Deligne's Proof of the Weil-conjecture

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0 Introduction

The Riemann zeta-function is defined by the sum and product

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}} \quad (s \in \mathbb{C})$$

which converge for $Re(s) > 1$.
The expression as a product, where p runs over the rational prime numbers, is generally attributed to Euler, and is therefore known as Euler product formula with terms the Euler factors. Formally the last equation is easily achieved by the unique decomposition of natural numbers as a product of prime numbers and by the geometric series expansion

$$\frac{1}{1-p^{-s}} = \sum_{m=0}^{\infty} p^{-ms} \quad .$$

The – to this day unproved – Riemann hypothesis states that all non-trivial zeros of $\zeta(s)$ should lie on the line $Re(s) = \frac{1}{2}$. This is more generally conjectured for the Dedekind zeta functions

$$\zeta_K(s) = \sum_{\mathfrak{a}\subset\mathcal{O}_K} \frac{1}{N\mathfrak{a}^s} = \prod_{\mathfrak{p}} \frac{1}{1-N\mathfrak{p}^{-s}} .$$

Here K is a number field, i.e., a finite extension of $\mathbb{Q}$, $\mathfrak{a}$ runs through the ideals $\neq 0$ of the ring $\mathcal{O}_K$ of the integers of K, $\mathfrak{p}$ runs through the prime ideals $\neq 0$, and $N\mathfrak{a} = |\mathcal{O}_K/\mathfrak{a}|$, where $|M|$ denotes the cardinality numbers of a finite set M.

Artin examined the analogue for global function fields. Let $\mathbb{F}_q$ be a finite field with q elements, q a power of a prime p

$\mathbb{Q}$ corresponds to $\mathbb{F}_q(t)$ (the rational function field),

$\mathbb{Z}$ corresponds to $\mathbb{F}_q[t]$ (the polynomial ring in a variable),

and on considers the analogous functions:

$$\sum_{\mathfrak{a}\subset\mathbb{F}_q[t]} \frac{1}{N\mathfrak{a}^s} = \prod_{\mathfrak{p}} \frac{1}{1-N\mathfrak{p}^{-s}} \quad ,$$

where again $\mathfrak{a}$ and $\mathfrak{p}$ runs through the non-trivial ideals and the prime ideals of $\mathbb{F}_q[t]$ respectively, and where $N\mathfrak{a} = |\mathbb{F}_q[t]/\mathfrak{a}|$. Similarly one can examine global function fields K, i.e., finite extensions to $\mathbb{F}_q(t)$. However the ring $\mathbb{F}_q[t]$ is no longer defined by the field $\mathbb{F}_q(t)$, as it was for $\mathbb{Z}$ in $\mathbb{Q}$; one could also consider $\mathbb{F}_q[\frac{1}{t}] \subseteq \mathbb{F}_q(t)$. This is even more the case for the general fields K, because these no longer contain $\mathbb{F}_q(t)$ canonically. It is better and more canonical to consider the uniquely determined smooth projective curve X over $\mathbb{F}_q$ with function field K and to define

$$\zeta_K(s) = \zeta(X,s) = \prod_{x\in X_0} \frac{1}{1-(Nx)^{-s}} = \prod_{x\in X_0} \frac{1}{1-q^{-deg(x)s}} \quad .$$

Here X_0 denotes the set of the closed points of X, and for $x \in X_0$ $Nx = |k(x)|$ is the (finite) cardinality of the residue field $k(x)$ of x. With $\deg(x) = [k(x) : \mathbb{F}_q]$ we apparently have $Nx = q^{\deg(x)}$, and therefore the last equality. These points here are meant in a scheme-theoretic sense: Observe that for an affine open part $U = SpecR \subset X$ the points $x \in U$ correspond to the prime ideals $\mathfrak{p}$ of R, and that $k(x)$ is the quotient field of $R/\mathfrak{p}$. The finite points x correspond to the maximal ideals; for these one has $Nx = |R/\mathfrak{p}|$, and one obtains a similar setting as above.

