Étale cohomology

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Contents

1 Basic theory of commutative rings

In the following, all rings will be commutative rings with unit, and will be simply called rings.

Let R be a commutative ring with 1.

Definition 1.1 A subset $\mathfrak{a} \subseteq R$ is called an ideal of R, if the following holds:

(i) $\mathfrak{a} \subseteq R$ is an additive subgroup.

(ii) For each $a \in \mathfrak{a}$ and $r \in R$ one has $r \cdot a \in \mathfrak{a}$.

Definition 1.2 (a) An element $r \in R$ is called a zero divisor, if there is a non-zero $s \in R$ with $r \cdot s = 0$.

(b) R is called an integral domain, if R does not contain any zero divisor.

Definition 1.3 An ideal $\mathfrak{p} \subseteq R$ is called a prime ideal, if it satisfies the following equivalent conditions:

(a) $\mathfrak{p} \neq R$, and if a and b are elements of R such that $a \cdot b \in \mathfrak{p}$, then $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.

(b) $R/\mathfrak{p}$ is an integral domain.

Definition 1.4 $\mathfrak{m} \subseteq R$ is called a maximal ideal, if it satisfies the following equivalent conditions:

(a) $\mathfrak{m} \neq R$, and there is no ideal $\mathfrak{a} \subseteq R$ with $\mathfrak{m} \subsetneqq \mathfrak{a} \subsetneqq R$.

(b) $R/\mathfrak{m}$ is a field.

Corollary 1.5 (a) R is an integral domain if and only if (0) is a prime ideal.

(b) R is a field if and only if (0) is a maximal ideal.

Definition 1.6 R is called a local ring, if the following equivalent conditions hold:

(a) R has exactly one maximal ideal.

(b) The set $R \smallsetminus R^\times$ of non-units is an ideal (Here $R^\times$ denotes the group of units in R).

Definition 1.7 The set $\mathrm{Spec}(R)$ of all prime ideals $\mathfrak{p} \subseteq R$ is called the spectrum of R.

Definition 1.8 For an ideal $\mathfrak{a} \subseteq R$ define

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{a} \subseteq \mathfrak{p}\}$$

Definition 1.9 One has

(a) $V(\{0\}) = \mathrm{Spec}(R), V(R) = \emptyset$.

(b) For any family $(\mathfrak{a}_i)_{i\in I}$ of ideals in R one has $V(\sum\limits_{i\in I} \mathfrak{a}_i) = \bigcap\limits_{i\in I} V(\mathfrak{a}_i)$.

(c) For ideals $\mathfrak{a}, \mathfrak{b}$ one has $V(\mathfrak{a}\cap\mathfrak{b}) = V(\mathfrak{a}\cdot\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.

Proof (a) and (b) are obvious.

(c) Obviously we have

$$\mathfrak{a} \subseteq \mathfrak{b} \Rightarrow V(\mathfrak{b}) \subset V(\mathfrak{a}) .$$

Since $\mathfrak{a}\cdot\mathfrak{b} \subseteq \mathfrak{a}\cap\mathfrak{b} \subseteq \mathfrak{a}, \mathfrak{b}$, we therefore have

$$V(\mathfrak{a}) \cup V(\mathfrak{b}) \subseteq V(\mathfrak{a}\cap\mathfrak{b}) \subseteq V(\mathfrak{a}\cdot\mathfrak{b}) .$$

Suppose that the right hand side is strictly bigger. Then there is a prime ideal $\mathfrak{a}$ which contains $\mathfrak{a}\cdot\mathfrak{b}$, but neither $\mathfrak{a}$ nor $\mathfrak{b}$. Choose $s \in \mathfrak{a} \smallsetminus \mathfrak{p}$ and $t \in \mathfrak{b} \smallsetminus \mathfrak{p}$. Then we have $s \cdot t \in \mathfrak{a}\cdot\mathfrak{b} \subseteq \mathfrak{p}$, in contradiction to the fact that $\mathfrak{p}$ is a prime ideal.

Lemma 1.9 implies that the sets $V(\mathfrak{a})$ ($\mathfrak{a}$ ideal in R) form the closed sets of a topology on $\mathrm{Spec}(R)$.

Definition 1.10 This topology is called the Zariski topology.

Definition 1.11 For $f \in R$ define

$$D(f) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid f \notin \mathfrak{p}\} .$$

Lemma 1.12 The sets $D(f)$ are open and form a basis of the Zariski topology. They are called standard open (or elementary open) sets.

Proof $D(f) = \mathrm{Spec}(R) \smallsetminus V(f)$ is open, and if $\mathfrak{a}$ is an ideal, then

$$\mathrm{Spec}(R) \smallsetminus V(\mathfrak{a}) = \bigcup_{f\in\mathfrak{a}} D(f) .$$

Lemma/Definition 1.13 (a) For an ideal $\mathfrak{a} \subseteq R$ the set

$$\sqrt{\mathfrak{a}} := \{f \in R \mid \exists n \in \mathbb{N} \text{ with } f^n \in \mathfrak{a}\}$$

is an ideal, and is called the radical of $\mathfrak{a}$.

(b) The ideal

$$\mathrm{nil}(R) := \{f \in R \mid \exists n \in \mathbb{N} \text{ with } f^n = 0\} = \sqrt{(0)}$$

is called the nil radical of R, and its elements are the nilpotent elements in R. R is called reduced if $\mathrm{nil}(R) = 0$.

Proof that $\sqrt{\mathfrak{a}}$ is an ideal: Let $a, b \in \sqrt{\mathfrak{a}}$ and $r \in R$; then there are $m, n \in \mathbb{N}$ with $a^m, b^n \in \mathfrak{a}$. This implies $(ra)^m = r^m a^m \in \mathfrak{a}$, i.e., $ra \in \sqrt{\mathfrak{a}}$. Furthermore one has

$$(a+b)^{m+n} = \sum_{k=0}^{m+n} \binom{m+n}{k} a^k b^{m+n-k} \in \mathfrak{a}\,,$$

since $b^{m+n-k} \in \mathfrak{a}$ for $0 \le k \le m$ and $a^k \in \mathfrak{a}$ for $m \le k$. Hence $a+b \in \sqrt{\mathfrak{a}}$.

Remark 1.14 Obviously one has

(a) $\mathfrak{a} \subseteq \sqrt{\mathfrak{a}}$.

(b) $\mathfrak{a} \subseteq \mathfrak{b} \Rightarrow \sqrt{\mathfrak{a}} \subseteq \sqrt{\mathfrak{b}}$.

(c) $\sqrt{\sqrt{\mathfrak{a}}} = \sqrt{\mathfrak{a}}$.

(d) $\mathrm{nil}(R/\mathfrak{a}) = \sqrt{\mathfrak{a}}/\mathfrak{a}$.

(e) $\sqrt{\mathfrak{p}} = \mathfrak{p}$ for a prime ideal.

Lemma/Definition 1.15 For an ideal $\mathfrak{a}$ in a ring R the following are equivalent:

(a) $\mathfrak{a} = \sqrt{\mathfrak{a}}$.

(b) $R/\mathfrak{a}$ is reduced.

If these properties hold, then $\mathfrak{a}$ is called a radical ideal.

The equivalence follows from 6.14 (d).

Lemma 1.16 Let R be a commutative ring with unit. Then $\mathrm{Spec}(R)$ is quasi-compact.

