ over $\mathbb{Z}[G]$ again inductively by the length. We make the simplifying assumption that the complements of the images of the differentials are free. Assume that the complex $\mathcal{X}$ starts with $X_n$. We consider the short exact sequence of $\mathbb{Z}[G]$-modules

$$0 \to X_n \to X_{n+1} \overset{p}{\to} X_{n+1}/X_n \to 0$$

and set

$$\mathcal{X}' : 0 \to X_{n+1}/X_n \overset{i}{\to} X_{n+2} \to \ldots$$

Let $c_n$ be the chosen basis of $X_n$. We choose a basis $c'_{n+1}$ of $X_{n+1}/X_n$. Lifting its elements and combining it with the images of the elements of $c_n$ and get a basis $b'$ of $X_{n+1}$. Note that $\mathcal{X}'$ is again based (by $c'_k := c_k$ for $k \geq n + 2$ and the basis $c'_{n+1}$ chosen above) and starts at $n + 1$. Then by definition of the Whitehead torsion

$$\tau(\mathcal{X}) = [c_{n+1}/b'](-1)^{n+1} \tau(\mathcal{X}') \in Wh(\mathcal{G}).$$

Note that

$$T(\mathcal{C}, h^C) = T(\mathcal{D}, h^D) T(\mathcal{C}', h^{C'})$$

where

$$\mathcal{D} : 0 \to X_n \otimes_{\mathbb{Z}[G]} \mathbb{C}^N \to X_{n+1} \otimes_{\mathbb{Z}[G]} \mathbb{C}^N \overset{p}{\to} X_{n+1}/X_n \otimes_{\mathbb{Z}[G]} \mathbb{C}^N \to 0$$
starting at $n$, and we use the metrics induced by $c_n, c_{n+1}$ and $c'_n$. Here we use (2) and that $d_{n+1} = i \circ p$ and hence $|\det d_{n+1}| = |\det p||\det i|$. Therefore we must check that

$$\chi_p([c_{n+1}/b'])^{(-1)^{n+1}} = T(\mathcal{D}, h^D).$$

The choice of the lift in the definition of $b'$ induces a split of this sequence $\mathcal{D}$. On its middle vector space we have two metrics, one defined by the split, and the other defined by the basis $c_{n+1}$. If we take the split metric, then its torsion is trivial. Hence $T(\mathcal{D}, h^D)$ is equal to the determinant of the base change from $b'$ to $c_{n+1}$, i.e. (5) holds true, indeed. $\square$

**Example 1.12.** We consider the group $\mathbb{Z}/5\mathbb{Z}$ and the complex

$$\mathcal{X} : \mathbb{Z}[\mathbb{Z}/5\mathbb{Z}]^{1-[-1]}[-4] \to \mathbb{Z}[\mathbb{Z}/5\mathbb{Z}]$$

starting at 0. This complex is acyclic since $1 - [2] - [3]$ is an inverse of the differential. We consider the representation $\mathbb{Z}/5\mathbb{Z} \to U(1)$ which sends $[1]$ to $\exp(\frac{2\pi i}{5})$. Its Whitehead torsion is represented by $1 - [1] - [4] \in \mathbb{Z}[\mathbb{Z}/5\mathbb{Z}]^*$. Then

$$\tau_p([1 - [1] - [4]]) = \|1 - \exp(\frac{2\pi i}{5}) - \exp(\frac{8\pi i}{5})\| = 2\cos(\frac{2\pi}{5}) - 1 \neq 1.$$

### 2 Zeta regularized determinants of operators - Ray-Singer torsion

#### 2.1 Motivation

Let $(\mathcal{C}, h^\mathcal{C})$ be a finite chain complex over $\mathbb{R}$ or $\mathbb{C}$ with a metric. Then by Lemma 1.9 we have the following formula for its analytic torsion

$$T(\mathcal{C}, h^\mathcal{C}, h^H_{\text{Hodge}}) = \sqrt{\prod_k (\det \Delta_k)^{(-1)^k}}.$$

Let now $(M, g^{TM})$ be a closed Riemannian manifold manifold. Then we can equip the de Rham complex $\Omega(M)$ with a metric $h^{\Omega(M)}_{L^2}$ given by

$$h^{\Omega(M)}_{L^2}(\alpha, \beta) = \int_M \alpha \wedge *_{g^{TM}} \beta,$$

where $*_{g^{TM}}$ is the Hodge-* operator associated to the metric.

More generally, let $(V, \nabla^V, h^V)$ be a vector bundle with a flat connection and a metric. Then we can form the twisted de Rham complex $\Omega(M, V)$. We consider the sheaf $\mathcal{V}$ of parallel sections of $(V, \nabla)$. The Rham isomorphism relates the sheaf cohomology of $\mathcal{V}$ with the cohomology of the twisted de Rham complex:

$$H(M, \mathcal{V}) \cong H(\Omega(M, V)).$$
The metric $h^V$ together with the Riemannian metric $g^{TM}$ induce a metric $h^{O(M,V)}_{L^2}$ on the twisted de Rham complex.

Note that to give $(V,\nabla)$ is, up to isomorphism, equivalent to give a representation of the fundamental group

$$\pi_1(M) \to \text{End}(\mathbb{C}^{\dim(V)})$$

(we assume $M$ to be connected, for simplicity). Hence $(M, V, \nabla^V)$ is differential-topological data, while the metrics $g^{TM}$ and $h^V$ are additional geometric choices.

In this section we discuss the definition of analytic torsion

$$T(M, \nabla^V, g^{TM}, h^V) := \sqrt[\prod_k \{\det \Delta_k^s\}(-1)^{\kappa_k}}$$

which is essentially due to Ray-Singer [RS71]. To this end we must define the determinant of the Laplace operators properly. We will also discuss in detail, how the torsion depends on the metrics.

### 2.2 Spectral zeta functions

We consider a finite-dimensional vector space with metric $(V, h^V)$ and a linear, invertible, selfadjoint and positive map $\Delta : V \to V$. Then the endomorphism $\log(\Delta)$ is defined by spectral theory and we have the relation

$$e^{\text{Tr} \log \Delta} = \det(\Delta) .$$

The spectral zeta function of $\Delta$ is defined by

$$\zeta_\Delta(s) = \text{Tr} \Delta^{-s} , \quad s \in \mathbb{C} .$$

It is an entire function on $\mathbb{C}$ and satisfies

$$-\zeta'_\Delta(0) = \text{Tr} \log(\Delta) .$$

So we get the formula for the determinant of $\Delta$ in terms of the spectral zeta function

$$\det \Delta = e^{-\zeta_\Delta(0)} .$$

The idea is to use this formula to define the determinant in the case where $\Delta$ is a differential operator.

We now consider a closed Riemannian manifold $(M, g^{TM})$ and a vector bundle $(V, \nabla^V, h^V)$ with connection and metric. The metrics induce $L^2$-scalar products on $\Omega^k(M, V)$ so that we can form the adjoint

$$\nabla^V^* : \Omega^1(M, V) \to \Omega^0(M, V)$$
of the connection $\nabla^V$. The **Laplace operator** is the differential operator

$$\Delta := \nabla^V \cdot \nabla^V : \Gamma(M, V) \to \Gamma(M, V).$$

It is symmetric with respect to the metric $\| \cdot \|_{L^2} := h^{q(M, V)}_{L^2}$.

More generally, a second order differential operator $A$ on $\Gamma(M, V)$ is called of **Laplace type** of it is of the form $A = \Delta + R$, for $\Delta$ defined for certain choices of $h^TM$, $h^V$, $\nabla^V$ such that $R \in \Gamma(M, \text{End}(V))$ is a selfadjoint bundle endomorphism with respect to the same metrics.

Assume that $A$ is a Laplace type differential operator on $\Gamma(M, V)$ and symmetric with respect to a metric $\| \cdot \|_{L^2}$. We consider (possibly densely defined unbounded) operators on the Hilbert space closure $\overline{\Gamma(M, V)}^{\| \cdot \|_{L^2}}$. The following assertions are standard facts from the analysis of elliptic operators on manifolds.

1. **A is an elliptic differential operator.** Indeed, its principal symbol is that of the Laplace operator and given by $\sigma_A(\xi) = \|\xi\|^2_{g^TM}$.