By the last formula one has $\zeta(X,s) = Z(X,q^{-s})$, where

$$Z(X,T) = \prod_{x\in X_0} \frac{1}{1-T^{\deg(x)}} \in \mathbb{Z}[[T]] \quad .$$

From this one obtains the equality of formal power series

$$\begin{aligned}\log Z(X,T) &= \sum_{x\in X} -\log(1-T^{\deg(x)}) = \sum_{x\in X}\sum_{n=1}^{\infty} \frac{T^{\deg(x)\cdot n}}{n} \\ &= \sum_{m=1}^{\infty}\Big(\sum_{\deg(x)|m} \deg(x)\Big)\frac{T^m}{m} = \sum_{m=1}^{\infty} |X(\mathbb{F}_{q^m})|\frac{T^m}{m} \quad ,\end{aligned}$$

where $X(\mathbb{F}_{q^m})$ is the set of the $\mathbb{F}_{q^m}$-rational points of X over $\mathbb{F}_q$: In fact for every $x \in X_0$ with $\deg(x)|m$ there exist exactly as many $\mathbb{F}_{q^m}$-rational points, as there exist $\mathbb{F}_q$-linear embeddings $\kappa(x) \hookrightarrow \mathbb{F}_{q^m}$, and their quantity is $\deg(x)$.

We consider an example. The smooth projective curve with function field $\mathbb{F}_q(t)$ is $\mathbb{P}^1_{\mathbb{F}_q}$, the one-dimensional projective space over $\mathbb{F}_q$. Geometrically, i.e., scheme theoretically we have $\mathbb{P}^1_{\mathbb{F}_q} = U_1 \cup U_2$, with $U_1 = \mathrm{Spec}\mathbb{F}_q[t] = \mathbf{A}^1_{\mathbb{F}_q}$ (the one-dimensional affine space over $\mathbb{F}_q$) and $U_2 = \mathrm{Spec}\mathbb{F}_q[t^{-1}]$ (the affine space with the coordinate t^{-1}). Then we have $U_1 \cap U_2 = \mathrm{Spec}\mathbb{F}_q[t,t^{-1}]$ and $U_1 \setminus U_2 =$ point $t = 0$ and $U_2 \setminus U_1 =$ point $t^{-1} = 0$ ("$t = \infty$"). Since $\mathbf{A}^1_{\mathbb{F}_q}(\mathbb{F}_{q^m}) = \mathrm{Hom}_{\mathbb{F}_q}(\mathrm{Spec}\mathbb{F}_{q^m}, \mathbf{A}^1_{\mathbb{F}_q}) = \mathrm{Hom}_{\mathbb{F}_q}(\mathbb{F}_q[t], \mathbb{F}_{q^m}) \cong \mathbb{F}_{q^m}$ (the last bijection sends a ring homomorphism φ to $\varphi(t)$), one obtains

$$|\mathbb{P}^1_{\mathbb{F}_q}(\mathbb{F}_{q^m})| = q^m + 1 \quad .$$

This also follows from the known description of points

$$\begin{aligned}\mathbb{P}^1_{\mathbb{F}_q}(\mathbb{F}_{q^m}) &= ((\mathbb{F}_{q^m})^2 \setminus \{0\})/\mathbb{F}_{q^m}^{\times} \\ &= \{[a_0 : a_1] | a_i \in \mathbb{F}_{q^m},\ \text{not both zero}\} \\ &= \{[1 : a_1] | a_1 \in \mathbb{F}_{q^m}\} \cup \{[0:1]\} \quad .\end{aligned}$$

By choosing the coordinate $t = \frac{a_1}{a_0}$, the first set of the union is of course $U_1(\mathbb{F}_{q^m})$, and $[0:1]$ is the point "$t = \infty$". With this we now calculate

$$\begin{aligned}Z(\mathbb{P}^1_{\mathbb{F}_q}, t) &= exp\Big(\sum_{m=1}^{\infty}(1+q^m)\frac{T^m}{m}\Big) \\ &= exp\Big(\sum_{m=1}^{\infty}\frac{T^m}{m}\Big)\cdot exp\Big(\sum_{m=1}^{\infty}\frac{(qT)^m}{m}\Big) = \frac{1}{(1-T)(1-qT)} \quad .\end{aligned}$$

In particular, this is a rational function!

More generally one can show the following result which goes back to E. Artin and F.K. Schmidt: for a smooth projective (geometrically irreducible) curve X of genus g over $\mathbb{F}_q$ one has:

$$Z(X,T) = \frac{P(T)}{(1-T)(1-qT)} \ ,$$

where $P(T)$ is a polynomial of degree $2g$ in $\mathbb{Z}[T]$, with constant coefficient 1. Furthermore Hasse (for $g = 1$, as well as for elliptic curves) and Weil (for arbitrary g) proved that the zeros of $P(q^{-s})$ lie on the line $Re(s) = \frac{1}{2}$. Applied to $\zeta(X, s) = Z(X, q^{-s})$ this proves the analogoue (conjectured by Artin) of the Riemann hypothesis in the case of function fields.