Proof Let $(U_i)_{i\in I}$ be an open covering of $\mathrm{Spec}(R)$. Since the standard opens $D(f)$ from a basis of the toplogy, we can (by refinement) assume that $U_i = D(f_i)$ for some $i \in I$. Since

$$\mathrm{Spec}(R) = \bigcup_{i\in I} D(f_i)\,,$$

we have, by taking complements,

$$\emptyset = \bigcap_{i\in I} V(f_i) = V(\langle f_i \mid i \in I\rangle)\,.$$

This implies $1 \in \sqrt{\langle f_i \mid i \in I\rangle}$, and hence $1 \in \langle f_i \mid i \in I\rangle$. This implies

$$1 = \sum_{\nu=1}^{n} r_\nu f_{i_\nu}$$

with $r_\nu \in R$ and $i_\nu \in I$. Conversely this implies

$$\mathrm{Spec}(R) = \bigcup_{\nu=1}^{n} D(f_{i_\nu})\,.$$

Now we consider morphisms between spectra.

Lemma 1.17 A morphism of rings $\varphi : A \to B$ induces a continuous map

$$\begin{array}{rrcl} \varphi^* : & \operatorname{Spec} B & \to & \operatorname{Spec}(A) \\ & \mathfrak{q} & \mapsto & \varphi^{-1}(\mathfrak{q}) \end{array}$$

Proof of the claims: The induced map

$$A/\varphi^{-1}(\mathfrak{q}) \hookrightarrow B/\mathfrak{q}$$

is injective. Hence $A/\varphi^{-1}(\mathfrak{q})$ is an integral domain, since $B/\mathfrak{q}$ is; hence $\varphi^{-1}(\mathfrak{q})$ is a prime ideal (Corollary 6.5). Furthermore, for $f \in A$ and $\mathfrak{q} \in \operatorname{Spec}(B)$ we have

$$\mathfrak{q} \in D(\varphi(f)) \Leftrightarrow \varphi(f) \notin \mathfrak{q} \Leftrightarrow f \notin \varphi^{-1}(\mathfrak{q}) \Leftrightarrow \varphi^{-1}(\mathfrak{q}) \in D(f)\,.$$

This implies

$$(\varphi^*)^{-1}(D(f)) = D(\varphi(f))\,.$$

In particular, $(\varphi^*)^{-1}(D(f))$ is open. Since the sets $D(f)$ form a basis of the topology of $\operatorname{Spec}(A)$, this implies that φ^* is continuous.

Now we consider this situation in particular cases.

Lemma 1.18 Let $\mathfrak{a} \subseteq R$ be an ideal. The canonical surjection $\varphi : R \to R/\mathfrak{a}$ induces an injective map

$$\varphi^* : \operatorname{Spec}(R/\mathfrak{a}) \hookrightarrow \operatorname{Spec}(R)$$

with image $V(\mathfrak{a})$. The induced bijection

$$\begin{array}{rrcl} \varphi^* : & \operatorname{Spec}(R/\mathfrak{a}) & \to & V(\mathfrak{a}) \\ & \overline{\mathfrak{p}} & \mapsto & \varphi^{-1}(\overline{\mathfrak{p}}) \end{array}$$

is a homeomorphism, if $V(\mathfrak{a}) \subseteq \operatorname{Spec}(R)$ is endowed with the subspace topology.

Proof One has bijections

$$\begin{array}{rcl} \operatorname{Spec}(R/\mathfrak{a}) & \underset{\psi}{\overset{\varphi^*}{\rightleftarrows}} & V(\mathfrak{a}) \\ \overline{\mathfrak{p}} & \mapsto & \varphi^{-1}(\overline{\mathfrak{p}}) \\ \mathfrak{p}/\mathfrak{a} & \leftarrow\!\shortmid & \mathfrak{p} \end{array}$$

which are inverse to each other. For an ideal $\mathfrak{b}' \subseteq R/\mathfrak{a}$ and a prime ideal $\mathfrak{p}' \subseteq R/\mathfrak{a}$ we obviously have $\mathfrak{b}' \subseteq \mathfrak{p}'$ if and only if $\varphi^{-1}(\mathfrak{b}') \subseteq \varphi^{-1}(\mathfrak{p}')$, and therefore

$$\varphi^*(V(\mathfrak{b}')) = V(\varphi^{-1}(\mathfrak{b}'))\,.$$

Hence φ^* is a closed map (the images of closed sets are closed). But a bijective continuous map is a homeomorphism.

Definition 1.19 A set $S \subset R$ in a ring R is called multiplcative, if for $a, b \in S$ also $a \cdot b \in S$.

Examples 1.20 (a) For any $f \in R$ the set $\{f^n \mid n \in \mathbb{N}_0\}$ is multiplicative.

(b) Let $\mathfrak{a} \subseteq R$ be an ideal. The set $R \smallsetminus \mathfrak{a}$ is multiplicative is and only if $\mathfrak{a}$ is a prime ideal.

(c) The set of non-zero divisors is multiplicative.

Let $S \subseteq A$ be multiplicative. Then consider the following relation on the set $A \times S$:

$$(a, s) \sim (a', s') :\Leftrightarrow \text{ there is a } t \in S \text{ with } ts'a = tsa'$$

Then $\sim$ is an equivalence relation: It is clear that the relation is reflexive and symmetric, and for the transitivity one "needs the t" in the definition, if S contains zero-divisors:

$$t(s'a - sa') = 0,\, t'(s''a' - s'a'') = 0 \Rightarrow tt's'(s''a - sa'') = 0\,.$$

For $(a, s) \in A \times S$ let $\frac{a}{s}$ be the equivalence class with respect to $\sim$. The set $A \times S/\sim$ of the equivalence classes is denoted by A_S (or $S^{-1}A$, or $A[S^{-1}]$).

Theorem 1.21 (a) With the operations

$$\begin{array}{rcl} \frac{a}{s} + \frac{b}{t} & := & \frac{at+bs}{st} \\ \frac{a}{s} \cdot \frac{b}{t} & := & \frac{ab}{st}\,, \end{array}$$

A_S becomes a commutative ring with 1, and is called the localization of A with respect to S.

(b) The map

$$\varphi_{\text{univ}} : A \to A_S\,,\ a \mapsto \frac{a}{1}$$

is a ring homomorphism, and all elements in $\varphi_{\text{univ}}(S)$ are invertible in A_S.

(c) (universal property) If $\varphi : A \to B$ is a ring homomorphism such that all elements in $\varphi(S)$ are invertible, then there is a unique ring homomorphism $\tilde{\varphi} : A_S \to B$ which makes the diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi_{\text{univ}}} & A_S \\ & {\scriptstyle \varphi}\searrow \quad \swarrow{\scriptstyle \exists! \tilde{\varphi}} & \\ & B & \end{array}$$

commutative.

2 Presheaves and sheaves

Definition 2.1 Let X be a topological space.

(a) A presheaf P (of abelian groups) on X is an association which to any open set $U \subseteq X$ associates an abelian group $P(U)$, and to any inclusion $V \subseteq U$ of open sets associates a group homomorphism

$$\mathrm{res}_{U,V} : P(U) \to P(V)$$

called the restriction from U to V, such that the following holds:

(i) For $W \subseteq V \subseteq U$ we have $\mathrm{res}_{U,W} = \mathrm{res}_{V,W} \circ \mathrm{res}_{U,V}$.

(ii) $\mathrm{res}_{U,U}$ is the identity.

(b) For $x \in P(U)$ we also call $\mathrm{res}_{U,V}(x)$ the restriction of x to V and denote it by $x|_V$.

A presheaf F on X is called a **sheaf**, if for every open set $U \subseteq X$ and any open covering $(U_i)_{i\in I}$ of U $(\bigcup\limits_{i\in I} U_i = U)$ the following two properties hold:

(1) (locality) If $s, t \in F(U)$ and $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$.

