31 Smooth aspects of algebraic $K$-theory

The goal of this talk is to explain some constructions with the algebraic $K$-theory of the ring of smooth functions on manifolds.

31.1 Algebraic $K$-theory of rings

We describe algebraic $K$-theory as a functor

$$K : \text{Rings} \to \text{Spectra}.$$ 

It is the composition of various intermediate functors which we describe in the following:

$$\text{Proj} : \text{Rings} \to \text{CMon}(\text{Cat}[W^{-1}]),$$

It maps a ring $R$ to the symmetric monoidal category of projective modules over $R$ with respect to $\oplus$. Weak equivalences in the category of categories $\text{Cat}$ are categorical equivalences. We must invert weak equivalences since the functoriality of $\text{Proj}(-)$ is only up to natural isomorphism. Furthermore, the monoidal operations on $\text{Proj}(R)$ are not strictly associative or commutative.

$$\text{Nerve} : \text{CMon}(\text{Cat}[W^{-1}]) \to \text{CMon}(\text{sSet}[W^{-1}]),$$

It maps a category to its nerve, a simplicial set, and a symmetric monoidal category to a commutative monoid in simplicial sets. Weak equivalences in $\text{sSet}$ are maps which induce isomorphisms in homotopy groups. The $\text{Nerve}$-functor descends to the localizations since it maps categorical equivalences to homotopy equivalences. It induces a map between commutative monoids since it preserves products.

$$\text{GrpCompl} : \text{CMon}(\text{sSet}[W^{-1}]) \cong \text{CGrp}(\text{sSet}[W^{-1}]) : \text{incl},$$

The group completion functor is the left adjoint of the inclusion of the full subcategory of groupe-like commutative monoids, i.e. those $X \in \text{CMon}(\text{sSet}[W^{-1}])$ with $\pi_0(X)$ a group.

$$\text{sp} : \text{CGrp}(\text{sSet}[W^{-1}]) \cong \text{connective Spectra} \hookrightarrow \text{Spectra}$$

maps a commutative group $X \in \text{CGrp}(\text{sSet}[W^{-1}])$ to the corresponding connective spectrum $\text{sp}(X)$ with infinite loop space $\Omega^\infty(\text{sp}(X)) \cong X$. 55
Definition 31.1. We define the algebraic $K$-theory functor

$$K := sp \circ GrpCompl \circ Nerve \circ Proj : \text{Rings} \to \text{Spectra}$$

and the algebraic $K$-theory groups of a ring $R$ by

$$K_n(R) := \pi_n(K(R)) , \quad n \in \mathbb{N}.$$ 

If $R$ is commutative, then the tensor product of modules turns $\text{Proj}$ and all the other monoids or groups appearing above into rig or ring objects. In particular we get $K(R) \in \text{CAlg}(\text{Spectra})$ and $K_*(R)$ is a graded commutative ring.

31.2 Smooth functions - first examples

We consider a smooth manifold $M$ and its ring of smooth functions. Then we can study the algebraic $K$-theory groups

$$K_n(C^\infty(M)).$$

Example 31.2. If $M$ is a point, then $C^\infty(M) \cong \mathbb{C}$ and $K_n(\mathbb{C})$ is the algebraic $K$-theory of $\mathbb{C}$. It has been calculated by Suslin.

$$K_0(\mathbb{C}) \cong \mathbb{Z} , \quad K_1(\mathbb{C}) \cong \mathbb{C}^* , \quad K_n(\mathbb{C}) \cong \left\{ \begin{array}{ll} \mathbb{Q}/\mathbb{Z} & n \text{ odd} \\ 0 & n \text{ even} \end{array} \right\} \oplus \mathbb{Q}\text{-vector space} , \quad n \geq 2 \tag*{\lozenge}$$

Example 31.3. Let $V \to M$ be a vector bundle. By Swan’s theorem $\Gamma(M, V)$ is a finitely generated projective module over $C^\infty(M)$, i.e. an object of $\text{Proj}(C^\infty(M))$. It represents a class

$$[V] \in K_0(C^\infty(M)).$$

\[ \square \]