2. **A is essentially selfadjoint on the domain $\Gamma(M, V)$.** It is a general fact for a symmetric elliptic operator $A$ on a closed manifold that its closure $\overline{A}$ coincides with the adjoint $A^*$. The proof uses elliptic regularity.

3. **The spectrum of $A$ is real, discrete of finite multiplicity and accumulates at $+\infty$.** Since $A$ is essentially selfadjoint the operator $A + i$ is invertible on $L^2(M, V)$. Using again elliptic regularity in the quantitative form

$$\|\phi\|_{H^2} \leq C(\|A\phi\|_{L^2} + \|\phi\|_{L^2})$$

we see that its inverse can be factored as a composition of a bounded operator and an inclusion

$$L^2(M, V) \xrightarrow{(A+i)^{-1}} H^2(M, V) \xrightarrow{\text{incl}} L^2(M, V).$$

For a closed manifold the inclusion of the second Sobolev space into the $L^2$-space is compact by Rellich’s theorem. This shows that $(A+i)^{-1}$ is compact as an operator on $L^2$. We conclude discreteness of the spectrum. Furthermore, using the positivity of the Laplace operator and the fact that $R$ is bounded, we see that the spectrum accumulates at $\infty$.

4. **The number (with multiplicity) of eigenvalues of $A$ less than $\lambda \in (0, \infty)$ grows as $\lambda^{\dim(M)/2}$. This is called Weyl’s asymptotic.** This follows from the heat asymptotics stated in Proposition 2.3.

5. **A preserves an orthogonal decomposition of**

$$\Gamma(M, V) = N \oplus P,$$

**such that** $\dim(N) < \infty$, $A|_N \leq 0$ and $A' := A|_P > 0$. This follows from 3.
6. \( \zeta_A'(s) := \text{Tr } A^{t-s} \) is holomorphic for \( \text{Re}(s) > \dim(M)/2 \). This is a consequence of Weyl’s asymptotic stated in 4.

In order to define \( \zeta'_A(0) \) we need an **analytic continuation** of the zeta function. Note that for \( \lambda > 0 \) and \( s > 0 \) we have the equality

\[
\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t\lambda}t^{s-1}dt.
\]

(6)

We define the **Mellin transform** of a measurable function \( \theta \) of \( t \in (0, \infty) \rightarrow \mathbb{C} \) for \( s \in \mathbb{C} \) by

\[
M(\theta)(s) := \int_0^\infty \theta(t)t^{s-1}dt,
\]

provided the integral converges.

**Example 2.1.** For \( \lambda \in (0, \infty) \) we have \( M(e^{\lambda t})(s) = \Gamma(s)\lambda^{-s} \).

**Lemma 2.2.** Assume that \( \theta \) is exponentially decreasing for \( t \rightarrow \infty \), and that it has an asymptotic expansion

\[
\theta(t) \overset{t \rightarrow 0}{\sim} \sum_{n \in \mathbb{N}} a_n t^{\alpha_n}
\]

for a monotoneously increasing, unbounded sequence \((\alpha_n)_{n \in \mathbb{N}} \) in \( \mathbb{R} \). Then \( M(\theta)(s) \) is defined for \( \text{Re}(s) > -\alpha_0 \) and has a meromorphic continuation to all of \( \mathbb{C} \) with first order poles at the points \( s = -\alpha_n, \ n \in \mathbb{N}, \) such that

\[
\text{res}_{s=-\alpha_n} M(\theta)(s) = a_n.
\]

**Proof.** This is an exercise. The idea is to split the integral in the Mellin transformation as \( \int_0^1 + \int_1^\infty \). The second summand yields an entire function. In order to study the first summand one decomposes \( \theta(t) \) as a sum of the first \( n \) terms of its expansion and a remainder. The integral of the asymptotic expansion term can be evaluated explicitly and contributes the singularities for \( \text{Re}(s) > -\alpha_{n+1} \), and the remainder gives a holomorphic function on this domain. Since we can choose \( n \) arbitrary large we get the assertion. \( \square \)

In view of Weyl’s asymptotics the **heat trace** of a Laplace-type operator \( A \) on a closed manifold is defined for \( t > 0 \) as

\[
\theta_A(t) := \text{Tr } e^{-tA}.
\]

**Proposition 2.3.** Assume that \( A \) is a Laplace-type operator on a closed manifold.

1. We have an asymptotic expansion

\[
\theta_A(t) \overset{t \rightarrow 0}{\sim} \sum_{n \geq 0} a_n(A)t^{n-\dim(M)/2}.
\]

(7)

2. \( \theta_A(t) \) vanishes exponentially as \( t \rightarrow \infty \).
Proof. For a proof of 1. we refer e.g. to [BGV04, Thm. 2.30]. The second assertion is an exercise. \hfill \qed

Note that the numbers $a_n(A)$ are integrals over $M$ of local invariants of $A$. Further note that $\theta_{A|N}(t)$ is smooth at $t = 0$ and therefore

$$\theta_{A'}(t) = \theta_{A}(t) - \theta_{A|N}(t)$$

also has an asymptotic expansion at $t \to 0$ whose singular part coincides with the singular part of $[i]$. But also note that in the odd-dimensional case the positive part of the expansion for $\theta_{A'}(t)$ in general has terms with $t^{m/2}$ for all $m \in \mathbb{N}$ (not only for odd $m$).

By \ref{eq:asym_exp}, the spectral zeta function of $A'$ can be written in the form

$$\zeta_{A'}(s) = \frac{1}{\Gamma(s)} M(\theta_{A'})(s).$$

By Proposition \ref{prop:mero_cont} and Lemma \ref{lem:mero_cont} it has a meromorphic continuation. Since $\Gamma(s)$ has a pole at $s = 0$ we further see that $\zeta_{A'}(s)$ is regular if $s = 0$.

**Definition 2.4.** We define the **zeta-regularized determinant** of a Laplace-type operator $A$ on a closed manifold by

$$\det A' := e^{-\zeta_{A'}(0)}.$$ 

**Remark 2.5.** The **value** $\zeta_{A'}(0)$ of the zeta function at zero can be calculated. It is given by the coefficient of the constant term of the asymptotic expansion of $\theta_{A'}(t)$. We get

$$\zeta_{A'}(0) = a_{\dim(M)/2}(A) - \dim(N).$$

It is a combination of a locally computable quantity $a_{\dim(M)/2}$ and information about finitely many eigenvalues. Note that the determinant is a much more difficult quantity.

**Example 2.6.** For $R > 0$ let $M$ be $S^1_R := \mathbb{R}/R\mathbb{Z}$, i.e. the circle of volume $R$, and $A := -\partial^2_t$.

**Lemma 2.7.** We have $\det A' = R^2$.

**Proof.** Then the eigenvalues of $A$ are given by

$$4\pi^2 R^{-2} n^2, \quad n \in \mathbb{Z}.$$ 

We can express the spectral zeta function in terms of the Riemann zeta function as

$$\zeta_{A'}(s) = 2^{1-2s} R^{2s} \pi^{-2s} \zeta(2s).$$

We get

$$\zeta_{A'}(0) = -4 \log(2\pi R^{-1}) \zeta(0) + 4 \zeta'(0).$$
Using the formulas
\[ \zeta(0) = -\frac{1}{2}, \quad \zeta'(0) = -\frac{\log(2\pi)}{2}, \]
we get
\[ \zeta_{A'}'(0) = 2\log(2\pi R - 1) - 2\log(2\pi) = -2\log(R). \]
The final formula
\[ \det A' = R^2 \]
now follows. \(\square\)
Observe the dependence of the determinant on the geometry.

### 2.3 Analytic torsion

We consider a closed Riemannian manifold \((M, g^{TM})\) and a vector bundle \((V, \nabla^V, h^V)\) with a flat connection and metric. The connection \(\nabla^V : \Omega^0(M, V) \to \Omega^1(M, V)\) extends uniquely to a derivation of \(\Omega(M)\)-modules
\[ d^V : \Omega(M, V) \to \Omega(M, V) \]
of degree one and square zero. Then the Laplace operator
\[ \Delta := (d^{V*} + d^V)^2 \]  
(8)
preserves degree and its components
\[ \Delta_k : \Omega^k(M, V) \to \Omega^k(M, V), \quad k \in \mathbb{N}, \]  
(9)
are of Laplace type.

