31 Smooth aspects of algebraic K-theory

The goal of this talk is to explain some constructions with the algebraic K-theory of the ring of smooth functions on manifolds.

31.1 Algebraic K-theory of rings

We describe algebraic K-theory as a functor

$$\mathbf{K} : \mathbf{Rings} \to \mathbf{Spectra} .$$

It is the composition of various intermediate functors which we describe in the following:

$$\mathbf{Proj} : \mathbf{Rings} \to \mathbf{CMon}(\mathbf{Cat}[W^{-1}])$$

It maps a ring R to the symmetric monoidal category of projective modules over R with respect to $\oplus$. Weak equivalences in the category of categories $\mathbf{Cat}$ are categorical equivalences. We must invert weak equivalences since the functoriality of $\mathbf{Proj}(-)$ is only up to natural isomorphism. Furthermore, the monoidal operations on $\mathbf{Proj}(R)$ are not strictly associative or commutative.

$$\mathbf{Nerve} : \mathbf{CMon}(\mathbf{Cat}[W^{-1}]) \to \mathbf{CMon}(\mathbf{sSet}[W^{-1}])$$

It maps a category to its nerve, a simplicial set, and a symmetric monoidal category to a commutative monoid in simplicial sets. Weak equivalences in **sSet** are maps which induce isomorphisms in homotopy groups. The **Nerve**-functor descends to the localizations since it maps categorical equivalences to homotopy equivalences. It induces a map between commmutative monoids since it preserves products.

$$\mathbf{GrpCompl} : \mathbf{CMon}(\mathbf{sSet}[W^{-1}]) \leftrightarrows \mathbf{CGrp}(\mathbf{sSet}[W^{-1}]) : \mathrm{incl} .$$

The group completion functor is the left adjoint of the inclusion of the full subcategory of groupe-like commutative monoids, i.e. those $X \in \mathbf{CMon}(\mathbf{sSet}[W^{-1}])$ with $\pi_0(X)$ a group.

$$\mathbf{sp} : \mathbf{CGrp}(\mathbf{sSet}[W^{-1}]) \simeq \mathbf{connective\ Spectra} \stackrel{\mathrm{incl}}{\to} \mathbf{Spectra}$$

maps a commutative group $X \in \mathbf{CGrp}(\mathbf{sSet}[W^{-1}])$ to the corresponding connective spectrum $\mathbf{sp}(X)$ with infinite loop space $\Omega^\infty(\mathbf{sp}(X)) \simeq X$.

Definition 31.1. *We define the algebraic K-theory functor*

$$\mathbf{K} := \mathbf{sp} \circ \mathbf{GrpCompl} \circ \mathbf{Nerve} \circ \mathbf{Proj} : \mathbf{Rings} \to \mathbf{Spectra}$$

and the algebraic K-theory groups of a ring R by

$$K_n(R) := \pi_n(\mathbf{K}(R)) \ , \quad n \in \mathbb{N} \ .$$

If R is commutative, then the tensor product of modules turns **Proj** and all the other monoids or groups appearing above into rig or ring objects. In particular we get $\mathbf{K}(R) \in \mathbf{CAlg}(\mathbf{Spectra})$ and $K_*(R)$ is a graded commutative ring.

31.2 Smooth functions - first examples

We consider a smooth manifold M and its ring of smooth functions. Then we can study the algebraic K-theory froups

$$K_n(C^\infty(M)) \ .$$

Example 31.2. If M is a point, then $C^\infty(M) \cong \mathbb{C}$ and $K_n(\mathbb{C})$ is the algebraic K-theory of $\mathbb{C}$. It has been calculated by Suslin.

$$K_0(\mathbb{C}) \cong \mathbb{Z} \ , \quad K_1(\mathbb{C}) \cong \mathbb{C}^* \ ,$$

$$K_n(\mathbb{C}) \cong \left\{ \begin{array}{cc} \mathbb{Q}/\mathbb{Z} & n\ odd \\ 0 & n\ even \end{array} \right\} \oplus \mathbb{Q}\text{-vector space} \ , \quad n \geq 2$$