Example 31.4. A map $g : M \to GL(n, \mathbb{C})$ represents an automorphism of $C^\infty(M) \otimes \mathbb{C}^n$ and therefore simplicial loop in $\text{Nerve}(\text{Proj}(C^\infty(M)))$. We get a class

$$[g] \in K_1(C^\infty(M)).$$

\[ \square \]

Example 31.5. Since $C^\infty(M)$ is commutative we can use the ring structure in $K_n(C^\infty(M))$ in order to produce higher degree classes, e.g.

$$[g_1] \cup [g_2] \in K_2(C^\infty(M)).$$

56
There are complicated relations. An example is the Steinberg relation. For \( g \in C^\infty(M)^x \) such that \( 1 - g \in C^\infty(M)^x \) we have

\[
[g] \cup [1 - g] = 0 .
\]

\( \square \)

### 31.3 \( K \)-theory of locally convex algebras

In general, for a locally convex algebra \( A \) one defines the two-period \( K \)-groups

\[
K_{2n}^{\text{lev}}(A) := \pi_0(\text{space of projections in } A\hat{\otimes}_\pi K)
\]

and

\[
K_{2n+1}^{\text{lev}}(A) := \pi_0(\text{space of invertible operators in } A\hat{\otimes}_\pi K) .
\]

There are natural comparison maps

\[
K_n(A) \to K_{n}^{\text{lev}}(A)
\]

for \( n \geq 0 \).

**Example 31.6.** Karoubi’s conjecture claims that they are isomorphism provided \( A \) is a \( C^* \)-algebra which is stable, i.e. satisfies \( A \cong A\hat{\otimes}_\pi K \). This is now a theorem and admits various generalizations.

\( \square \)

The functors

\[
A \mapsto K_{n}^{\text{lev}}(A)
\]

are obviously stable and homotopy invariant in the sense that the natural maps

\[
K_{n}^{\text{lev}}(A) \to K_{n}^{\text{lev}}(A\hat{\otimes}_\pi K) , \quad K_{n}^{\text{lev}}(A) \to K_{n}^{\text{lev}}(C^\infty(1)\hat{\otimes}_\pi A)
\]

are isomorphism.

**Example 31.7.** Note that the ring \( C^\infty(M) \) is a Fréchet algebra so that we can apply this theory. In this case we know that

\[
K_n(C^\infty(M)\hat{\otimes}_\pi K) \cong K_{n}^{\text{lev}}(C^\infty(M)) \cong K_{n}^{\text{lev}}(C(M)) \cong \textbf{KU}^{-n}(M) .
\]

The first isomorphism is a generalization of Karoubis’s conjecture, the second by spectral invariance, and the last is Swan’s theorem.

\( \square \)
31.4 Diffeological algebras

Let $\mathbf{Mf}$ be the category of manifolds with the open covering topology.

**Definition 31.8.** A diffeological structure on an algebra $A$ is a concrete sheaf of algebras $A^\infty$ on $\mathbf{Mf}$ with underlying algebra $A$.

**Example 31.9.** If $A$ is a locally convex algebra, then we get a diffeological structure by $A^\infty(M) := C^\infty(M, A)$. We consider this as the standard diffeological structure on a locally convex algebra.

**Example 31.10.** If $R$ is a ring, then we can consider the sheaf $R^\infty$ of locally constant functions to $R$. This is also a diffeological structure.

**Example 31.11.** For a manifold $X$ we define the functor

$$i_X : \mathbf{Mf} \to \mathbf{Mf}, \quad i_X(M) := M \times X.$$ 

If $(A, A^\infty)$ is a diffeological algebra, then get a new diffeological algebra by $(A^\infty(X), i_X^*A^\infty)$.

**Example 31.12.** We consider the locally convex algebra $\mathbb{C}$. The associated diffeological structure is given by $\mathbb{C}^\infty(M) = C^\infty(M)$. The construction 31.11 produces a diffeological structure in $C^\infty(X)$.