**Definition 2.8.** We define the analytic torsion of \((M, \nabla, h^{TM}, h^V)\) by
\[ T_{an}(M, \nabla^V, h^{TM}, h^V) := \sqrt{\prod_{k \in \mathbb{N}} (\det \Delta_k)^{(-1)^k}}. \]
It is the analog of \(T(\Omega(M, V), h^{TM}_{L^2}, h^{H(M,V)}_{Hodge})\).

As a consequence of Poincaré duality the analytic torsion for unitary flat bundles on even-dimensional manifolds is trivial. We say that a hermitean bundle with connection \((V, \nabla^V, h^V)\) is unitary, if \(\nabla^V\) preserves \(h^V\). Unitary flat bundles correspond to unitary representations of the fundamental group.

**Proposition 2.9.** If \(M\) is even-dimensional and \(\nabla^V\) is unitary, then
\[ T_{an}(M, \nabla^V, g^{TM}, h^V) = 1. \]
Proof. We have
\[ \Delta_k = d^*_k d_k^V + d^*_{k-1} d^V_{k-1} \]
and therefore, with appropriate definitions and (3),
\[ T_{an}(M, \nabla^V, g^{TM}, h^V) = \sqrt{\prod_k \det(d_k^* d_k^V)'^{-1}} . \]
Using that \( \nabla^V \) is unitary we get the identity
\[ *_{g^{TM}} (d_k^* d_k^V) *^{-1}_{g^{TM}} = d^V_{n-k-1} d^*_{n-k-1} . \]
It implies
\[ \det(d_k^* d_k^V)' = \det(d^V_{n-k-1} d^*_{n-k-1})' . \]
If \( \dim(M) \) is even, then we see that the factors for \( k \) and \( \dim(M) - k - 1 \) in (10) cancel each other. \( \square \)

2.4 Ray-Singer torsion

Let \((M, g^{TM})\) be a closed odd-dimensional Riemannian manifold, \((V, \nabla^V, h^V)\) be a vector bundle on \(M\) with flat connection and metric, and \(h^{H(M,V)}\) be a metric on the cohomology.

**Definition 2.10.** We define the Ray-Singer torsion of \((M, \nabla^V, h^{H(M,V)})\) by
\[ T_{RS}(M, \nabla^V, h^{H(M,V)}) := v(h^{H(M,V)}, h_{Hodge}^{H(M,V)}) T_{an}(M, \nabla^V, g^{TM}, h^V) . \]
It is interesting because of the following theorem (which also justifies the omission of the metrics in the notation).

**Theorem 2.11.** The Ray-Singer torsion \( T_{RS}(M, \nabla^V, h^{H(M,V)}) \) is independent of the choices of metrics \( g^{TM} \) and \( h^V \).

**Proof.** We give a sketch. We first assume that \( H(M,V) = 0 \). As a consequence all integrands below vanish exponentially at \( t \to \infty \). Moreover, the Ray-Singer torsion coincides with the analytic torsion. Let \( N \) denote the \( \mathbb{Z} \)-grading operator on \( \Omega(M,V) \).

We define
\[ F(s) := \frac{1}{\Gamma(s)} \int_0^\infty \text{Tr} (-1)^N Ne^{-t\Delta} t^{s-1} dt . \]
This integral converges for \( \Re(s) >> 0 \) and by Proposition 2.3 and Lemma 2.2 has, as a function of \( s \), a meromorphic continuation to \( \mathbb{C} \). Then
\[ \log T_{an}(M, \nabla^V, h^{TM}, h^V) = -\frac{d}{ds}_{|s=0} F(s) . \]
Since any two metric data can be connected by a path, it suffices to discuss the variation formula. The derivative of \( F(s) \) with respect to the metric data is given by
\[ \delta F(s) = \int_0^\infty \text{Tr} (-1)^N N \delta(e^{-t\Delta}) t^{s-1} dt = -\int_0^\infty \text{Tr} (-1)^N N \delta(\Delta) e^{-t\Delta} t^s dt . \]
Here we use that $\delta(\Delta)$ commutes with $N$ and the cyclicity of the trace. In order to calculate $\delta(\Delta)$ we encode the metric data into a duality map

$$I : \Omega(M, V) \to \Omega(M, V)' , \quad \langle \alpha, \omega \rangle = I(\alpha) (\omega) .$$

We further define its logarithmic derivative $L := I^{-1} \delta(I) \in \Gamma(M, \text{End}(\Lambda^* T^* M \otimes V))$. We write $d^V \kappa = I^{-1} d^V I$, where the adjoint $d^V$ of $d^V$ does not depend on the metrics. Consequently, $\delta(d^V \kappa) = -[L, d^V \kappa]$. Inserting this into (8) we get

$$\delta(\Delta) = -Ld^V \kappa + d^V (d^V \kappa) = -Ld^V \kappa + d^V \kappa + d^V \kappa + d^V \kappa$$

Using $[\Delta, d^V] = 0$, $[\Delta, d^V \kappa] = 0$, $[d^V, N] = -d^V$ and $[d^V \kappa, N] = d^V \kappa$ and the cyclicity of the trace we get

$$\text{Tr}(-1)^N N \delta(\Delta) e^{-t\Delta} = \text{Tr}(-1)^N L \Delta e^{-t\Delta} = -\frac{d}{dt} \text{Tr}(-1)^N Le^{-t\Delta} .$$

Here are some more details of the calculation in which we move all differential operators on the left of $L$ to the right. In this process we must commute them with $(-1)^N N$, the heat operator, and we use the cyclicity of the trace.

$$\text{Tr}(-1)^N N \delta(\Delta) e^{-t\Delta} = \text{Tr}(-1)^N N (-Ld^V \kappa + d^V d^V \kappa + d^V \kappa + d^V \kappa) e^{-t\Delta}$$

$$= \text{Tr}(-1)^N N (-Ld^V \kappa + d^V d^V \kappa + d^V \kappa + d^V \kappa) e^{-t\Delta}$$

$$+ \text{Tr}(-1)^N LD^V \kappa e^{-t\Delta} + \text{Tr}(-1)^N d^V d^V \kappa e^{-t\Delta}$$

We get by partial integration for $\text{Re}(s) >> 0$ (in order to avoid a boundary term at $t = 0$)

$$\delta F(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{d}{dt} \left[ -\text{Tr}(-1)^N Le^{-t\Delta} \right] t^s dt$$

$$= \frac{s}{\Gamma(s)} \int_0^\infty \text{Tr}(-1)^N Le^{-t\Delta} t^{s-1} dt .$$

We now use the asymptotic expansion (a generalization of Proposition 2.3.1. to traces of the form $\text{Tr} Le^{-t\Delta}$, where $L$ is some bundle endomorphism)

$$\text{Tr}(-1)^N Le^{-t\Delta} \sim \sum_{n \in \mathbb{N}} b_n t^{n - \text{dim}(M)/2} .$$

In particular it has no constant term. Therefore, by Lemma 2.2 we have

$$\delta F(s) = \frac{s}{\Gamma(s)} \kappa(s) ,$$

where $\kappa$ is meromorphic on $\mathbb{C}$ and regular at $s = 0$. In order to get the logarithmic derivative of the Ray-Singer torsion we must apply $-\frac{d}{ds}|_{s=0}$ to this function. Since $\frac{s}{\Gamma(s)}$ has a second order zero at $s = 0$ we conclude that

$$\delta \log T_{RS}(M, \nabla^V, h^{H(M,V)}) = 0 .$$
In the presence of cohomology one uses a similar argument. In the definition of $F(s)$ one replaces $\text{Tr}$ by $\text{Tr}(1 - P)$, where $P$ is the projection onto $\ker(\Delta)$. Then (12) has a constant term given by $-\text{Tr}(-1)^NPL$. Using that $\text{res}_{s=0}\Gamma(s) = 1$ we get

$$\delta \log T_{an}(M, \nabla^V, h^T, h^V) = -\text{Tr}(-1)^NPL.$$ 

This is exactly the negative of the logarithmic variation of the volume on the cohomology induced by the Hodge metric, i.e.

$$\delta \log \nu(h^H(M, V), h^H(M, V)_{\text{Hodge}}) = \text{Tr}(-1)^NPL.$$ 

These two terms cancel in the product defining the Ray-Singer torsion.