(2) (glueing) If $s_i \in F(U_i)$ are given for all $i \in I$, and we have

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

for all $i, j \in I$, then there is an $s \in F(U)$ with

$$s|_{U_i} = s_i \quad \text{for all } i \in I\,.$$

(c) A morphism of presheaves $\varphi : P \to Q$ on X is a family of maps $\varphi_U : P(U) \to Q(U)$ for all $U \subseteq X$ open such that for $V \subseteq U$ open the diagram

$$\begin{array}{ccc} P(U) & \xrightarrow{\varphi_U} & Q(U) \\ \downarrow{\scriptstyle \mathrm{res}_{U,V}} & & \downarrow{\scriptstyle res_{U,V}} \\ P(V) & \xrightarrow{\varphi_V} & Q(V) \end{array}$$

commutes.

(d) An element $s \in F(U)$ is also called a section of F over U, and an element in $F(X)$ is called a global section.

(e) Let x be a point of X. The **stalk** of a presheaf P at a point $x \in X$ is the inductive limit

$$P_x := \varinjlim_{x \in U} P(U)$$

over all open sets U containing x. In other words, P_x is the set of equivalence classes of pairs (U, s), where U is an open neighborhood of x, and $s \in \mathcal{F}(U)$, and where (U, s) is equivalent to (V, t), if there is an (W, r) with $W \subseteq U, V$ and $s|_W = t|_W$.

The stalk of a sheaf F at x is the stalk of the underlying presheaf.

A morphism of sheaves is a morphism of presheaves.

Lemma 2.2 Let $\varphi, \psi : \mathcal{F} \to \mathcal{G}$ be two morphisms of presheaves.

(a) Assume that $\mathcal{F}$ is a sheaf. Then the induced maps on stalks $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ are injective for all $x \in X$ if and only if $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all open subsets $U \subseteq X$.

(b) If $\mathcal{F}$ and $\mathcal{G}$ are both sheaves, the maps φ_x are bijective for all $x \in X$ if and only if φ_U is bijective for all open subsets $U \subseteq X$.

(c) If $\mathcal{F}$ and $\mathcal{G}$ are both sheaves, the morphisms φ and ψ are equal if and only if $\varphi_x = \psi_x$ for all $x \in X$.

Proof

Claim: For $U \subseteq X$ open the map

$$\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x \quad , \; s \mapsto (s_x)_{x \in U}$$

is injective if and only if $\mathcal{F}$ is a sheaf. In fact, let $s, t \in \mathcal{F}(U)$ such that $s_x = t_x$ for all $x \in U$. Then for all $x \in U$ there exists an open neighborhood $V_x \subseteq U$ of x such that $s|_{V_x} = t|_{V_x}$. Since $U = \bigcup\limits_{x \in U} V_x$ we get $s = t$ by sheaf condition (1).

Using the commutative diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \prod\limits_{x \in U} \mathcal{F}_x \\ \downarrow & & \downarrow \\ \mathcal{G}(U) & \longrightarrow & \prod\limits_{x \in U} \mathcal{G}_x \, , \end{array}$$

we see that (c) and the necessity of the condition in (a) are implied by the above claim. Moreover, a filtered inductive limit of injective map is injective again (as can be checked easily); therefore the condition in (a) is also sufficient.

Moreover, a filtered inductive limit of injective maps is injective, so that the condition in (a) is also sufficient.

Hence we are done if we show that the bijectivity of φ_x for all $x \in U$ implies the surjectivity of φ_U.

Let $t \in \mathcal{G}(U)$. For all $x \in U$ we choose an open neighborhood U^x of x in U and $s^x \in \mathcal{F}(U^x)$ such that $(\varphi_{U^x}(s^x))_x = t_x$. Then there exists an open neighborhood $V^x \subseteq U^x$ of x with $\varphi_{V^x}(s^x|_{V^x}) = t|_{V^x}$. Then $(V^x)_{x \in V}$ is an open covering of U, and for $x, y \in U$

$$\varphi_{V^x \cap V^y}(s^x|_{V^x \cap V^y}) = t|_{V^x \cap V^y} = \varphi_{V^x \cap V^y}(s^y|_{V^x \cap V^y})$$

As we already know that $\varphi_{V^x \cap V^y}$ is injective, this shows $s^x|_{V^x \cap V^y} = s^y|_{V^x \cap V^y}$, and the sheaf condition (2) assures that we find $s \in \mathcal{F}(V)$ such that $s|_{V^x} = s^x$ for all $x \in U$. Clearly, we have $\varphi_V(s)_x = t_x$ for all $x \in U$ and hence $\varphi_U(s) = t$.

We call a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves injective (resp. surjective, resp. bijective), if $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective (resp. surjective, resp. bijective) for all $x \in X$.

If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves on X, φ is surjective if and only if for all open subsets $U \subseteq X$ and every $t \in \mathcal{G}(V)$ there exists an open covering $U = \bigcup\limits_i U_i$ (depending on t) and sections $s_i \in \mathcal{F}(U_i)$ such that $\varphi_{U_i}(s_i) = t|_{U_i}$, i.e., such that locally we can find a preimage of t. (But the surjectivity does **not** in general imply hat $\mathcal{F}(U) \to \mathcal{G}(V)$ is surjective for all open subsets U of X).

Examples 2.3 (a) Let A be an abelian group. Then one obtains the associated constant presheaf A^P with value A by

$$\begin{array}{rcll} A^P(U) & = & A & \text{for all open } U \subseteq X, \\ \mathrm{res}_{U,V} & = & \mathrm{id}_A & \text{for all } V \subseteq U\,. \end{array}$$

This is not in general a sheaf (exercise!).

Lemma 2.4 Let X be a topological space, and let P be a presheaf of abelian groups on X. Then P is a sheaf if and only if for all open sets $U \subseteq X$ and all open coverings $(U_i)_{i \in I}$ of U the sequence

$$0 \longrightarrow P(U) \xrightarrow{\alpha} \prod_{i \in I} P(U_i) \xrightarrow{\beta} \prod_{i,j \in I} P(U_i \cap U_j)$$

$$s \longmapsto (s_i|_{U_i})_{i \in I}$$

$$(s_i)_{i \in I} \longmapsto (s_i|_{U_i \cap U_j} - s_j|_{V_i \cap V_j})$$

is exact

Proof The injectivity of α is just the sheaf condition (1) from 6.1, and the sheaf condition (2) is just the exactness $\mathrm{im}(\alpha) = \ker(\beta)$.

To any presheaf we can associate a canonical sheaf.

Theorem 2.5 Let P be a presheaf on a topological space X. Then there exists a sheaf $\tilde{P}$ on X and a morphism $i_P : P \to \tilde{P}$ of presheaves, such that the following holds: If $\mathcal{G}$ is a sheaf on X and $\varphi : P \to \mathcal{G}$ is a morphism of presheaves, then there exists a unique morphism of sheaves $\tilde{\varphi} : \tilde{P} \to \mathcal{G}$ making the diagram

$$\begin{array}{ccc} P & \xrightarrow{i_P} & \tilde{P} \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\tilde{\varphi}} & \\ & \mathcal{G} & \end{array}$$

commutative.

The pair $(\tilde{P}, I_P)$ is unique up to unique isomorphism.

Proof For $U \subseteq X$ open we let $\tilde{P}(U)$ be the set of elements $(s_x)_{x\in U} \in \prod_{x\in U} P_x$ such that there exists an open neighborhood $W \subseteq U$ of x and a $t \in P(W)$ such that $s_x = t_x$ for all $x \in W$.

For $U \subseteq V$ the restriction map $\tilde{P}(V) \to \tilde{P}(U)$ is induced by the natural projection

$$\prod_{x\in V} P_x \to \prod_{x\in U} P_x \,.$$

Then it is easy to check that $\tilde{P}$ is a sheaf.