A diffeological algebra in particular provides a functor

$$A^\infty : \mathbf{Mf}^{\text{op}} \to \mathbf{Rings}$$

which can be composed with the $K$-theory functor. We therefore get a presheaf of spectra

$$\mathcal{K}_A := K \circ A^\infty \in \mathbf{PShSpectra}(\mathbf{Mf}).$$

31.5 Operations with presheaves of spectra

The functor $L$ in the adjunction

$$L : \mathbf{PShSpectra}(\mathbf{Mf}) \rightleftarrows \mathbf{ShSpectra}(\mathbf{Mf}) : \text{incl}$$

is called the sheafification functor. We define

$$K_A := L(\mathcal{K}_A).$$

One would like to know what the functors

$$M \mapsto \pi_n(K_A(M))$$
are. They are like a cohomology theory, i.e. have a Mayer-Vietoris sequence for an open covering \( M = U \cup V \),
\[
\pi_{n+1}(K_A(U \cap V)) \to \pi_n(K_A(M)) \to \pi_n(K_A(V)) \oplus \pi_n(K_A(U)) \to \pi_n(K_A(U \cap V)),
\]
but they are not homotopy invariant. This is a typical example of a differential cohomology theory.

**Problem 31.13.** Understand \( K_C \) as a differential cohomology theory.

From every sheaf of spectra one can extract a homotopy invariant piece by applying the functor \( \Gamma \) in the adjunction

\[
\Gamma : \text{ShSpectra}(Mf) \rightleftarrows \text{Spectra} : L \circ \text{const}.
\]

It is called realization. For a diffeological algebra \((A, A^\infty)\) we define

\[
K^{\text{top}}(A) := \Gamma(K_A), \quad K_n^{\text{top}}(A) := \pi_n(K^{\text{top}}(A)).
\]

For a complete locally convex algebras the functor

\[
A \mapsto K_n^{\text{top}}(A)
\]

is homotopy invariant. It coincides with other definitions of topological \( K \)-theory in the literature.

**Problem 31.14.** How does \( K_n^{\text{top}}(A) \) compare to \( K_n^{\text{lc}}(A) \) for complete locally convex algebras. In particular, under which conditions \( K_n^{\text{top}}(A) \) is stable?

We know that \( K_n^{\text{top}}(C) \cong K_n^{\text{lc}}(C) \) for all \( n \geq 0 \).

*We do not know what happens for \( A = C^\infty(X) \).*

**Problem 31.15.** If \( M \) is compact and of dimension \( d \). Is \( K_n(C^\infty(M)) \to K_n^{\text{top}}(C^\infty(M)) \) trivial for \( n > d \)? I do not know any example of a topologically non-trivial class in \( K_n(C^\infty(M)) \).

For example, if \( M = \star \), then \( K_n(C) \to K_n^{\text{top}}(C) \) vanishes for \( n \geq 1 \).
31.6 Maps to cohomology

All algebras are now over $\mathbb{Q}$. The Godwillie-Jones Chern character is a transformation

$$\text{ch}^{GJ} : K(A) \to \text{CC}^-(A)$$

where the target is a spectrum which represents the negative cyclic cohomology of $A$. The negative (algebraic) cyclic cohomology of $C^\infty(M)$ is not understood. But it can further be mapped to differential forms.

We define

$$\text{DD}^- := \prod_{p \in \mathbb{Z}} \text{DD}^-(p) \ , \ \text{DD}^-(p) := H(\sigma^p \Omega[2p]) \in \text{Sh}_{\text{Spectra}}(\text{Mf})$$

where

$$H : \text{Chain complexes}[W^{-1}] \to \text{Spectra}$$

is the Eilenberg-MacLane functor. There is natural a map of spectra

$$\sigma : \text{CC}^-(C^\infty(X)) \to \text{DD}^-(X)$$

We define

$$\text{ch}^{alg} := \sigma \circ \text{ch}^{GJ} : K(C^\infty(M)) \to \text{DD}^-(M)$$

Example 31.16. We have

$$\pi_0(\text{DD}^-(p)(M)) \cong \begin{cases} 0 & p < 0 \\ \Omega^p_d(M) & p = 0 \\ H^p_dR(M) & p \geq 0 \end{cases}$$

For the classes of a bundle $V \to M$ we get:

$$\text{ch}^{alg}([V])(0) = \dim(V) \ , \ \text{ch}^{alg}([V])(p) = \text{ch}_{2p}(V) \ , p \geq 1$$

We have

$$\pi_1(\text{DD}^-(p)(M)) \cong \begin{cases} 0 & p \leq 0 \\ \Omega^p_d(M) & p = 1 \\ H^p_dR(M) & p \geq 2 \end{cases}$$

For $g : M \to GL(n, \mathbb{C})$ we get

$$\text{ch}^{alg}([g])(1) = \frac{1}{2\pi i} d\log \det(g) \ , \ \text{ch}^{alg}([g])(p) = \text{ch}_{2p-1}(g) \ , p \geq 2$$

We see that $[g]$ is not homotopy invariant and non-trivial if $\det g$ is not constant. In particular, the functor $M \to K(C^\infty(M))$ is not homotopy invariant.