**Remark 2.12.** The arguments $(M, \nabla^V, h^H(M, V))$ of the Ray-Singer torsion are differential-topological data. The right-hand side is of global analytic nature and apriori depends on additional geometric choices. The interesting fact is that its actually does not depend on these choices. This is a typical situation in which the natural question is now to provide an explicit description of this quantity in terms of differential topology.

### 2.5 Torsion for flat bundles on the circle

In this example we give an explicit calculation of the Ray-Singer torsion for $M = S^1$ and the flat line $(V, \nabla^V)$ bundle with holonomy $1 \neq \lambda \in U(1)$. We have $H(S^1, V) = 0$ so that we can drop the metric on the cohomology from the notation.

**Proposition 2.13.** We have

$$T_{RS}(S^1, \nabla^V) = \frac{1}{2 \sin(\pi q)}.$$ 

**Proof.** We represent $S^1 := \mathbb{R}/\mathbb{Z}$ in order to fix the geometry. In order to calculate the spectrum of the Laplace operator we work on the universal covering $\mathbb{R}$ and trivialize the bundle $T^*\mathbb{R}$ using the section $dt$. We further trivialize the pull-back of the flat line bundle using parallel sections. Under these identifications

$$\Omega^1(S^1, L) = \{ f \in C^\infty(\mathbb{R}) \mid (\forall t \in \mathbb{R} \mid f(t + 1) = \lambda f(t)) \}.$$ 

The Laplace operator $\Delta_1$ acts as $-\partial^2$.

We now calculate its spectrum. We choose $q \in (0, 1)$ such that $\lambda = e^{2\pi i q}$. The eigenvectors of $\Delta_1$ are the functions $t \mapsto e^{2\pi i (q+n)t}$ for $n \in \mathbb{Z}$, and the corresponding eigenvalues are given by

$$4\pi^2(q+n)^2.$$ 

The zeta function of the Laplace operator is now

$$\zeta_{\Delta_1} = 4^{-s}\pi^{-2s} \sum_{n \in \mathbb{Z}} (q+n)^{-2s}.$$ 

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In order to calculate $\det \Delta_1$ we express this zeta function in terms of the Hurwitz zeta function

$$\zeta(s, q) := \sum_{n \in \mathbb{N}} (q + n)^{-s}$$

and then use known properties of the latter. We have

$$\zeta_{\Delta_1}(s) = 4^{-s} \pi^{-2s} [\zeta(2s, q) + \zeta(2s, 1 - q)] .$$

We have the relation

$$\partial_q \zeta(s, q) = -s \zeta(s + 1, q) .$$

This gives

$$\partial_s \partial_q [\zeta(s, q) + \zeta(s, 1 - q)] = [\zeta(s + 1, 1 - q) - \zeta(s + 1, q)] + s[\partial_s \zeta(s + 1, 1 - q) - \partial_s \zeta(s + 1, q)] .$$

We now use the expansion of the Hurwitz zeta function at $s = 1$

$$\zeta(s, q) = (s - 1)^{-1} - \psi(q) + O(s - 1)$$

with

$$\psi(q) := \frac{\Gamma'(q)}{\Gamma(q)} .$$

We see that the two differences are regular at $s = 1$. The evaluation of the second term at $s = 0$ vanishes because of the prefactor $s$. Hence

$$\partial_{s|s=0} \partial_q [\zeta(s, q) + \zeta(s, 1 - q)] = \psi(q) - \psi(1 - q) .$$

Consequently, integrating from $q = 1/2$ we get using

$$\Gamma(q) \Gamma(1 - q) = \frac{\pi}{\sin(\pi q)}$$

that

$$\partial_{s|s=0} [\zeta(s, q) + \zeta(s, 1 - q)] = \gamma + \log(\Gamma(q) \Gamma(1 - q)) = \gamma + \log\left(\frac{\pi}{\sin \pi q}\right) ,$$

where

$$\gamma := 2\partial_{s|s=0} \zeta(s, 1/2) - \log(\pi) .$$

In order to determine this number we express the Hurwitz zeta function in terms of the Riemann zeta function

$$\zeta(s, 1/2) = \sum_{n \in \mathbb{N}} (n + 1/2)^{-s} = 2^s \sum_{n \in \mathbb{N}} (2n + 1)^{-s}$$

$$= 2^s \sum_{n \in \mathbb{N}} (2n + 1)^{-s} + 2^s \sum_{n \in \mathbb{N}} (2n)^{-s} - 2^s \sum_{n \in \mathbb{N}_+} (2n)^{-s}$$

$$= (2^s - 1) \zeta(s) .$$
We get, using $\zeta(0) = -\frac{1}{2}$,

$$
\partial_{s|s=0}\zeta(0, 1/2) = (\log(2)2^s\zeta(s) + (2^s - 1)\zeta'(s))|_{s=0} = -\frac{1}{2} \log(2) .
$$

Finally,

$$
\gamma = -\log(2) - \log(\pi) .
$$

So

$$
\partial_{s|s=0}[\zeta(s, q) + \zeta(s, 1 - q)] = -\log(2) - \log(\sin(\pi q)) .
$$

We have $\zeta(0, q) = 1/2 - q$. This implies $[\zeta(2s, q) + \zeta(2s, 1 - q)]|_{s=0} = 0$. We now calculate

$$
\zeta'_{\Delta_1}(0) = \partial_{s|s=0} \left( 4^{-s}\pi^{-2s}[\zeta(2s, q) + \zeta(2s, 1 - q)] \right) = -2 \log(2) - 2 \log(\sin(\pi q)) .
$$

We thus get

$$
det \Delta_1 = 4 \sin^2(\pi q) .
$$

We finally get

$$
T_{RS}(S^1, \nabla^V) = \frac{1}{2 \sin(\pi q)} .
$$

We now compare this result with an evaluation of the Reidemeister-Franz torsion (to be defined later) $T_{RF}(S^1, \nabla^V)$. Using the standard cell decomposition $S^1 \cong \Delta^1/\partial \Delta^1$ the Reidemeister-Franz torsion is the analytic torsion $T(\mathcal{C}, h^C)$ of the chain complex $\mathcal{C}$ given by

$$
\mathcal{C} : \mathbb{C} \xrightarrow{1-\lambda} \mathbb{C}
$$

starting at 0, where $h^C$ is the canonical metric. We have

$$
det \Delta_1 = |(1 - \lambda)(1 - \lambda^{-1})| = 2 - \lambda - \lambda^{-1} = 2 - 2 \cos(2\pi q) .
$$

We calculate

$$
2 - 2 \cos(2\pi q) = 2 - 2(1 - 2 \sin^2(\pi q)) = 4 \sin^2(\pi q) .
$$

This gives

$$
T_{RF}(S^1, \nabla^V) = T(\mathcal{C}, h^C) = \frac{1}{2 \sin(\pi q)} .
$$

We observe that

$$
T_{RF}(S^1, \nabla^V) = T_{RS}(S^1, \nabla^V) .
$$

The equality $T_{RF} = T_{RS}$ in general is the contents of the Cheeger-Müller theorem.
3 Morse theory, Witten deformation, and the Müller-Cheeger theorem

3.1 Morse theory

Consider a function $f \in C^\infty(M)$. A point $x \in M$ is called critical if $df(x) = 0$. If $x$ is critical, then we have a well-defined symmetric bilinear form $\text{Hess}_f(x)$ on $T_xM$ which is called the Hessian. For tangent vectors $X, Y \in T_xM$ choose extensions to vector fields which are denoted by the same symbols. Then the Hessian is defined by

$$\text{Hess}_f(x) := X(Y(f))(x).$$

Since $df(x) = 0$ this does not depend on the choice of the extensions.

**Definition 3.1.** A function $f \in C^\infty(M)$ is called a Morse function if $\text{Hess}_f(x)$ is non-degenerate at every critical point $x \in M$ of $f$. The index of a critical point $i_f(x)$ is the number of negative eigenvalues of $\text{Hess}_f(x)$.

For $i \in \mathbb{N}$ we let $\text{Crit}_f(i)$ denote the set of critical points of index $i$ so that $\text{Crit}_f = \bigcup_{i \in \mathbb{N}} \text{Crit}_f(i)$ is the set of critical points of $f$.