For $U \subseteq X$ there is a map $i_{P,U} : \mathcal{F}(U) \to \tilde{\mathcal{F}}(U)$ by $s \mapsto (s_x)_{x\in U}$.

The definition of $\tilde{P}$ shows that, for $x \in X$, $\tilde{P}_x = P_x$, and that $i_{P,x}$ is the identity.

Now let Q be a presheaf on X, and let $\varphi : P \to Q$ be a morphism of presheaves. Then sending $(s_x)_{x\in U} \in \tilde{P}(U)$ to $(\varphi_x(s_x))_{x\in U} \in \tilde{Q}(U)$ defines a morphism $\tilde{\varphi} : \tilde{P} \to \tilde{Q}$ which is the unique morphism making

$$\begin{array}{ccc} P & \longrightarrow & \tilde{P} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\tilde{\varphi}} \\ Q & \longrightarrow & \tilde{Q} \end{array}$$

commutative.

Definition 2.6 Let $f : X \to Y$ be a continuous map of topological spaces.

(a) If $\mathcal{F}$ is a presheaf on X, one defines a presheaf $f_*\mathcal{F}$ on Y by defining

$$(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$$

for any open subset $V \subseteq X$, with the restriction maps given by the restriction maps for $\mathcal{F}$. We call $f_*\mathcal{F}$ the direct image of $\mathcal{F}$ under f.

This association is functorial: If $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$ is a morphism of presheaves, the family of maps $f_*(\varphi)_V := \varphi_{f^{-1}(V)}$ for $V \subseteq Y$ open is a morphism $f_*(\varphi) : f_*\mathcal{F}_1 \to f_*\mathcal{F}_2$.

Therefore f_* is a functor from the category of presheaves on X to the presheaves on Y.

Remark 2.7 (1) If $\mathcal{F}$ is a sheaf on X, $f_*\mathcal{F}$ is a sheaf on Y. Therefore f_* also defines a functor $f_* : (\mathrm{Sh}(X) \to \mathrm{Sh}(Y))$.

(2) If $g : Y \to Z$ is a second continuous map, there exists an identity $g_*(f_*\mathcal{F}) = (g \circ f)_*\mathcal{F}$, which is functorial in $\mathcal{F}$.

2.8 Now we come to the definition of the inverse image of a presheaf. Let $f : X \to Y$ be a continuous map and let $\mathcal{G}$ be a presheaf on Y. Define a presheaf on X by

$$U \mapsto \varinjlim_{\substack{V \supset f(U) \\ Y \subseteq Y \text{ open}}} \mathcal{G}(V)\,, \tag{2.8.1}$$

the restriction maps being induced by the restriction maps of $\mathcal{G}$. Denote this presheaf by $f^+\mathcal{G}$. Let $f^{-1}\mathcal{G}$ be the sheaf associated to the presheaf $f^+\mathcal{G}$. We call $f^{-1}\mathcal{G}$ the inverse image of $\mathcal{G}$ under f. Note that even if $\mathcal{G}$ is a sheaf, $f^+\mathcal{G}$ is not a sheaf in general.

If f is the inclusion of a subspace X of Y, we also write $\mathcal{G}|_X$ instead of $f^{-1}\mathcal{G}$.

Again the construction of $f^+\mathcal{G}$ and hence of $f^{-1}\mathcal{G}$ is functorial in $\mathcal{G}$. Therefore we obtain a functor f^{-1} from the category of presheaves on Y to the category of sheaves on X.

Let $g : Y \to Z$ be a second continuous map, and let $\mathcal{H}$ be a presheaf on Z. Fix an open subset U in X. An open subset $W \subseteq Z$ contains $g(f(U))$ if and only if it contains a subset of the form $g(V)$, where $V \subseteq Y$ is an open subset containing $f(U)$.

This implies that $f^+(g^+\mathcal{H}) = (g \circ f)^+(\mathcal{H})$, and we can deduce an isomorphism

$$f^{-1}(g^{-1}\mathcal{H}) \cong (g \circ f)^{-1}\mathcal{H}\,, \tag{2.8.2}$$

which is functorial in $\mathcal{H}$.

If $x \in X$ and $i : \{x\} \hookrightarrow X$ is an inclusion, the definition (2.8.1) shows that

$$i^{-1}\mathcal{F} = \mathcal{F}_x$$

for every presheaf $\mathcal{F}$ on X. In particular, (2.8.2) implies for each presheaf $\mathcal{G}$ on Y the identity

$$(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}\,. \tag{2.8.3}$$

Direct image and inverse image are functors adjoint to each other. More precisely:

Proposition 2.9 Let $f : X \to Y$ be a continuous map, let $\mathcal{F}$ be a sheaf on X, and let $\mathcal{G}$ be a presheaf on Y. Then there is a bijection

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}\mathcal{G}, \mathcal{F}) & \cong & \mathrm{Hom}_{\mathrm{PreSh}(Y)}(\mathcal{G}, f_*\mathcal{F}) \\ \varphi & \mapsto & \varphi^b \\ \psi^\sharp & \leftarrow\!\shortmid & \psi\,, \end{array} \tag{2.9.1}$$

which is functorial in $\mathcal{F}$ and $\mathcal{G}$.

Proof Let $\varphi : f^{-1}\mathcal{G} \to \mathcal{F}$ be a morphism of sheaves on X, and let $t \in \mathcal{G}(V), V \subseteq Y$ open. Since $f(f^{-1}(V)) \subseteq V$, we have a map $\mathcal{G}(V) \to f^{+}\mathcal{G}(f^{-1}(V))$, and we define $\varphi^{\flat}_{V}(t)$ as the image of t under the map

$$\mathcal{G}(V) \to f^{+}\mathcal{G}(f^{-1}(V)) \to f^{-1}\mathcal{G}(f^{-1}(V)) \stackrel{\varphi_{f^{-1}(V)}}{\longrightarrow} \mathcal{F}(f^{-1}(V)) = f_{*}\mathcal{F}(V)\,.$$

Conversely, let $\psi : \mathcal{G} \to f_{*}\mathcal{F}$ be a morphism of sheaves on Y. To define the morphism $\psi^{\sharp}$ it suffices to define a morphism of presheaves $f^{+}\mathcal{G} \to \mathcal{F}$ which we call again $\psi^{\sharp}$. Let $U \subseteq X$ be open, and $s \in f^{+}\mathcal{G}(U)$. If V is some open neighborhood of $f(U)$, U is contained in $f^{-1}(V)$. Let V be such a neighborhood such that there exists $s_V \in \mathcal{G}(V)$ representing s. Then $\psi_V(s_V) \in f_{*}\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$. Let $\psi^{\sharp}_{U}(s) \in \mathcal{F}(V)$ be the restriction of the section $\psi_V(s_V)$ to U.

Clearly, these two maps from (2.9.1) are inverse to each other. One then can check that the constructed maps are functorial in $\mathcal{F}$ and $\mathcal{G}$.

3 Ringed and locally ringed spaces

Definition 3.1 (a) A ringed space is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of commutative rings on X.

(b) If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are ringed spaces, we define a morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ to be a pair (f, f^b), where $f : X \to Y$ is a continuous map and $f^b : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a homomorphism of sheaves of rings on Y.

Note that the datum of f^b is equivalent to the datum of a homomorphism of sheaves of rings $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ by Proposition 2.9. Often we just write f instead of $(f, f^\sharp)$ or (f, f^b). Often we simply write f for (f, f^b) or $(f, f^\sharp)$.

The composition of morphisms of ringed spaces is defined in the obvious way, noting that for a sheaf $\mathcal{F}$ on X, $f_*\mathcal{F}$ is a sheaf on Y.

Note that for a second continuous map $g : Y \to Z$ we have an identity $g_*(f_*\mathcal{F}) = (g \circ f)_*\mathcal{F}$, which is functorial in $\mathcal{F}$.