□

Example 31.3. Let $V \to M$ be a vector bundle. By Swan's theorem $\Gamma(M, V)$ is a finitely generated projective module over $C^\infty(M)$, i.e. an object of $\mathbf{Proj}(C^\infty(M))$. It represents a class

$$[V] \in K_0(C^\infty(M)) \ .$$

□

Example 31.4. A map $g : M \to GL(n, \mathbb{C})$ represents an automorphism of $C^\infty(M) \otimes \mathbb{C}^n$ and therefore simplicial loop in $\mathbf{Nerve}(\mathbf{Proj}(C^\infty(M)))$. We get a class

$$[g] \in K_1(C^\infty(M)) \ .$$

□

Example 31.5. Since $C^\infty(M)$ is commutative we can use the ring structure in $K_n(C^\infty(M))$ in order to produce higher degree classes, e.g.

$$[g_1] \cup [g_2] \in K_2(C^\infty(M)) \ .$$

There are complicated relations. An example is the Steinberg relation. For $g \in C^\infty(M)^\times$ such that $1-g \in C^\infty(M)^\times$ we have

$$[g] \cup [1-g] = 0 \ .$$

$\square$

31.3 K-theory of locally convex algebras

In general, for a locally convex algebra A one defines the two-period K-groups

$$K^{lcv}_{2n}(A) := \pi_0(\text{space of projections in } A\hat{\otimes}_\pi K)$$

and

$$K^{lcv}_{2n+1}(A) := \pi_0(\text{space of invertible operators in } A\hat{\otimes}_\pi K) \ .$$

There are natural comparison maps

$$K_n(A) \to K^{lcv}_n(A)$$

for $n \geq 0$.

Example 31.6. Karoubi's conjecture claims that they are isomorphism provided A is a C^*-algebra which is stable, i.e. satisfies $A \cong A\hat{\otimes}_\pi K$. This is now a theorem and admits various generalizations. $\square$

The functors

$$A \mapsto K^{lcv}_n(A)$$

are obviously stable and homotopy invariant in the sense that the natural maps

$$K^{lcv}_*(A) \mapsto K^{lcv}_*(A\hat{\otimes}_\pi K) \ , \quad K^{lcv}_*(A) \to K^{lcv}_*(C^\infty(I)\hat{\otimes}_\pi A)$$

are isomorphism.

Example 31.7. Note that the ring $C^\infty(M)$ is a Fréchet algebra so that we can apply this theory. In this case we know that

$$K_n(C^\infty(M)\hat{\otimes}_\pi K) \stackrel{\cong}{\to} K^{lcv}_n(C^\infty(M)) \stackrel{\cong}{\to} K^{lcv}_n(C(M)) \stackrel{\cong}{\to} \mathbf{KU}^{-n}(M) \ .$$

The first isomorphism is a generalization of Karoubis's conjecture, the second by spectral invariance, and the last is Swan's theorem. $\square$

31.4 Diffeological algebras

Let **Mf** be the category of manifolds with the open covering topology.

Definition 31.8. *A diffeological structure on an algebra A is a concrete sheaf of algebras A^∞ on* **Mf** *with underlying algebra A.*

Example 31.9. If A is a locally convex algebra, then we get a diffeological structure by $A^\infty(M) := C^\infty(M, A)$. We consider this as the standard diffeological structure on a locally convex algebra. □

Example 31.10. If R is a ring, then we can consider the sheaf R^∞ of locally constant functions to $\mathbb{R}$. This is also a diffeological structure. □

Example 31.11. For a manifold X we define the functor

$$i_X : \mathbf{Mf} \to \mathbf{Mf} \ , \quad i_X(M) := M \times X \ .$$

If (A, A^∞) is a diffeological algebra, then get a new diffeological algebra by $(A^\infty(X), i_X^* A^\infty)$. □

Example 31.12. We consider the locally convex algebra $\mathbb{C}$. The associated diffeological structure is given by $\mathbb{C}^\infty(M) = C^\infty(M)$. The construction 31.11 produces a diffeological structure in $C^\infty(X)$. □

A diffeological algebra in particular provides a functor

$$A^\infty : \mathbf{Mf}^{op} \to \mathbf{Rings}$$

which can be composed with the K-theory functor. We therefore get a presheaf of spectra

$$\check{\mathbf{K}}_A := \mathbf{K} \circ A^\infty \in \mathbf{PSh}_{\mathbf{Spectra}}(\mathbf{Mf}) \ .$$