We change the interpretation. Write

$$P(T) = \prod_{i=1}^{2g}(1 - \alpha_i T), \text{ with } \alpha_i \in \overline{\mathbb{Q}} \subset \mathbb{C} ,$$

where $\overline{\mathbb{Q}}$ denotes the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. For a complex number s we obviously have $P(q^{-s}) = 0$ if and only if there is an i with $\alpha_i \cdot q^{-s} = 1$. In this case we furthermore have

$$Re(s) = \frac{1}{2} \Longleftrightarrow \mid \alpha_i \mid = q^{\frac{1}{2}} .$$

A. Weil now observed, that the definition of the zeta-function makes sense for arbitrary varieties over $\mathbb{F}_q$, and after calculation of these in several non-trivial cases ([Weil]) stated the following conjectures.

Weil-conjecture (proved by Deligne in 1973): Let X be a geometric irreducible smooth projective variety $\mathbb{F}_q$. Define

$$Z(X, T) = exp(\sum_{n=1}^{\infty} |X(\mathbb{F}_{q^n})| \frac{T^n}{n}) \in \mathbb{Q}[[T]] .$$

Then the following holds

I: $Z(X, T)$ is rational, i.e., in $\mathbb{Q}(T)$.

(In particular, this implies the existence of a meromorphic continuation of the zeta-function $\zeta(X, s) = Z(X, q^{-s})$, for which the series initially only converges for $Re(s) >> 0$).

II: One has a functional equation

$$Z(X, \frac{1}{q^d T}) = \pm q^{\frac{dE}{2}} T^E Z(X, T) ,$$

where $d = \dim X$ is the dimension of X and $E = (\Delta \cdot \Delta)$ is the selfintersection number of the diagonal Δ on $X \times X$.

For the zeta-function in s this means

$$\zeta(X, d - s) = \pm q^{E(\frac{d}{2} - s)} \zeta(X, s) .$$

For a curve of genus g it is easily shown that $E = 2 - 2g$, and one obtains the classical type of functional equation, which relates s and $1 - s$).

III: One has

$$Z(X, T) = \frac{P_1(T) P_3(T) \ldots \qquad P_{2d-1}(T)}{P_0(T) P_2(T) \ldots \qquad P_{2d}(T)} ,$$

where $P_0(T) = 1 - T$, $P_{2d}(T) = 1 - q^d T$, and generally $P_i(X) \in \mathbb{Z}[T]$ with constant coefficient 1. Moreover one has

$$P_i(T) = \prod_{j=1}^{b_i}(1 - \alpha_j^{(i)} T) \quad \text{in} \quad \mathbb{C}[T] ,$$

with

$$\mid \alpha_j^{(i)} \mid = q^{\frac{i}{2}} \qquad \text{for all } j$$

This is the most difficult part - the analogue of the Riemann hypothesis for arbitrary dimensions.

IV: If X is obtained by reduction mod p (i.e., mod $\mathfrak{p}$ for a prime ideal $\mathfrak{p}|p$) of a variety over a number field $K \subseteq \mathbb{C}$, then $b_i = \deg P_i$ is equal to the i-th Bettinumber of $X(\mathbb{C})$ (the dimension of the i-th singular homology group of $X(\mathbb{C})$).

We add two remarks. Property III implies that there is no cancellation between the $P_i(T)$, so they are determined by $Z(X,T)$. In IV arithmetic properties are linked with topological invariants in an interesting way. For example, if X is a curve of genus g over $\overline{\mathbb{Q}}$, then $X(\mathbb{C})$ is a Riemannian surface with "g handles", and therefore $b_0 = 1 = b_2, b_1 = 2g$ (this coincides with the results of Hasse and Weil). The number of handles thus has consequences for the number of points mod p.

Indeed, for his conjectures Weil was guided strongly by topological considerations. In particular he noticed, that a big part of the conjecture (namely I, II and IV) would follow from the existence of a "good" cohomology theory, which suffices the usual topological formalism, such as the Lefschetz-fixpoint-formula, Poincaré-duality and so on. Such a cohomology theory was then discovered by M. Artin and A. Grothendieck with the étale cohomology, and this forms the basis for Deligne's proof, which we will study below.