Definition 3.2 (a) A locally ringed space is a ringed space $(X, \mathcal{O}_X)$, where for all $x \in X$ the stalk $\mathcal{O}_{X,x}$ is a local ring.

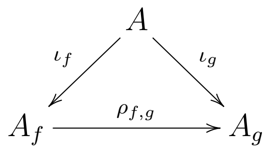
(b) A morphism of locally ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces such that for all $x \in X$ the induced homomorphism on stalks

$$f_x^\sharp : (f^{-1}\mathcal{O}_Y)_x = \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

is a homomorphism of local rings (i.e., such that $f_x^\sharp$ maps the maximal ideal of $\mathcal{O}_{Y,f(x)}$ into the maximal ideal of $\mathcal{O}_{X,x}$).

4 Affine schemes

Lemma 4.1 Let $f, g \in A$ with $D(g) \subseteq D(f)$. Then there is a unique ring homomorphism $\rho_{f,g} : A_f \to A_g$ which makes the diagram



commutative.

Proof We only have to show that f is invertible in A_g, then the claim follows from the universal property of the localization A_f. But we have

$$
\begin{aligned}
& D(g) \subseteq D(f) \\
\Leftrightarrow\ & V(f) \subseteq V(g) \text{ by taking complements} \\
\Leftrightarrow\ & \sqrt{\langle g \rangle} \subseteq \langle f \rangle \text{ (Corollary 5.22 (a))} \\
\Leftrightarrow\ & g^n \in \langle f \rangle \text{ for some } n \in \mathbb{N}_0 \\
\Leftrightarrow\ & g^n = rf \text{ with } m \in \mathbb{N}_0 \text{ and } r \in A \\
\Leftrightarrow\ & f \text{ invertible in } A_g
\end{aligned}
$$

(where the last equivalences are elementary)

Whenever we have $D(f) \subseteq D(g) \subseteq D(h)$ we have $\rho_{f,g} \circ \rho_{g,h} = \rho_{f,h}$. In particular, if $D(f) = D(g)$, then $\rho_{f,g}$ is an isomorphism, which we use to identify A_g and A_f.

Therefore we can define

$$\mathcal{O}_X(D(f)) := A_f\,,$$

and obtain a presheaf of rings on the basis $B := \{D(f) \mid f \in A\}$ for the topology of $\mathrm{Spec}(A)$.

Theorem 4.2 This presheaf $\mathcal{O}_X$ is a sheaf on B.

Proof Let $D(f)$ be a principal open set, and let $D(f) = \bigcup_{i \in I} D(f_i)$ be a covering by principal open sets. We have to show:

(1) Let $s \in \mathcal{O}_X(D(f))$ be such that $s|_{D(f_i)} = 0$ for all i. Then $s = 0$.

(2) For $i \in I$ let $s_i \in D(f_i)$ be such that

$$s_i|_{D(f_i) \cap D(f_j)} = s_j|_{D(f_i) \cap D(f_j)}$$

for all $, j \in I$. Then there exists an $s \in \mathcal{O}_X(D(f))$ such that $s|_{D(f_i)} = s_i$ for all $i \in I$.

As $D(f)$ is quasi-compact, we can assume that I is finite. Restricting the presheaf to $D(f)$ and replacing A by A_f, we can assume that $f = 1$ and hence $D(f) = X$.

The relation $X = U_i D(f_i)$ is equivalent to $(f_i \mid i \in I) = A$. As $D(f_i) = D(f_i^n)$ for all integers $n \geq 1$, there exist elements $b_i \in A$ (depending on n) such that

$$\sum_{i \in I} b_i f_i^n = 1\,. \tag{4.2.1}$$

Proof of (1): Let $s = a \in A$ be such that the image of a in A_{f_i} is zero for all i. As I is finite, there exists an integer $n \geq 1$, independent of i, such that $f_i^n a = 0$. Then, by (4.2.1)

$$a = (\sum_{i \in I} b_i f_i^n) a = 0$$

Proof of (2): As I is finite, we can write

$$s_i = \frac{a_i}{f_i^n}$$

for some n independent of i. By hypothesis, the images of $\frac{a_i}{f_i^n}$ and $\frac{a_j}{f_j^n}$ is $A_{f_i f_j}$ are equal for all $i, j \in I$.

Therefore there exists an integer $m \geq 1$ (which again we can choose independent of i and j). such that $(f_i f_j)(f_j^n a_i - f_i^n a_j) = 0$. Replacing a_i by $f_i^m a_i$ and n by $n + m$ (which does not change s_i), we see that

$$f_j^n d_i = f_i^n a_j \tag{4.2.2}$$

for all $i, j \in I$.

We set $s = \sum_{j \in I} b_j \cdot a_j \in A$, where b_j are the elements in (4.2.1). Then

$$f_i^n s = f_i^n \sum_{j \in I} b_j a_j = \sum_{j \in I} b_j (f_i^n a_j) \overset{(4.2.2)}{=} (\sum_{j \in J} b_j f_j^n) a_i \overset{(4.2.1)}{=} a_i$$

This means that the image of s in A_{f_i} is s_i.

Definition 4.3 A locally ringed space $(X, \mathcal{O}_X)$ is called an affine scheme, if there exists a ring A such that $(X, \mathcal{O}_X)$ is isomorphic to $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$.

A morphism of affine schemes is a morphism of locally ringed spaces. We obtain the category of affine schemes.

Theorem 4.4 The functors

$$\begin{array}{rcl} \text{Spec}: \ (\text{commutative rings}) & \to & (\text{affine schemes}) \\ A & \mapsto & (\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)}) \\ (\text{affine schemes}) & \to & (\text{commutative rings}) \\ \text{Spec}(A) & \mapsto & A \end{array}$$

define an anti-equivalence between the category of rings and the category of affine schemes.

Proof Let $\varphi : A \to B$ be a homomorphism of commutative rings with 1, and let $X = \text{Spec}(B)$ and $Y = \text{Spec}(A)$. Let ${}^a\varphi : \text{Spec}(B) \to \text{Spec}(A)$ be the associated continuous map. We will now define a morphism $(f, f^b) : X \to Y$ of locally ringed spaces such that $f =^a \varphi$ and

$$f^b_Y : A = \mathcal{O}_Y(Y) \to (f_*\mathcal{O}_X)(Y) = B \tag{4.4.1}$$

equals φ. Let $f = {}^a\varphi$. For $s \in A$ we have $f^{-1}(D(s)) = D(\varphi(s))$ by definition, and we define

$$f^b_{D(s)} : \mathcal{O}_Y(D(s)) = A_s \to B_{\varphi(s)} = (f_*\mathcal{O}_X)(D(s)) \tag{4.4.2}$$

as the ring homomorphism induced by φ. This is compatible with restrictions to principal open subsets $D(t) \subseteq D(s)$. As the principal open subsets form a basis for the topology, this defines a homomorphism $f^b : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings. Choosing $s = 1$ in (4.4.2) we obtain (4.4.1). This proves the claim of Theorem 4.4.

5 Schemes

Definition 5.1 (a) A scheme is a locally ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to an affine scheme, i.e., there is an open covering $(U_i)_{i\in I}$ of X such that $(U_i, \mathcal{O}_{X|_{U_i}})$ is an affine scheme.

(b) A morphism of schemes is a morphism of locally ringed spaces.

Example 5.2 Let $\mathbb{P}^1_k$ be the one-dimensional projective space over a field k. Then $\mathbb{P}^1_k$ can be covered by two affine spaces $\mathbb{A}^1_k = \mathrm{Spec}(k[t])$ and $\mathrm{Spec}(k[t^{-1}])$, but $\mathbb{P}^1_k$ is not an affine scheme.