31.5 Operations with presheaves of spectra

The functor L in the adjunction

$$L : \mathbf{PSh}_{\mathbf{Spectra}}(\mathbf{Mf}) \leftrightarrows \mathbf{Sh}_{\mathbf{Spectra}}(\mathbf{Mf}) : \mathrm{incl}$$

is called the sheafification functor. We define

$$\mathbf{K}_A := L(\check{\mathbf{K}}_A) \ .$$

One would like to know what the functors

$$M \mapsto \pi_n(\mathbf{K}_A(M))$$

are. They are like a cohomology theory, i.e. have a Mayer-Vietoris sequence for a open covering $M = U \cup V$,

$$\pi_{n+1}(\mathbf{K}_A(U \cap V)) \to \pi_n(\mathbf{K}_A(M)) \to \pi_n(\mathbf{K}_A(V)) \oplus \pi_n(\mathbf{K}_A(U)) \to \pi_n(\mathbf{K}_A(U \cap V)) \ ,$$

but they are not homotopy invariant. This is a typical example of a differential cohomology theory.

Problem 31.13. *Understand $\mathbf{K}_{\mathbb{C}}$ as a differential cohomology theory.* □

From every sheaf of spectra one can extract a homotopy invariant piece by applying the functor Γ in the adjunction

$$\Gamma : \mathbf{Sh}_{\mathbf{Spectra}}(\mathbf{Mf}) \leftrightarrows \mathbf{Spectra} : L \circ \mathrm{const} \ .$$

It is called realization. For a diffeological algebra (A, A^∞) we define

$$\mathbf{K}^{top}(A) := \Gamma(\mathbf{K}_A) \ , \quad K_n^{top}(A) := \pi_n(\mathbf{K}^{top}(A)) \ .$$

For a complete locally convex algebras the functor

$$A \mapsto K_n^{top}(A)$$

is homotopy invariant. It coicides with other definitions of topological K-theory in the literature.

Problem 31.14. *How does $K_*^{top}(A)$ compare to $K_n^{lcv}(A)$ for complete locally convex algebras. In particular, under which conditions $K_*^{top}(A)$ is stable?*

We know that $K_n^{top}(\mathbb{C}) \cong K_n^{lcv}(\mathbb{C})$ for all $n \geq 0$.

We do not know what happens for $A = C^\infty(X)$. □

Problem 31.15. *If M is compact and of dimension d. Is $K_n(C^\infty(M)) \to K_n^{top}(C^\infty(M))$ trivial for $n > d$? I do not know any example of a topologically non-trivial class in $K_n(C^\infty(M))$.*

For example, if $M = *$, then $K_n(\mathbb{C}) \to K_n^{top}(\mathbb{C})$ vanishes for $n \geq 1$. □

31.6 Maps to cohomology

All algebras are now over $\mathbb{Q}$. The Godwillie-Jones Chern character is a transformation

$$\mathbf{ch}^{GJ} : \mathbf{K}(A) \to \mathbf{CC}^{-}(A) ,$$

where the target is a spectrum which represents the negative cyclic cohomology of A. The negative (algebraic) cyclic cohomology of $C^\infty(M)$ is not understood. But it can further be mapped to differential forms.

We define

$$\mathbf{DD}^{-} := \prod_{p\in\mathbb{Z}} \mathbf{DD}^{-}(p) , \quad \mathbf{DD}^{-}(p) := H(\sigma^{\ge p}\Omega[2p]) \in \mathbf{Sh}_{\mathbf{Spectra}}(\mathbf{Mf}) ,$$

where

$$H : \mathbf{Chain\,complexes}[W^{-1}] \to \mathbf{Spectra}$$

is the Eilenberg-MacLane functor. There is natural a map of spectra

$$\sigma : \mathbf{CC}^{-}(C^\infty(X)) \to \mathbf{DD}^{-}(X) .$$

We define

$$\mathbf{ch}^{alg} := \sigma \circ \mathbf{ch}^{GJ} : \mathbf{K}(C^\infty(M)) \to \mathbf{DD}^{-}(M) .$$

Example 31.16. We have

$$\pi_0(\mathbf{DD}^{-}(p)(M)) \cong \begin{cases} 0 & p<0 \\ \Omega^0_{cl}(M) & p=0 \\ H^{2p}_{dR}(M) & p\ge 0 \end{cases} .$$

For the classes of a bundle $V \to M$ we get:

$$\mathbf{ch}^{alg}([V])(0) = \dim(V) , \quad \mathbf{ch}^{alg}([V])(p) = \mathbf{ch}_{2p}(V) , p \ge 1$$