We can use this to construct classes in $K_n(C^\infty(T^n))$ by taking $n$-fold products of classes in $K_1(C^\infty(S^1))$ and detect their non-triviality.

One can transport such classes non-trivially to arbitrary manifolds of dimension $\geq n$ through they might become topologically trivial.

$\square$
31.7 Fredholm modules

We consider a closed $d$-dimensional manifold $M$ with a Dirac operator $\mathcal{D}$ acting on sections of $E \to M$. It gives rise to a $d + 1$-summable Fredholm module

$$(L^2(M,E), D(D^2 + 1)^{\frac{1}{2}})$$

over $C^\infty(M)$. There is a classifying algebra $\mathcal{M}_d$ for $d + 1$-summable Fredholm modules and a universal such module $(H, F)$ over $\mathcal{M}_d$. There is a unique (up to unitary equivalence) isomorphism homomorphism $b_{\mathcal{D}} : C^\infty(M) \to \mathcal{M}_d$ such that there is an unitary isomorphism $u : L^2(M,E) \overset{\cong}{\to} H$ with

$$u \circ f = b_{\mathcal{D}}(f) \circ u, \quad F \circ u = u \circ D(D^2 + 1)^{\frac{1}{2}}.$$

We get an induced map

$$b_{\mathcal{D},*} : K_*(C\infty) \to K_*(\mathcal{M}_d).$$

We now assume that $d$ is odd. In this case we have the Connes-Karoubi multiplicative character

$$\delta^{CK} : K_{d+1}(\mathcal{M}_d) \to \mathbb{C}^*.$$

We can use it to define the algebraic index map

$$\text{index}^{alg}_{\mathcal{D}} := \delta^{CK} \circ b_{\mathcal{D},*} : K_{d+1}(C\infty)(M) \to \mathbb{C}^*.$$

**Problem 31.17.** Understand this map $\text{index}^{alg}_{\mathcal{D}}$.

We have a good formula for $\text{index}^{alg}_{\mathcal{D}}(x)$ if $x$ is topologically trivial. \hfill \square

**Example 31.18.** We consider $M = S^1$ and let $f \in C^\infty(S^1)$ and $u \in C^\infty(S^1)^\times$. Then we can form the class $[e^f] \cup [u] \in K_2(C^\infty(S^1))$. We have (Kaad, Carey-Pincus)

$$\text{index}^{alg}_{\mathcal{D}}([e^f] \cup [u]) = \exp(\int_{S^1} f \, d\log u).$$

Note that $[e^f]$ and hence $[e^f] \cup [u]$ are topologically trivial.

Let us check the Steinberg relation by explicit calculation. We set $u = 1 - e^f$ and assume that $f$ is real and has no zeros.

$$\int_{S^1} f \, d\log u = - \int_{S^1} \frac{f e^f df}{1 - e^f} = - \frac{1}{2} \int_{S^1} \frac{e^{\sqrt{f}}}{1 - e^{\sqrt{f}}} dg = 0$$

where $g = \sqrt{f}$ is smooth since $f$ has no zeros. \hfill \square
For $u_1, u_2 \in C^\infty(S^1)^\times$ we have, using the cup product $\cup_{\text{Del}}$ in Deligne cohomology,

$$\text{index}_{B}^{\text{alg}}([u_1] \cup [u_2]) = \exp(2\pi i \langle u_1 \cup_{\text{Del}} u_2, [S^1]\rangle).$$

Note that $[u_1] \cup [u_2]$ may be topologically non-trivial (do not know). This is the edge of knowledge about $\text{index}_{B}^{\text{alg}}$ for (potentially) topologically non-trivial classes.