6 Differentials

Let A be a ring, let B be an A-algebra and let M be a B-module.

Definition 6.1 An A-derivation from B to M is a map

$$D: B \to M$$

which is A-linear and for which the Leibniz rule holds:

$$D(bb') = b'D(b) + bD(b') \tag{1.1.1}$$

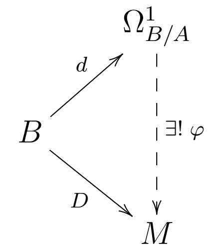
Let $D_A(B, M)$ be the set of the A-derivations D from B to M.

Obviously, $D_A(B, M) \subseteq \mathrm{Hom}_A(B, M)$ is an A-submodule.

Lemma/Definition 6.2 There is a B-module $\Omega^1_{B/A}$, the module of the (relative) (Kähler) differentials of B over A, and a derivation

$$d : B \to \Omega^1_{B/A}\,,$$

which is universal for all A-derivations into B-modules: If $D : B \to M$ is an A-derivation, then there is exactly one B-module homomorphism $\varphi : \Omega^1_{B/A} \to M$, which makes the diagram



commutative. Therefore we have an isomorphism

$$\begin{array}{rcl} \mathrm{Hom}_B(\Omega^1_{B/A}, M) & \xrightarrow{\sim} & \mathrm{Der}_A(B, M) \\ \varphi & \mapsto & \varphi d\,. \end{array} \tag{1.2.1}$$

Construction: Set

$$\Omega^1_{B/A} = \Big\{\text{free } B \text{ module over symbols } \tilde{d}b, b \in B\Big\}/N$$

where N is the submodule, which is generated of all elements

$$\begin{gathered} \tilde{d}(b + b') - \tilde{d}b - \tilde{d}b' \\ \tilde{d}(bb') - b'\tilde{d}b - b\tilde{d}b' \\ \tilde{d}a \end{gathered}$$

for $b, b' \in B$ and $a \in A$. Define $d : B \to \Omega^1_{B/A}$ by

$$d(b) = \text{ class of } \tilde{d}b$$

Lemma 6.3 Let $B = A[X_i \mid i \in I]$ be a polynomial ring in arbitrary many variables X_i. Then $\Omega^1_{B/A}$ is a free A-module with basis $dX_i, i \in I$.

Proof Obviously the dX_i are generators, as one proves by induction:

$$d(\prod_i X_i^{n_i}) \;=\; \sum_i n_i \prod_{\substack{j \\ j\neq i}} X_j^{n_j} X_i^{n_i-1} dX_i\,.$$

Assume $\sum P_i dX_i = 0$ with $P_i \in B$. The formal partial derivative

$$\begin{array}{rrcl} \frac{\partial}{\partial X_i}: & B & \to & B \\ & \prod_j X_j^{n_j} & \mapsto & n_i X_i^{n_i-1} \prod_{\substack{j \\ j\neq i}} X_j^{n_j} \end{array}$$

is a derivation; therefore there exists a B-module homomorphism

$$\varphi_i : \; \Omega^1_{B/A} \;\to\; B$$

with $\varphi_i(dX_j) = \frac{\partial X_j}{\partial X_i} = \delta_{ij}$. Applied to $\sum P_j dX_j$, $P_i = 0$ follows for all i.

Remark 6.4 For a polynomial P we get

$$dP \;=\; \sum_i \frac{\partial P}{\partial X_i} dX_i\,,$$

as it should be.

Proposition 6.5 Let $\mu : B \otimes_A B \to B$ be defined by $\mu(b \otimes b') = b \cdot b'$ and let $I = \ker(\mu)$. Consider $B \otimes_A B$ as B-module via the multiplication from the *left*, and regard I/I^2 as a B-module by the induced structure. Let

$$\begin{array}{rrcl} d: & B & \to & I/I^2 \\ & b & \mapsto & 1 \otimes b - b \otimes 1 \quad \bmod I^2\,. \end{array}$$

Then $(I/I^2, d)$ is isomorphic to $(\Omega^1_{B/A}, d)$.

Proof (a) d is derivation: Left to the reader.

(b) We obtain a homomorphism

$$\begin{array}{rrcl} \varphi: & \Omega^1_{B/A} & \to & I/I^2 \\ & db & \mapsto & 1 \otimes b - b \otimes 1 \end{array}$$

(c) In $B \otimes_A B$ we have $x \otimes y = xy \otimes 1 + x(1 \otimes y - y \otimes 1)$, hence I is generated as a B-module by elements of the form db.

(d) Let M be an arbitrary B-module. Then, on the abelian group $B \oplus M$, one can define a ring-structure by

$$(b, m)(b', m') \;=\; (bb', bm' + b'm)$$

(left to the reader). Denote this ring by $B * M$. The map

$$\begin{array}{rccc} \pi : & B * M & \to & B \\ & (b,m) & \mapsto & b \end{array}$$

is a surjective ring homomorphism; the kernel is M, and as an ideal we have $M^2 = 0$.

In our situation, $B * M$ is also an A-algebra (even an B-algebra by the section $s : B \to B * M, b \mapsto (b, 0)$) of π.

Now, let $D : B \to M$ be an A-derivation. Then the map

$$\begin{array}{rccc} \psi : & B \otimes_A B & \to & B * M \\ & b \otimes b' & \mapsto & (bb', bD(b')) \end{array}$$

is a well-defined ring homomorphism:

$$\begin{aligned} b_1 b_2 \otimes b_1' b_2' \ \mapsto \ (b_1 b_2 b_1' b_2', b_1 b_2 D(b_1' b_2')) &= (b_1 b_2 b_1' b_2', b_1 b_2 (b_2' D(b_1') + b_1' D(b_2')) \\ &= (b_1 b_1', b_1 D(b_1')) \cdot (b_2 b_2', b_2 D(b_2'))\,. \end{aligned}$$

By the commutativity of

$$\begin{array}{ccc} B \otimes_A B & \xrightarrow{\psi} & B * M \\ \downarrow \mu & & \downarrow \pi \\ B & = & B\,, \end{array}$$

we have $\psi(I) \subseteq M$, and since $M^2 = 0$ we have $\psi(I^2) = 0$. Hence we obtain a well-defined B-module-homomorphism

$$\overline{\psi} : \ I/I^2 \ \to \ M\,,$$

which sends $1 \otimes b - b \otimes 1 \mod I^2$ to $D(b)$, therefore commutes with the derivations. Applied to $M = \Omega^1_{B/A}$ we obtain an inverse for φ.

Proposition 6.6 (a) If A' is an A-algebra, then we have

$$\Omega^1_{B \otimes_A A'/A'} \ \cong \ \Omega^1_{B/A} \otimes_A A' \ \cong \ \Omega^1_{B/A} \otimes_B B' \quad \text{, where } B' = B \otimes_A A'\,.$$

(b) If $S \subseteq B$ is a multiplicative subset, then we have

$$\Omega^1_{S^{-1}B/A} \ \cong \ S^{-1}\Omega^1_{B/A}\,.$$

Proof Left to the reader (follows from the universal properties).