**Lemma 3.2.** As a subset of $M$ the set $\text{Crit}_f$ of a Morse function $f$ is discrete.

**Example 3.3.** We consider the function $f : \mathbb{R}^n \to \mathbb{R}$ given by

$$f(x) := \frac{1}{2}(-x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2).$$

This is a Morse function which has one critical point at $x = 0$ of index $i$.

**Fact 3.4.** Being a Morse function is genericity condition on $f$. In particular, if $f$ is any smooth function, then there exists Morse functions in every $C^\infty$-neighbourhood of $f$. Moreover, the condition of being Morse is open.

Given a Morse function $f$ we get an increasing filtration $(M^{\leq a})_{a \in \mathbb{R}}$ of $M$ by closed subsets

$$M^{\leq a} := \{ f \leq a \}.$$

If $a$ is a regular value of $f$, then $M^{\leq a} \subseteq M$ is a closed submanifold with boundary $M^a := \{ f = a \}$. For $a, b \in \mathbb{R}$ with $a \leq b$ we can study the inclusion of $M^{\leq a}$ into $M^{\leq b}$. We distinguish two cases.

- If the interval $[a, b]$ does not contain a critical value, then the inclusion $M^{\leq a} \to M^{\leq b}$ is a deformation retract. To this end one chooses a Riemannian metric on $M$ with product structure near the boundaries of $M^{\leq a}$ and $M^{\leq b}$. Then the retraction can be built from the gradient flow $(\Phi_t)_{t \in \mathbb{R}}$ determined by $x' = -\text{grad}(f)(x)$.

- If $[a, b]$ contains critical points, then $M^{\leq b}$ admits a deformation retract to a space obtained from $M^{\leq a}$ by attaching cells.
We shall discuss a particular nice situation. A Morse function Morse is called self-indexing if
\[ f(x) = \text{index}_f(x) \]
for every critical point \( x \in \text{Crit}_f \).

**Fact 3.5.** Self-indexing Morse functions exist in abundance.

**Example 3.6.** The function \( f : \mathbb{R}^n \to \mathbb{R} \) given by
\[ f(x) := i + \frac{1}{2}(-x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2) \]
is self-indexing.

**Definition 3.7.** The **stable manifold** \( W^s(x) \) of a critical point \( x \in \text{Crit}_f \) is the subset of \( M \) of all points \( y \in M \) such that \( \lim_{t \to \infty} \Phi_t(y) = x \). The **unstable manifold** \( W^u(x) \) is defined in a similar manner replacing \( \infty \) by \( -\infty \).

**Example 3.8.** In the example 3.3 the unstable manifold is \( W^u(0) = \mathbb{R}^i \subseteq \mathbb{R}^n \) embedded in the standard way and the stable manifold is the linear subspace \( W^s(x) = \mathbb{R}^{n-i} \subseteq \mathbb{R}^n \) generated by the standard basis vectors \( e_{i+1}, \ldots, e_n \).

This example describes the local picture of the stable and unstable manifolds near a critical point in general. But also the global structure is understood.

**Lemma 3.9.** The stable manifold \( W^s(x) \) is the image of an injective immersion
\[ \mathbb{R}^{\dim(M) - i_f(x)} \to M. \]

Similarly, the unstable \( W^u(x) \) is the image of an injective immersion of
\[ \mathbb{R}^{i_f(x)} \to M. \]

**Fact 3.10.** We assume that \( f \) is a self-indexing Morse function. We now describe the structure of the inclusion
\[ M^{i-1/2} \subseteq M^{i+1/2}. \]

For \( x \in \text{Crit}_f(i) \) the gradient flow induces a diffeomorphism
\[ (D^u(x), S^u(x)) : = (W^u(x) \cap M^{\geq i-1/2}, W^u(x) \cap M^{i-1/2}) \cong (D^{i_f(x)}, S^{i_f(x)-1}) . \]

Using the gradient flow further one can show that
\[ M^{\leq i-1/2} \cup_{x \in \text{Crit}_f(i)} D^u(x) \subseteq M^{\leq i+1/2} \]
is a deformation retract. So up to homotopy \( M^{\leq i+1/2} \) is obtained from \( M^{\leq i-1/2} \) by attaching a collection of \( i \)-cells, one for every critical point of \( f \) of index \( i \).
We will now show that the choice of a Riemannian metric and a Morse function determines a cell structure on $M$. In order to get a simple structure we assume the Morse-Smale condition.

**Definition 3.11.** We say that the pair of $f$ of a Morse function and the Riemannian metric on $M$ satisfies the **Morse-Smale** condition, if for every pair of critical points $x, y \in M$ the intersection of $W^u(x)$ and $W^s(y)$ is transversal.

**Fact 3.12.** The Morse-Smale condition depends on the gradient flow of $f$ and hence also on the choice of the metric. If we fix the Morse function $f$ and consider a Riemannian metric, then in every $C^\infty$-neighbourhood of the metric there exists metrics such that the pair satisfies the Morse-Smale condition. Moreover, for fixed $f$ the Morse-Smale condition is an open condition on the metric.

In the following we choose a parametrization of $W^u(x)$ by the interior of the unit disc $D^{index_f(x)}$. For the following we refer to the appendix of [BZ92] by Laudenbach.

**Theorem 3.13.** The immersion $W^u(x) \to M$ extends to a continuous map
\[
\kappa(x) : D^{i_f}(x) \to M .
\]

We let
\[
M_i := \bigcup_{x \in \text{Crit}_f(i), i \leq k} W^u(x) .
\]

**Theorem 3.14.** The space $M_{i+1}$ is obtained from $M_i$ by attaching a collection of $i + 1$-cells, one for every critical point of index $i + 1$ with characteristic maps $\kappa(x)$.

In particular we get a $CW$-decomposition of $M$.

### 3.2 The Morse-Smale complex

We assume that $M$ is a closed manifold which is equipped with a self-indexing Morse function $f$ and a Riemannian metric such that the Morse-Smale condition is satisfied. We now describe the associated cellular chain complex $C_*(M)$ which is also called the **Morse-Smale** complex. The vector space of degree $i$-chains is given by
\[
C_i := \mathbb{R}[\text{Crit}_f(i)] .
\]

The chosen parametrizations of the unstable manifolds induce orientations. The same choices induce coorientations of the stable manifolds. For a pair
\[
x, y \in \text{Crit}_f \text{ with } i_f(y) = i_f(x) + 1
\]
the intersection
\[
W^u(y) \cap W^s(x)
\]
is transversal and defines a smooth one-dimensional submanifold of $M$. Every component
\[
\gamma \in \pi_0(W^u(y) \cap W^s(x))
\]
is an orbit of the gradient flow and hence diffeomorphic to \( \mathbb{R} \). As an intersection of an oriented and a cooriented manifold it is oriented. We define the multiplicity of \( \gamma \)

\[
m(\gamma) \in \{1, -1\}
\]
such that the gradient \( m(\gamma) \) \( \text{grad}(f) \) is positively oriented on \( \gamma \). We can now define the differential of the Morse-Smale complex by

\[
\partial : C_{i+1} \to C_i, \quad y \mapsto \sum_{x \in \text{Crit}_f(i)} \sum_{\gamma \in \pi_0(W^u(y) \cap W^s(x))} m(\gamma) x .
\]

Here we consider the critical points \( x \) and \( y \) as basis vectors in the chain groups \( C_i \) and \( C_{i+1} \), respectively.

**Theorem 3.15.** The construction described above defines a chain complex \( (C_i, \partial) \) which is naturally isomorphic to chain complex associated to the cell decomposition of \( M \) given in Theorem 3.14.

In the following we extend this construction to a local coefficient system \( \mathcal{V} \) given by the parallel sections of a flat vector bundle \( (V, \nabla_V) \) on \( M \). Since we consider homology, we will actually work with the adjoint bundle \( (V^*, \nabla_V^*) \) and the associated local coefficient system \( V^* \).

For \( x \in M \) let \( V^*_x \) be the fibre of \( V^* \). For a smooth curve \( \gamma : [0, 1] \to M \) we have a parallel transport \( P_\gamma : V^*_y(0) \to V^*_y(1) \). The value of \( P_\gamma \) on \( v_0^* \in V^*_y(0) \) is given by the value at \( t = 1 \) of the solution of the initial value problem

\[
\nabla_{\gamma'(t)} v^*(\gamma(t)) = 0 , \quad v^*(0) = v_0^* .
\]

**Lemma 3.16.** If \( \gamma \) is an \( \mathbb{R} \)-orbit from the critical point \( y \) of index \( i + 1 \) to the critical point \( x \), then we can define a parallel transport \( P_\gamma : V^*_y \to V^*_x \).