We start with a couple of comments about the applications. The Weil-conjectures (i.e. Deligne's theorems) found many, totally different applications and are a central tool for many results in modern arithmetic geometry. Here we only sketch three applications that are very elementary and quite typical.

Application 1: (Weil) Is X a geometric irreducible smooth (projective) curve of genus g over $\mathbb{F}_q$, then

$$|X(\mathbb{F}_{q^n})| \leq q^n + 1 + 2g(\sqrt{q})^n \ .$$

Proof: By comparing coefficients of the power series for $\log Z(X,T)$ with the terms above, one gets

$$|X(\mathbb{F}_{q^n})| = 1 + q^n - \sum_{j=1}^{2g} \alpha_j^n \leq 1 + q^n + 2g(\sqrt{q})^n \ .$$

Generalizations for higher dimensional varieties are left to the reader, compare also [De 1] 8.1.

Application 2: (Hasse, Weil) For the Kloostermann-sum

$$K(p,a) := \sum_{x \in \mathbb{F}_p^\times} e^{\frac{2\pi i}{p}(x+\frac{a}{x})} \in \mathbb{C} \qquad (p \ prim, a \in \mathbb{Z})$$

one has the bound

$$|K(p,a)| \leq 2 \cdot \sqrt{p} \quad .$$

This follows by examination of the curve

$$T^p - T = x + \frac{a}{x} \quad .$$

More generally one gets estimates of the type

$$|\sum_{x_1,\ldots,x_n\in\mathbb{F}_q}\Psi(Q(x_1,\ldots,x_n))|\leq(d-1)^n q^{\frac{n}{2}}\quad ,$$

where Q is a polynomial of degree d in n variables and $\Psi:\mathbb{F}_q\to\mathbb{C}^\times$ is an additive character, see $[De1]$ and $[Ka1]$.

Application 3: (Deligne) The Ramanujan-conjecture: Let

$$\Delta=q\prod_{n=1}^{\infty}(1-q^n)^{24}=\sum_{n=1}^{\infty}\tau(n)q^n$$

be the Ramanujan-Δ-function. Then one has

$$(*)\qquad \tau(n)=O(n^{\frac{11}{2}+\epsilon})\qquad\text{for all }\epsilon>0\quad .$$

First a few words on the history. The following estimates were obtained before Deligne - all by analytical methods:

Ramanujan (1916)	$O(n^7)$
Hardy/Littlewood (1918)	$O(n^6)$
Kloostermann (1927)	$O(n^{\frac{47}{8}+\epsilon})$
Davenport/Salié (1933)	$O(n^{\frac{35}{6}+\epsilon})$
Rankin (1939)	$O(n^{\frac{29}{5}+\epsilon})$

For clarification: the last fractions are $6-\frac{1}{8}, 6-\frac{1}{6}$ and $6-\frac{1}{5}$; the conjecture requires $6-\frac{1}{2}$.

More precisely Ramanujan conjectured ([Ra]):

(A) τ is multiplicative, i.e., for $(n,n')=1$ one has $\tau(nn')=\tau(n)\tau(n')$.

(B) $|\tau(n)|\leq n^{\frac{11}{2}}\cdot d(n)$, where $d(n)$ is the sum of the divisors of n.

(C) For the associated Dirichlet series there is a product expansion of the form

$$\sum_{n=1}^{\infty}\frac{\tau(n)}{n^s}=\prod_p\frac{1}{1-\tau(p)p^{-s}+p^{11-2s}}\quad .$$

Further he noticed:

(i) (C) implies (A) (generally the coefficients of a Dirichelet series $\sum a_n n^{-s}$ are multiplicative, if they have an Euler product expansion).

(ii) (B) implies the conjecture $(*)$ above, by the known estimates for $d(n)$.

(iii) If (C) is known, it suffices to show (B) for prime numbers, i.e., to show that for prime numbers p one has

(B') $\qquad |\tau(p)|\leq 2\cdot p^{\frac{11}{2}}$,

because the Euler product also produces a recursion formula for $\tau(p^m)$.

(iv) Property (B') is equivalent to the fact that the zeros of the polynomial $1-\tau(p)T+p^{11}T^2$ are complex conjugates (the discriminant of the corresponding monic polynomial is $p^{-22}(\tau(p)^2-4p^{11})$).