Theorem 6.7 (First fundamental sequence) If C is a B-algebra, then one has an exact sequence

$$\Omega^1_{B/A} \otimes_B C \ \xrightarrow{\alpha} \ \Omega^1_{C/A} \ \xrightarrow{\beta} \ \Omega^1_{C/B} \ \to 0$$

Proof Let $\alpha(db \otimes c) = cdb$ and let $\beta(c_1 dc_2) = c_1 dc_2$; then these homomorphisms are well-defined, and β is surjective and $\beta\alpha = 0$. For the exactness in the middle it suffices to show that for every C-module M the sequence

$$(1.7.1) \qquad \mathrm{Hom}_C(\Omega^1_{C/B}, M) \to \mathrm{Hom}_C(\Omega^1_{C/A}, M) \to \mathrm{Hom}_C(\Omega^1_{B/A} \otimes_B C, M)$$

is exact (universal property of the cokernel of α). By the universal property (1.2.1), (1.7.1) is identified with the sequence

$$(1.7.2) \qquad \mathrm{Der}_B(C, M) \to \mathrm{Der}_A(C, M) \to \mathrm{Der}_A(B, M) = \mathrm{Hom}_B(\Omega^1_{B/A}, M)$$

which is exact by definition (An A-derivation d is a B-derivation if and only if we have $db = 0$ for all $b \in B$).

Theorem 6.8 (Second fundamental sequence) Let J be an ideal of B and let $C = B/J$. Then there is an exact sequence of C-modules

$$J/J^2 \xrightarrow{\delta} \Omega^1_{B/A} \otimes_B C \xrightarrow{\alpha} \Omega^1_{C/A} \to 0$$

where $\delta(b \mod J^2) = db \otimes 1$ for $b \in J$.

Proof (a) For $b \in B$ and $b' \in J$ we have $d(bb') = b'db + bdb'$. Thus $b' \mapsto db' \otimes 1$ is B-linear and maps J^2 to 0.

(b) Obviously, α is surjective and $\alpha\delta = 0$. Therefore it suffices to show the exactness of

$$\mathrm{Hom}_C(\Omega^1_{C/A}, M) \to \mathrm{Hom}_C(\Omega^1_{B/A} \otimes_B C, M) \to \mathrm{Hom}_C(J/J^2, M)$$

for every C-module M. But this sequence can be identified with

$$\mathrm{Der}_A(C, M) \to \mathrm{Der}_A(B, M) \to \mathrm{Hom}_B(J, M)$$

where the second map maps D to $D|_J$. This sequence is obviously exact.

Corollary 6.9 If B is a finitely generated A-algebra or a localization of it, then $\Omega^1_{B/A}$ is finitely generated.

Proof If $B = A[X_1, \dots, X_n]/J$, then, by 6.3, we have

$$B^n \cong \Omega^1_{A[X_1,\dots,X_n]/A} \otimes B \twoheadrightarrow \Omega^1_{B/A},$$

and the second case follows with 6.6 (b).

Example 6.10 Let $L = K(\alpha)/K$ be a field extension, generated by one element.

(1) Let α be transcendental over K, i.e., $L \cong K(t)$, via $\alpha \mapsto t \in (K[t] \smallsetminus \{0\})^{-1}K[t]$. By 6.3 and 6.6 (b) we then have

$$\Omega^1_{L/K} = (K[t] \smallsetminus \{0\})^{-1} K[\alpha] d\alpha = L d\alpha,$$

which is one-dimensional over L.

(2) Let α be algebraic over K. Let $f(x)$ be the minimal polynomial of α over K. Then we have $L \cong K[x]/\langle f(x)\rangle, \alpha \mapsto \overline{x}$. By the second fundamental sequence we have an exact sequence

$$\begin{array}{rcl} \langle f(x)\rangle/\langle f(x)^2\rangle & \to & K[x]/\langle f(x)\rangle \cong L\,dx \;\to\; \Omega^1_{L/K} \;\to\; 0 \\ f(x) \mapsto \;\; df & = & f'(x) \mod \langle f(x)\rangle dx \\ & = & f'(\alpha)dx \in Ldx \end{array}$$

(2a) If α is separable over K, then $f'(\alpha) \neq 0$ and thus

$$\Omega^1_{L/K} \;=\; 0\,.$$

(2b) If α not separable over K, then $f'(x) \equiv 0$ and

$$\Omega^1_{L/K} \;=\; Ld\alpha\,.$$

(3) **Addition:** If α is separable over K and if $A \subseteq K$ is a subring, then

$$\Omega^1_{K/A} \otimes L \;\xrightarrow{\sim}\; \Omega^1_{L/A}\,.$$

Note: We already know the surjectivity (by 6.7 and (2a)), and we have to show that for every L-module M the map

$$\begin{array}{ccccc} \alpha : \mathrm{Hom}_L(\Omega^1_{L/A}, M) & \longrightarrow & \mathrm{Hom}_L(\Omega^1_{K/A} \otimes_K L, M) & \cong & \mathrm{Hom}_K(\Omega^1_{K/A}, M) \\ \| & & & & \| \\ \mathrm{Der}_A(L, M) & & \longrightarrow & & \mathrm{Der}_A(K, M) \end{array}$$

is surjective. This follows from the diagram

$$\begin{array}{ccccc} & & \mathrm{Der}_K(K[x], M) & \overset{\beta}{\searrow} & \\ & & \downarrow & & \\ \mathrm{Der}_A(L, M) & \longrightarrow & \mathrm{Der}_A(K[x], M) & \longrightarrow & \mathrm{Hom}_{K[x]}(\langle f\rangle, M) \\ & \underset{\alpha}{\searrow} & \downarrow & & \\ & & \mathrm{Der}_A(K, M) & & \end{array}$$

with exact rows and columns, where β is surjective. Then β maps a K-derivation $D : K[X] \to M$, which is determined only by $Dx \in M$, to the homomorphism $\varphi : \langle f\rangle \to M$ with $\varphi(f) = f'(\alpha) \cdot Dx$, where we have $f'(\alpha) \neq 0$ by assumption.

Corollary 6.11 A field extension L/K is algebraic and separable if and only if $\Omega^1_{L/K} = 0$.

Proof For $\alpha \in L$ we have the first fundamental sequence

$$\Omega^1_{K(\alpha)/K} \otimes L \;\to\; \Omega^1_{L/K} \;\to\; \Omega^1_{L/K(\alpha)} \;\to\; 0\,.$$

Now, if L is finitely generated over K (as field) and $\Omega^1_{L/K} = 0$, then it follows inductively that $L/K(\alpha)$ is finitely separable, and that the first map is an isomorphism, therefore $\Omega^1_{K(\alpha)/K} = 0$. Therefore by 6.10 (3), $K(\alpha)/K$ is separable. The converse follows easily.

If L is arbitrarily generated, then $L = \bigcup\limits_i L_i$, where L_i is finitely generated, and we have

$$\Omega^1_{L/K} \;=\; \varinjlim_i \Omega^1_{L_i/K}\,,$$

so that the claim follows for L from the case of the finitely generated L_i.

Theorem 6.12 Let k be a field, let A be a local k-algebra and let the residue field $A/\mathfrak{m}$ be isomorphic to k. Then the map

$$\delta:\; \mathfrak{m}/\mathfrak{m}^2 \;\to\; \Omega^1_{A/k} \otimes_A k$$

from the second fundamental sequence 6.8 is an isomorphism.

Proof By 6.8 we have $\mathrm{Coker}(\delta) = \Omega^1_{k/k} = 0$, therefore δ is surjective. For the injectivity of δ it suffices to show that the map

$$\begin{array}{ccccc} \mathrm{Hom}_k(\Omega^1_{A/k} \otimes_A k, k) & \longrightarrow & \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k) & & D/\mathfrak{m} \\ \Vert & \nearrow & & \nearrow & \\ \mathrm{Der}_k(A, K) & & D & & \end{array}$$

of the dual spaces is surjective. Let $f : \mathfrak{m}/\mathfrak{m}^2 \to k$, and let $\pi : A \to k$ be the projection, so that we have $a - \pi(a) \in \mathfrak{m}$ for all $a \in A$. Define

$$\begin{array}{rccl} D: & A & \to & k \\ & a & \mapsto & f(a - \pi a \mod \mathfrak{m}^2)\,. \end{array}$$

Then D is a k-derivation: The additivity is obvious, and for the Leibniz rule we have:

$$aa' \mapsto f(aa' - \pi(aa') \mod \mathfrak{m}^2) = f(a'(a - \pi(a)) + a(a' - \pi(a')))$$

$$-(a - \pi(a))(a' - \pi(a')) \mod \mathfrak{m}^2) = a'D(a) + aD(a')\,.$$

Furthermore we have $D|_k = 0$ (therefore D k-linear) and $D|_{\mathfrak{m}} = f$.