We have

$$\pi_1(\mathbf{DD}^{-}(p)(M)) \cong \begin{cases} 0 & p\le 0 \\ \Omega^1_{cl}(M) & p=1 \\ H^{2p-1}_{dR}(M) & p\ge 2 \end{cases} .$$

For $g : M \to GL(n,\mathbb{C})$ we get

$$\mathbf{ch}^{alg}([g])(1) = \frac{1}{2\pi i} d\log\det(g) , \quad \mathbf{ch}^{alg}([g])(p) = \mathbf{ch}_{2p-1}(g) , p\ge 2 .$$

We see that $[g]$ is not homotopy invariant and non-trivial if $\det g$ is not constant. In particular, the functor $M \to \mathbf{K}(C^\infty(M))$ is not homotopy invariant.

We can use this to construct classes in $K_n(C^\infty(T^n))$ by taking n-fold products of classes in $K_1(C^\infty(S^1))$ and detect their non-triviality.

One can transport such classes non-trivially to arbitrary manifolds of dimension $\ge n$ through they might become topologically trivial.

$\square$

31.7 Fredholm modules

We consider a closed d-dimensional manifold M with a Dirac operator $\not{D}$ acting on sections of $E \to M$. It gives rise to a $d+1$-summable Fredholm module

$$(L^2(M,E), D(D^2+1)^{\frac{1}{2}})$$

over $C^\infty(M)$. There is a classifying algebra $\mathcal{M}_d$ for $d+1$-summable Fredholm modules and a universal such module (H,F) over $\mathcal{M}_d$. There is a unique (up to unitary equivalence) isomorphism homomorphism $b_{\not{D}} : C^\infty(M) \to \mathcal{M}_d$ such that there is an unitary isomorphism $u : L^2(M,E) \xrightarrow{\cong} H$ with

$$u \circ f = b_{\not{D}}(f) \circ u \ , \quad F \circ u = u \circ D(D^2+1)^{\frac{1}{2}} \ .$$

We get an induced map

$$b_{\not{D},*} : K_*(C^\infty) \to K_*(\mathcal{M}_d) \ .$$

We now assume that d is odd. In this case we have the Connes-Karoubi multiplicative character

$$\delta^{CK} : K_{d+1}(\mathcal{M}_d) \to \mathbb{C}^* \ .$$

We can use it to define the algebraic index map

$$\mathtt{index}^{alg}_{\not{D}} := \delta^{CK} \circ b_{\not{D},*} : K_{d+1}(C^\infty)(M) \to \mathbb{C}^* \ .$$

Problem 31.17. *Understand this map* $\mathtt{index}^{alg}_{\not{D}}$.
We have a good formula for $\mathtt{index}^{alg}_{\not{D}}(x)$ if x is topologically trivial. □

Example 31.18. We consider $M = S^1$ and let $f \in C^\infty(S^1)$ and $u \in C^\infty(S^1)^\times$. Then we can form the class $[e^f] \cup [u] \in K_2(C^\infty(S^1))$. We have (Kaad, Carey-Pincus)

$$\mathtt{index}^{alg}_{\not{D}}([e^f] \cup [u]) = \exp(\int_{S^1} f \, d \log u) \ .$$

Note that $[e^f]$ and hence $[e^f] \cup [u]$ are topologically trivial.
Let us check the Steinberg relation by explicit calculation. We set $u = 1 - e^f$ and assume that f is real and has no zeros.

$$\begin{aligned} \int_{S^1} f d \log u &= -\int_{S^1} \frac{f e^f df}{1-e^f} \\ &= -\frac{1}{2} \int_{S^1} \frac{e^{\sqrt{g}}}{1-e^{\sqrt{g}}} dg \\ &= 0 \end{aligned}$$

where $g = \sqrt{f}$ is smooth since f has no zeros. □

For $u_1, u_2 \in C^\infty(S^1)^\times$ we have, using the cup product $\cup_{Del}$ in Deligne cohomology,

$$\mathtt{index}^{alg}_{\not{D}}([u_1] \cup [u_2]) = \exp(2\pi i \langle u_1 \cup_{Del} u_2, [S^1]\rangle) \ .$$

Note that $[u_1] \cup [u_2]$ may be topologically non-trivial (do not know).
This is the edge of knowledge about $\mathtt{index}^{alg}_{\not{D}}$ for (potentially) topologically non-trivial classes.