**Remark 3.17.** This requires an argument since the curve \( \gamma \) is parametrized by \( \mathbb{R} \) and not by a finite interval.

We define

\[
C_i(V^*) := \bigoplus_{i \in \text{Crit}_f(i)} V^*_x .
\]

We further define the differential

\[
C_{i+1}(V^*) \to C_i(V^*) , \quad \partial_{V^*_y} := \sum_{x \in \text{Crit}_f(i)} \sum_{\gamma \in \pi_0(W^u(y) \cap W^s(x))} m(\gamma) P_\gamma
\]

**Theorem 3.18.** The construction described above defines a chain complex \( (C_i(V^*), \partial) \) which is naturally isomorphic to chain complex associated to the cell decomposition of \( M \) and the local coefficient system induced by \( (V^*, \nabla_{V^*}^*) \).
Example 3.19. If \((V, \nabla^V)\) is the trivial one-dimensional bundle, then \((C_i(V^*), \partial)\) is the Morse-Smale complex \((C_i, \partial)\) defined above.

We can now form the dual complex

\[(C^*(V), d) \,, \quad C^i(V) := C_i(V^*)^* \,, \quad d := \partial^* .\]

Note that

\[
C^i(V) := \bigoplus_{x \in \text{Crit}_f(i)} V_x .
\]

(13)

The complex \((C^*(V), d)\) will be called the Morse-Smale cochain complex associated to the flat bundle \((V, \nabla^V)\).

Remark 3.20. Though it is not reflected in the notation, the cochain spaces \(C^i(V)\) depend on \(f\), and the differential in addition depends on the Riemannian metric on \(M\).

3.3 Milnor metric and Ray-Singer metric

We keep the assumptions on \(M\) made in the beginning of Section 3.2. We further assume that \((V, \nabla)\) has a hermitean metric \(h^V\) which is preserved by \(\nabla^V\). Then we get a metric on the fibres \(V_x\) for \(x \in M\). In particular, in view of (13), we get a metric on the complex \(C^*(V)\). We let \(h^\text{det} C^*(V)\) be the induced metric on \(\text{det} C^*(V)\), see Definition 1.1.

The cohomology of the Morse-Smale cochain complex \((C^*(V), d)\) is the cohomology \(H_f(M, V)\) of \(M\) with coefficients in the locally constant sheaf \(V\) of parallel sections of \(V\). We use the subscript \(f\) to indicate this group depends on \(f\) (and more data).

Remark 3.21. Of course it is known that these groups \(H_f(M, V)\) are mutually isomorphic for different choices of \(f\) and the additional data, but the choice of the isomorphisms matters in the following discussion.

Recall the canonical map \(\tau_{C^*(V)}\) given in Proposition 1.2.

Definition 3.22. We define the Milnor metric on \(\text{det} H_f(M, V)\) by

\[h^\text{det} H_f(M, V)\text{Miln} := \tau_{C^*(V)} h^\text{det} C^*(V) .\]

We now consider the de Rham complex \(\Omega(M, V)\). Its cohomology will be denoted by \(H(M, V)\). The Riemannian metric on \(M\) and the hermitean metric \(h^V\) induce an \(L^2\)-metric \(h^H(M, V)\text{Hodge}\) on the cohomology. We get an induced metric \(h^\text{det} H^H(M, V)\text{Hodge}\) on \(\text{det} H(M, V)\) and the analytic torsion \(T_{an}(M, \nabla^V, g^T M, h^V)\), see Definition 2.8.

Definition 3.23. We define the Ray-Singer metric

\[h^\text{det} H^H(M, V)\text{RayS} := T_{an}(M, \nabla^V, g^T M, h^V)^{-1} h^\text{det} H^H(M, V)\text{Hodge} .\]
We will use the de Rham isomorphism

\[ H(I_f) : H(M, V) \to H_f(M, V) \]

in order to compare these two metrics. The de Rham isomorphism is induced by the integration map

\[ I_f : \Omega^i(M, V) \to C^*(V) \quad \text{(14)} \]

Recall that \( C^i(V) := C^i(V^*)^\dagger \). For \( \omega \in \Omega^i(M, V) \), \( x \in \text{Crit}_f(i) \), and \( v \in V_x \) we define

\[ I_f(\omega)(v) := \int_{W^u(x)} \tilde{v} \omega \]

Here \( \tilde{v} \) is the parallel extension of \( v \) along the unstable manifold \( W^u(x) \) and \( \tilde{v}(\omega) \in \Omega^i(W^u_x) \) is the scalar \( i \)-form on \( W^u(x) \) obtained by pairing the values of \( \omega \) with \( \tilde{v} \) and restricting the form to \( W^u(x) \). The integral uses the orientation of \( W^u \).

**Theorem 3.24.** The integration map (14) is a well-defined map of complexes and a quasi-isomorphism.

**Remark 3.25.** We discuss some of the steps of the proof. First of all, since the unstable manifold \( W^u(x) \) is not compact, one must give an argument that the integral is well-defined. Here one must improve Theorem 3.13 to the extent that the inclusion of \( W^u(x) \hookrightarrow M \) extends smoothly to a compactification to a manifold with corners. Then one must use Stokes’ theorem to show that \( I_f \) is a chain map. Here one must identify the boundary faces of the compactification of \( W^u(x) \) with a collection of unstable manifolds of critical points of index \( i - 1 \).

From now one we will always implicitly identify \( H_f(M, V) \) with \( H(M, V) \) using \( I_f \). In particular we get a Milnor metric \( h_{\text{Miln}}^{\text{det}H(M,V)} \).

**Lemma 3.26.** We assume that \( \dim(M) \) is odd. The quotient

\[ h_{\text{RayS}}^{\text{det}H(M,V)}/h_{\text{Miln}}^{\text{det}H(M,V)} \in \mathbb{R}^+ \quad \text{(15)} \]

is independent of the choice of the Riemannian metric \( g^TM \) on \( M \).

**Proof.** Indeed, let \( h^{H(M,V)} \) be the metric induced on \( H(M, V) \) through Hodge theory of the complex \( C^*(V) \) and \( H(I_f) \). Then this quotient is given by

\[ \frac{T_{\text{an}}(C^*(V), h^V, h^{H(M,V)})}{T_{\text{RS}}(M, \nabla^V, h^{H(M,V)})} = \frac{T_{\text{an}}(C^*(V), h^V, h^{H(M,V)})}{v(h^{H(M,V)}, h_{\text{Hodge}}^{H(M,V)})T_{\text{an}}(M, \nabla^V, g^TM, h^{V})}. \]

We apply Theorem 2.11 \( \square \)
Remark 3.27. In the theorem above we of course still require that the pair $f$ and $g^{TM}$ of the Morse function and the Riemannian metric satisfies the Morse-Smale condition. Note that the Ray-Singer metric does not depend on $f$. The Milnor metric does not change if we deform $f$ within Morse functions satisfying in addition the Morse-Smale condition. So we see that the quotient (15) is quite rigid.

We can now formulate the **Cheeger-M"uller theorem** [M78] [Che79]:

**Theorem 3.28.** If $(M, g^{TM})$ is a closed odd-dimensional Riemannian manifold and $(V, \nabla^V, h^V)$ is an unitary flat bundle, then we have the equality of metrics

$$h^{\det H(M,V)}_{RayS} = h^{\det H(M,V)}_{Miln}.$$  

This is not the most general version. We refer to [M93] for unimodular bundles and [BZ92] for the general case. An Extension to flat bundles of von Neumann algebras have been studied in [BFKM96].

4 The Witten deformation

4.1 The Witten deformation

In this section we give a sketch of the proof of the Cheeger-M"uller Theorem [3.28] given by Bismut-Zhang [BZ92], [BFKM96] using the **Witten deformation** [Wit82]. By now, the Witten deformation is a widely used tool in differential topology.