Definition 6.13 Let L/K be a field extension.

(a) A transcendence basis $(x_i)_{i\in I}$ of L/K is called separated, if $L/K(x_i; i \in I)$ is separable.

(b) L/K is called separable (separably generated), if there is a separated transcendental basis.

Remark 6.14 If K is complete, then every finitely generated field extension L/K is generated separably (s. Zariski-Samuel 'Commutative algebra, Vol 1, p. 105).

Proposition 6.15 Let L/K be a finitely generated field extension. Then L/K is separably generated if and only if we have

$$\dim_L \Omega^1_{L/K} \;=\; \mathrm{tr}\,.gr_K L\,.$$

Addition: In this case, if we have $\alpha_1, \ldots, \alpha_n \in L$ such that $d\alpha_1, \ldots, d\alpha_n$ form a basis of $\Omega^1_{L/K}$, then $\alpha_1, \ldots, \alpha_n$ is a separated transcendence basis of L/K.

Proof If L is a separable algebraic extension of $K(X_1, \ldots, X_n)$, then $\mathrm{tr}\,.\deg_K L = n$, and by 6.11 and 6.10 (3) we have

$$\Omega^1_{L/K} \;\cong\; \Omega^1_{K(X_1,\ldots,X_n)/K} \otimes_{K(X_1,\ldots,X_n)} L \;\cong\; L^n\,,$$

where the last isomorphism holds by 6.3 and 6.6 (b). If equality holds, then there are $\alpha_1, \ldots, \alpha_n \in L$, such that $d\alpha_1, \ldots, d\alpha_n$ form a basis of $\Omega^1_{L/K}$. Let $L_0 = K(\alpha_1, \ldots, \alpha_n)$. By the exact sequence

$$\Omega^1_{L_0/K} \otimes_{L_0} L \;\to\; \Omega^1_{L/K} \;\to\; \Omega^1_{L/L_0} \;\to\; 0\,,$$

we have $\Omega^1_{L/L_0} = 0$, and therefore L/L_0 is separably algebraic by 6.1. Since $\mathrm{tr}\,.\deg_K L = n$, $\alpha_1, \ldots, \alpha_n$ have to be transcendental over K.

Theorem 6.16 Let k be an algebraically closed field, let A be a finitely generated k-algebra and let $\mathfrak{m} \subseteq A$ be a maximal ideal. Then the following are equivalent:

(a) $A_\mathfrak{m}$ is regular.

(b) $\Omega^1_{A/k} \otimes_A A_\mathfrak{m} = \Omega^1_{A_\mathfrak{m}/k}$ is free of rank $\dim A_\mathfrak{m}$ ($= \dim A$, is A is irreducible).

Proof If (b) holds, then by 6.12 we have $\dim \mathfrak{m}/\mathfrak{m}^2 \;=\; \dim \mathfrak{m}A_\mathfrak{m}/\mathfrak{m}^2 A_\mathfrak{m} \;=\; \dim A_\mathfrak{m}$, therefore we have (a) by the definition of regularity. Conversely, from (a) it follows with 6.12 that $\dim_k(\Omega^1_{A_\mathfrak{m}/k} \otimes_{A_\mathfrak{m}} k) = r := \dim A_\mathfrak{m}$. On the other hand, let $K = \mathrm{Quot}(A_\mathfrak{m})$. Then, by 6.6 (b) we have

$$\Omega^1_{A_\mathfrak{m}/k} \otimes_{A_\mathfrak{m}} K \;=\; \Omega^1_{K/k}$$

and by 6.14 and 6.15 this has the dimension $\mathrm{tr}\,.gr_k K = \dim A' = \dim A_\mathfrak{m} = r$ for the (integral) irreducible component $\mathrm{Spec}(A')$ of $\mathrm{Spec}(A)$, where $\mathfrak{m}$ lies (see Alg. Geo I, Proposition 7.10). Now the claim follows from

Lemma 6.17 Let A be a local integral/domain with residue field k and quotient field K. If M is a finitely generated A-module with

$$\dim_k M \otimes_A k \;=\; r \;=\; \dim_K M \otimes_A K\,,$$

then M is free of rank r.

Proof If $\dim_k M \otimes_A k = r$, then, by the Nakayama-Lemma, M has r generators $m_1, \ldots, m_r$, and we obtain an exact sequence

$$0 \;\to\; N \;\to\; A^r \;\twoheadrightarrow\; M \;\to\; 0\,.$$

By tensoring with K, this sequence stays exact; it follows $N \otimes_A K = 0$, therefore $N = 0$, since N is torsion-free.

Definition 6.18 Let $f : X \to S$ be a morphism of schemes, and let $\Delta_X : X \to X \times_S X$ be the diagonal (defined by the two component maps (id_X, id_X) and the universal property of the fibre product). This is a closed immersion in a open subscheme W of $X \times_S X$ (see Lemma 2.3 below); let $J \subseteq \mathcal{O}_W$ be the associated ideal sheaf. Then we define

$$\Omega^1_{X/S} \;:=\; \Delta_X^*(J/J^2)\,;$$

this is the sheaf of the relative (Kähler) differentials of X over S.

Remarks 6.19 (a) If $U = \mathrm{Spec}\, A \subseteq S$ and $V = \mathrm{Spec}\, B \subseteq X$ are open and affine with $f(V) \subseteq V$, then we obviously have $J/J^2 = \widetilde{I/I^2}$ for $I = \mathrm{Ker}(B \otimes_A B \to B)$, considered as $B \otimes_A B/I$ module resp. $\mathcal{O}_X$-module; therefore by 6.5 we have $\Omega^1_{X/S|V} \cong \widetilde{\Omega^1_{B/A}}$.

(b) The local differentials glue together to a $\mathcal{O}_S$-derivation $d : \mathcal{O}_X \to \Omega^1_{X/S}$.

From the previous results for the affine case and Remark 6.19 we immediately obtain the following results:

Proposition 6.20 (a) If $S' \to S$ is a morphism, then we have

$$\Omega^1_{X\times_S S'/S'} \;\cong\; p_1^*\Omega^1_{X/S}\,,$$

where $p_1 : X \times_S S' \to X$ is the first projection.

(b) If $U \subseteq X$ is open, then we have

$$\Omega^1_{U/S} \;=\; \Omega^1_{X/S|U}\,.$$

Theorem 6.21 (First fundamental sequence) For morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ of schemes one has a canonical exact sequence of $\mathcal{O}_X$-module sheaves

$$f^*\Omega^1_{Y/Z} \;\to\; \Omega^1_{X/Z} \;\to\; \Omega^1_{X/Y} \;\to\; 0$$

Theorem 6.22 (Third fundamental sequence) If $i : Z \hookrightarrow X$ is a closed immersion with ideal sheaf $J \subseteq \mathcal{O}_X$, then one has an exact sequence

$$J/J^2 \to i^*\Omega^1_{X/S} \to \Omega^1_{Z/S} \to 0\,.$$

Example 6.23 If $X = \mathbb{A}^n_S$, then $\Omega^1_{X/S}$ is a free $\mathcal{O}_X$-module of rank n, with basis $dx_1, \ldots, dx_n$.