It is remarkable to see that Ramanujan, who is known to many for his work in analytic number theory, here reduces everything to purely algebraic questions. Moreover, the conjecture was indeed proved according to the observations (i), (ii), (iii) and (iv):

If one writes

$$1 - \tau(p)T + p^{11}T^2 = (1 - \alpha_1 T)(1 - \alpha_2 T) \quad ,$$

then the zeros are complex conjugates, if their reciprocals α_1 and α_2 are. Since $\alpha_1 \cdot \alpha_2 = p^{11}$, this holds if and only if

$$| \alpha_1 |=| \alpha_2 |= p^{\frac{11}{2}} .$$

This was proved by Deligne, and (C) was already shown by Mordell in 1917.

By the way, the function Δ interested Ramanujan, since it is the q-expansion of an especially important modular form, and he formed similar conjectures for certain families of these. These conjectures follow from Deligne's results as well, because he proved more generally the Petersson-conjecture, which we will formulate briefly here, without going into the theory of the modula forms. Hecke showed in 1936, that for a normed cusp formula of weight k for $SL_2(\mathbb{Z})$ with q-expansion

$$f(z) = \sum_{n=1}^{\infty} a_n q^n \qquad (q = e^{2\pi i z}) ,$$

the associated Dirichlet series has a product development of the form

$$\sum_{n=1} \frac{a_n}{n^s} = \prod_p \frac{1}{1 - a_p p^{-s} + p^{k-1} - p^{-2s}} ,$$

if and only if f is an eigenform for all Hecke operators (s.[Se 1]). In this case Petersson conjectured in 1939 [Pet], that

$$a_n = O(n^{\frac{k-1}{2}+\epsilon}) \qquad \text{for all } \epsilon > 0 \quad .$$

As above it suffices to show: If one writes

$$1 - a_p T + p^{k-1}T^2 = (1 - \alpha_1 T)(1 - \alpha_2 T)$$

then

$$| \alpha_1 |=| \alpha_2 |= p^{\frac{k-1}{2}} \quad .$$

After preliminary work of Eichler, Ihara and Shimura, in 1969 Deligne [De1] reduced this statement to the Weil conjecture, by showing that the polynomial above divides the polynomial $P_{k-1}(T)$ for a smooth projective variety X over $\mathbb{F}_p$ - for Δ one has $k = 12$. For higher forms see [De 1]and [De 2].

1 Rationality of the zeta function

The rationality of the zeta functions was proved by p-adic methods by B. Dwork in 1960. In 1964, A. Grothendieck gave another proof, based on the étale cohomology developed by himself and M. Artin, which also gives the functional equation.

Theorem 1.1 (Grothendieck) Let X be a geometrically irreducible smooth projective variety of dimension d over $\mathbb{F}_q$.

(a) For any prime $\ell \neq p = char(\mathbb{F}_q)$ one has

$$Z(X,T) = \frac{P_1(T) \cdot P_3(T) \dots P_{2d-1}(T)}{P_0(T)P_2(T) \dots P_{sd}(T)} \ ,$$

where $P_0(T) = 1 - T, P_{2d}(T) = 1 - q^d T$ and generally

$$P_i(T) = det(1 - F^*T \mid H^i(\overline{X}, \mathbb{Q}_\ell)) \ ,$$

where $\overline{X} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ for an algebraic closure $\overline{\mathbb{F}_q}$ of $\mathbb{F}_q$, $H^i(\overline{X}, \mathbb{Q}_\ell)$ denotes the i-th ℓ-adic cohomology and F^* is the endomorphism which is induced on it by the q-linear Frobenius endomorphism $F : X \to X$.

(b) In particular $Z(X,T)$ is rational, i.e., in $\mathbb{Q}(T)$.

(c) One has the functional equation

$$Z(\frac{1}{q^d T}) = \pm q^{\frac{dE}{2}} T^E Z(T) \ ,$$

with the Euler-Poincaré-characteristic

$$E = \chi(X, \mathbb{Q}_\ell) := \sum_{i=0}^{2d} (-1) \dim_{\mathbb{Q}_\ell} H^i(\overline{X}, \mathbb{Q}_\ell) \ .$$

This is also the self intersection number $(\Delta.\Delta)$ of the diagonal $X \overset{\Delta}{\hookrightarrow} X \times X$.