We consider the de Rham complex $\Omega(M,V)$ on a Riemannian manifold $M$ with a unitary flat bundle $(V, \nabla^V, h^V)$. If $f \in C^\infty(M)$ is any function, then we can consider the conjugated differential

$$d_f := e^{-f} \circ d \circ e^f = d + \epsilon(df),$$

where $\epsilon(\alpha)(\omega) := \alpha \wedge \omega$ is the operation of multiplication by $\alpha \in \Omega^1(M)$. The adjoint of this operation is given by

$$\epsilon(\alpha)^* = \iota(\alpha),$$

where $\iota(\alpha)$ is the insertion of the vector field dual to $\alpha$. Hence

$$d_f^* = d^* + \iota(df).$$

We form the Dirac operator

$$\mathcal{D}_f := d_f + d_f^*: \Omega(M,V) \to \Omega(M,V).$$

The square of the Dirac operator is called **Witten Laplacian**

$$\Delta_f := \mathcal{D}_f^2.$$  \hspace{1cm} (16)

Inserting the formulas for the constituents of the Dirac operator we get

$$\Delta_f = \Delta + \{d, \iota(df)\} + \{d^*, \epsilon(df)\} + \{\epsilon(df), \iota(df)\}. \hspace{1cm} (17)$$

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Remark 4.1. The multiplication by $e^f$ induces an isomorphism of chain complexes
\[(\Omega(M, V), d) \to (\Omega(M, V), d_f)\].
It in particular induces an isomorphism in cohomology.
Let $h_{L^2}$ denote the $L^2$-metric on $\Omega(M, V)$. Then the multiplication by $e^f$ induces an isometry
\[(\Omega(M, V), h_{L^2}) \to (\Omega(M, V), e^{-f}h_{L^2})\].
Since
\[e^f \circ d_f \circ e^{-f} = d\]
the pair of mutually adjoint operators $(d_f, d_f^*)$ is isometric to the pair $(d, d^*)$ which is mutually adjoint on $(\Omega(M, V), e^{-f}h)$.

In Theorem 2.11 we have seen, that in the odd-dimensional case the Ray-Singer torsion is independent of the choice of the metric. Hence we could use $\Delta_f$ instead of $\Delta$ in order to calculate the Ray-Singer torsion.

We now study the Witten Laplacian (17) in greater detail. Let us abbreviate the term which is linear in $f$ by $A(f)$. It is apriori a first order differential operator.

Lemma 4.2. $A(f)$ is a multiplication operator.

Proof. We must show that $A(f)$ commutes with multiplication by smooth functions. Since $\iota(df)$ commutes with the multiplication by functions for a $g \in C^\infty(M)$ we have, using $[d, g] = \epsilon dg$,
\[\{\{d, \iota(df)\}, g\} = \{\epsilon(dg), \iota(df)\} = < df, dg > .\]
The adjoint of this gives
\[\{\{d^*, \epsilon(df)\}, g\} = - < df, dg > .\]
Consequently, $[A(f), g] = 0$, hence $A(f)$ is a zero-order operator.

The term of the Witten Laplacian which is quadratic in $f$ is given by
\[\{\epsilon(df), \iota(df)\} = \|df\|^2 .\]
We thus have
\[\Delta_f = \Delta + A(f) + \|df\|^2 .\]
The idea is now to replace $f$ by $tf$ and study the limit $t \to \infty$. Then
\[\Delta_{tf} = \Delta + tA(f) + t^2\|df\|^2 .\] (18)
The dominating zero order term is $t^2\|df\|^2$. The operator (18) has the form of a family of Schrödinger operator with potential $t^2\|df\|^2 + O(t)$. In order to describe the cohomology of $(\Omega(M, V), d_f)$ we are interested in the kernel of $\Delta_f$. For the torsion it will turn out that also the small eigenvalues matter. One should expect that the form of the operator forces the eigenfunctions to small eigenvalues to localize at the zeros of $\|df\|^2$, i.e. critical points of $f$. 

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4.2 A model case

In order to understand spectrum of the Witten Laplacian $\Delta_{tf}$ and the behaviour of the eigenvectors for large $t$ better we consider a model case. We take $M = \mathbb{R}^n$ with the standard metric and for $i \in \{0, \ldots, n\}$ we consider

$$f(x) := i + \frac{1}{2}(-x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2).$$

This function has a Morse critical point in $x = 0$ of index $i$. Then

$$\|df\|^2 = \|x\|^2.$$

We further have

$$df = -x_1 dx_1 - \cdots - x_idx_i + x_{i+1} dx_{i+1} \cdots + x_n dx_n.$$

In order to understand the term $A(f)$ it suffices to calculate it on the basis forms $dx^J$. We calculate

$$\{d, \iota(x_k dx_k)\} dx^J = \epsilon(dx_k) \iota(dx_k) dx^J.$$

Similarly, using $[d^*, x_k] = -\iota(dx_k)$,

$$\{d^*, \epsilon(x_k dx_k)\} dx^J = -\iota(dx_k) \epsilon(dx_k) dx^J.$$

We introduce the abbreviations

$$\epsilon(dx_k) \iota(dx_k) =: N_k, \quad \iota(dx_k) \epsilon(dx_k) =: \bar{N}_k.$$

These operators are degree-preserving and act according to the following formulas

$$N_k(dx^J) = \begin{cases} dx^J & k \in J \\ 0 & k \notin J \end{cases}, \quad \bar{N}_k(dx^J) = \begin{cases} dx^J & k \notin J \\ 0 & k \in J \end{cases}.$$

From (19) we now get

$$A(f) = (-N_1 + \bar{N}_1) + \cdots + (-N_i + \bar{N}_i) + (N_{i+1} - \bar{N}_{i+1}) + \cdots + (N_n - \bar{N}_n).$$

The eigenvectors of $A(f)$ (on the constant forms on $\mathbb{R}^n$) are the $dx^J$ with eigenvalue

$$a(J) = -4p + 2i + 2k - n,$$

where $p \in \{0, \ldots, k\}$ is determined by the condition that the $k$-form $dx^J$ contains $p$-factors $dx_j$ with index $j \in \{1, \ldots, i\}$ and $k - p$-factors with index $j \in \{i + 1, \ldots, n\}$.

The minimal eigenvalue of $A(f)$ is $-n$ and occurs for $k = i = p$. It is of multiplicity 1 with eigenform $dx_1 \wedge \cdots \wedge dx_i$. 

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We have a decomposition of

\[ \Omega^k(\mathbb{R}^n) \cong \bigoplus_{|J|=k} C^\infty(\mathbb{R}^n)dx^J, \]

where the sum runs over the multi-indices \( 1 \leq j_1 < \cdots < j_k \leq n \). This decomposition is compatible with the \( L^2 \)-scalar products. The operator \( \Delta_{tf} \) acts on the summand with index \( J \) by

\[ \Delta + ta(J) + t^2|x|^2. \]

We now separate the variables and write the operator in the form

\[ \sum_{i=1}^{n} (-\frac{d^2}{dx_i^2} + t^2x_i^2) + ta(J). \tag{20} \]

The operator

\[ H := -\frac{d^2}{dx^2} + t^2x^2 \]

is called harmonic oscillator. Considered on the domain \( C^\infty_c(\mathbb{R}) \) it is essentially self-adjoint on the Hilbert space \( L^2(\mathbb{R}) \). It has a discrete spectrum consisting of eigenvalues.

We sketch this calculation. For the eigenfunctions we make the Ansatz

\[ \psi(x) = P(x)e^{-tx^2/2} \]

for a polynomial \( P \). These functions are clearly in \( L^2(\mathbb{R}) \). The eigenvector equation for an eigenvalue \( \lambda \) is then

\[ 0 = (H - \lambda)(\psi) = e^{-tx^2/2}(-P'' + 2txP' + (t - \lambda)P(x)). \]

Hence the polynomial must satisfy

\[ -P'' + 2txP' + (t - \lambda)P = 0 \tag{21} \]

Assume that the order of \( P \) is \( p \). Then considering the leading term in (21) we get the condition

\[ 2tp + t = \lambda. \]

If we normalize \( P \) by \( P(x) = x^p + \ldots \), then the lower terms are determined uniquely inductively.

**Example 4.3.** Here is a table of the first three \( P \)'s.

<table>
<thead>
<tr>
<th>( P )</th>
<th>( \lambda )</th>
<th>( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( t )</td>
<td>0</td>
</tr>
<tr>
<td>( x )</td>
<td>3( t )</td>
<td>1</td>
</tr>
<tr>
<td>( x^2 - \frac{1}{2}x )</td>
<td>5( t )</td>
<td>2</td>
</tr>
<tr>
<td>( x^3 - \frac{3}{2}x )</td>
<td>7( t )</td>
<td>3</td>
</tr>
</tbody>
</table>
Thus we have found for every $n \in \mathbb{N}$ an eigenfunction of $H$ of the form

$$\psi(x) := P(x)e^{-tx^2/2},$$

where $P(x) = x^n + O(x^{n-1})$. The linear subspace

$$\{e^{-tx^2/2}P(x) \mid P \in \mathbb{C}[x]\} \subset L^2(\mathbb{R})$$

is dense. Hence we have found a complete set eigenfunctions. We can therefore conclude that the spectrum of $H$ is the discrete set

$$\{2tp + t \mid p \in \mathbb{N}\} \subset \mathbb{R}$$

and of multiplicity one.

In view of (20) the eigenvalues of the Witten Laplacian

$$\Delta + ta(J) + t^2\|x\|^2$$

acting as essentially selfadjoint operator on $L^2(\mathbb{R}^n)$ with domain $\Omega_c(\mathbb{R}^n)$ are given by

$$\left\{2tp_1 + \cdots + 2tp_n + tn + ta(J) \mid (p_i) \in \mathbb{N}^n\right\}.$$

This set contains the value 0 if an only if $J = (1, \ldots, i)$. In this case it is represented by the parameter $(p_i) = (0, \ldots, 0) \in \mathbb{N}^n$ and has with multiplicity one. The corresponding normalized eigenvector is given by

$$\sqrt{\frac{t}{2\pi}} e^{-t\|x\|^2/2}dx_1 \wedge \cdots \wedge dx_i.$$  \hspace{1cm} (22)

If $t \to \infty$, then all non-zero eigenvalues tend linearly to $\infty$. Note that the second largest eigenvalue is $2t$.

### 4.3 The spectrum of the Witten Laplacian

We now consider a closed manifold $M$ with a self-indexing Morse function $f$. We can choose the Riemannian metric $g^{TM}$ such that the Morse-Smale condition is satisfied, and such that the pair $(f, g^{TM})$ near the critical points of $f$ is is isomorphic to the model case studied in Subsection 4.2. Let us call the domain of such an isomorphism a **standard neighbourhood**. Note that we can trivialize the bundle $V$ in the standard neighbourhoods using the parallel transport associated to the flat connection.

We observe that for large $t$ the eigenform (22) of the model operator concentrates at 0. It can be transplanted to a standard neighbourhoods of the critical points. After multiplication by a cut-off function which is constant near the critical point it can be extended to the whole manifold. Denote the result by $\phi_t$. Then

$$\|\phi_t\|_{L^2} = 1 + O(e^{-ct}), \quad \Delta_f \phi_t = O(e^{-ct})$$
exponentially. Moreover, these forms are mutually orthogonal for different critical points. We define a map

$$T_i : C^i(V) \to \Omega^i(M)$$

which for $x \in \text{Crit}_f(i)$ sends $v \in V_x$ to the form $T_i^i(v) = \phi_t \otimes v$. We further define $T_t := \sum T_i : C(V) \to \Omega(M, V)$.

We abbreviate $H := L^2(\Omega(M, V))$ and let

$$H_t[a, b] = P_t[a, b]H \subseteq H$$

be the spectral subspace of $\Delta_{tf}$ for the interval $[a, b] \subseteq \mathbb{R}$.

**Theorem 4.4.**

1. For every $u, v \in C(V)$ we have

$$\langle T_t(u), T_t(v) \rangle = \langle u, v \rangle + O(e^{-tc})$$

2. There exists positive constants $C, c \in \mathbb{R}$ such that for sufficiently large $t \in \mathbb{R}$ we have

$$\text{spec}(\Delta_{tf}) \cap [e^{-ct}, Ct] = \emptyset.$$  

3. We have

$$(1 - P_t[0, 1])T_t = O(e^{-ct}),$$

and

4. 

$$P_t[0, 1]T_t = P_t[0, 1] + O(e^{-ct}).$$

**Proof.** We have already explained the first and the third assertion. For the second we argue by contradiction. We assume that $\psi$ is a normalized eigenvector of $\Delta_{tf}$ for an eigenvalue $\lambda \in [1, C]$. We write

$$\lambda = \langle \Delta_{tf} \psi, \psi \rangle = \|d\psi\|^2 + \|d^*\psi\| + t\langle A(f)\psi, \psi \rangle + t^2 \|\|df\|\psi\|^2.\quad (23)$$

We conclude that most of the $L^2$-norm of $\psi$ must be concentrated near $\text{Crit}_f$. But here $\Delta_{tf}$ is equal to the model operator. One then decomposes $\psi = \psi_0 + \psi_1$, where $\psi_0 \in \text{Im}(T_t)$ and $\psi_1 \perp \text{im}(T_t)$. One can transplant $\psi_1$ to the model case making an error of order $O(t^{-1})$. The resulting function is still orthogonal to the kernel of the model operator up to an error of the same order. Using the known spectrum of the model operator one gets an estimate

$$\langle \Delta_{tf} \psi_1, \psi_1 \rangle \geq Ct.$$  

Furthermore

$$\langle \Delta_{tf} \psi_0, \psi_0 \rangle = O(e^{-ct}).$$

Finally, for the mixed terms we get

$$\langle \Delta_{tf} \psi_0, \psi_1 \rangle = O(e^{-ct}).$$
These three inequalities together contradict \([23]\).

Note that by Remark \([4.1]\) we have an isomorphism \(\ker(\Delta_{tf}) \cong H(M, \mathcal{V})\). We get the Morse inequalities

**Corollary 4.5.** For all \(i \in \mathbb{N}\) we have

\[
\dim H^i(M, \mathcal{V}) \leq \# \text{Crit}_f(i).
\]

The idea to show the Morse inequalities in this way is due to Witten \([\text{Wit82}]\).

### 4.4 The small eigenvalue complex

We now study the analytic torsion. We have a decomposition of the Hilbert space

\[
H = H_t[0, 1] \oplus H_t[C, \infty] = H_t^{\text{small}} \oplus H_t^{\text{large}}
\]

which is preserved by \(\Delta_{tf}\). As remarked before we define the analytic torsion as in Definition \((2.8)\) but using \(\Delta_{tf}\). We can decompose the analytic torsion correspondingly as

\[
T_{\text{an}}(M, \nabla^V, h^TM, h^V) = T_{\text{an}}^{\text{small}}(t)T_{\text{an}}^{\text{large}}(t)
\]

This induces a decomposition

\[
T_{\text{RS}}(M, \nabla^V, h^{H(M,V)}) := T_{\text{RS}}^{\text{small}}(t)T_{\text{an}}^{\text{large}}(t),
\]

where

\[
T_{\text{RS}}^{\text{small}}(t) := v(h^{H(M,V), h^{H(M,V)}(t)})T_{\text{an}}^{\text{small}}(t).
\]

**Theorem 4.6.** We have \(\lim_{t \to \infty} T_{\text{an}}^{\text{large}}(t) = 1\).

The differential \(d_{tf}\) preserves the small eigenvalue subspace \(H_t^{\text{small}}\). We can thus consider the finite-dimensional complex \((H_t^{\text{small}}, d_{tf})\). Its analytic torsion is \(T_{\text{an}}^{\text{small}}(t)\). The main point is now that this complex approximates \((C(V), d)\) for large \(t\). This is much more complicated than the previous results. Here we use the restriction of the integration map

\[
I_f : H_t^{\text{small}} \to C(V).
\]

At this point we must understand the eigenvectors for large \(t\) not only near the critical points, but also along the unstable manifolds \(W^u(x)\). At this point one uses insights from the quantum mechanical study of tunnel effects. The details are too complicated to be explained here. The original idea is due to Helffer-Sjstrand \([\text{HSS85}]\). The final result is

**Theorem 4.7.**

\[
\lim_{t \to \infty} v(h^{H(M,V), h^{H(M,V)}(t)})T_{\text{an}}^{\text{small}}(t) = T_{\text{an}}(C(V), h^V, h^{H(M,V)}).
\]
References