Notation: $\mathbb{Z}/m$ or $\mathbb{Z}/m\mathbb{Z}$ also denotes the constant sheaf with this value on a scheme S with respect to the étale topology. For an étale sheaf F on S, let $H^i(S, F)$ be the i-th cohomology ($i \geq 0$). Then by definition one has

$$\begin{aligned} H^i(S, \mathbb{Z}_\ell) &= \varprojlim_n H^i(S, \mathbb{Z}/\ell^n) \ , \\ H^i(S, \mathbb{Q}_\ell) &= H^i(S, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \ . \end{aligned}$$

Note that $H^i(S, \mathbb{Z}/\ell^n)$ is a $\mathbb{Z}/\ell^n$-module, and so $H^i(S, \mathbb{Z}_\ell)$ is a module over the ring $\mathbb{Z}_\ell = \varprojlim_{\leftarrow, n} \mathbb{Z}/\ell^n$ of ℓ-adic integers and $H^i(S, \mathbb{Q}_\ell)$ is a vector space over the quotient field $\mathbb{Q}_\ell$ of ℓ-adic numbers.

We need the following facts about ℓ-adic cohomology, where $A = \mathbb{Z}/m$, $\mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$.

COH 1: Functoriality: A morphism $f : S \to S'$ induces an A-module homomorphism

$$f^* : H^i(S', A) \to H^i(S, A) \ .$$

For $g : S' \to S''$ one has $(gf)^* = f^*g*$. The absolute Galois group $Gal(k_s/k)$ acts continously on $H^i(X \times_k k_s\ , A)$ for a scheme X over a field k with separable closure k_s: For $\sigma \in Gal(k_s/k)$ take the action induced by $id \times \mathrm{Spec}(\sigma) : X \times_k k_s \to X \times_k k_s$.

COH 2: Cupproduct: There are A-bilinear pairings

$$H^i(S, A) \times H^j(S, A) \to H^{i+j}(S, A)\ ,\ (x, y) \mapsto x \cdot y\ .$$

These are graded commutative ($y \cdot x = (-1)^{ij} x \cdot y$) and (in an obvious sense) associative.

COH 3: Künneth formula: If X and Y are smooth and proper over a separably closed field L, and if $\ell \neq char\,(L)$, then one obtains isomorphisms

$$\bigoplus_{i+j=k} H^i(X, \mathbb{Q}_\ell) \otimes H^j(Y, \mathbb{Q}_\ell) \to H^k(X \times_L Y, \mathbb{Q}_\ell)$$

$$x \otimes y \mapsto p_1^* x \cdot p_2^* y\ ,$$

where $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are the projections.

COH 4: Poincaré duality: If X is smooth, proper and purely d-dimensional over a field k, and if $\overline{X} = X \times_k k_s$ for a separable closure k_s of k, then there is a canonical Galois-equivariant $\mathbb{Z}/\ell^n$-homomorphism for $\ell \neq char(k)$

$$tr : H^{2d}(\overline{X}, \mathbb{Z}/\ell^n)(d) \to \mathbb{Z}/\ell^n\ ,$$

and the pairing

$$H^i(\overline{X}, \mathbb{Z}/\ell^n)(j) \times H^{2d-i}(\overline{X}, \mathbb{Z}/\ell^n)(d-j) \to H^{2d}(\overline{X}, \mathbb{Z}/\ell^n)(d) \stackrel{tr}{\to} \mathbb{Z}/\ell^n$$

is a perfect duality. Here $M(m)$ denotes the m-th Tate twist of a $\mathbb{Z}/\ell^n - Gal(k_s/k)$-module: $M(m) = M \otimes \mathbb{Z}/\ell^n(m)$, with

$$\mathbb{Z}/\ell^n(m) = \begin{cases} \mu_{\ell^n}^{\otimes m} & m \geq 0 \\ (\mu_{\ell^n}^{\otimes -m})^\vee & m \leq 0 \end{cases}\ .$$

Here μ_{ℓ^n} is the Galois module of the ℓ^n-th root of unity in $k_s^\times$, and $M^\vee = Hom(M, \mathbb{Z}/\ell^n)$ is the $\mathbb{Z}/\ell^n$-dual of a $\mathbb{Z}/\ell^n$-$Gal(k_s/k)$-module M.

COH 5: Finiteness: If X is proper over a separably closed field L, then $H^i(X, A)$ is a finitely generated A-module for all $i \geq 0$, $A = \mathbb{Z}/\ell^n$, $\mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$, $\ell \neq char(L)$.

COH 6: Frobenius endomorphisms: Let X be of a finite type over $\mathbb{F}_q$. The $\mathbb{F}_q$-linear Frobenius-endomorphism

$$F : X \to X$$

is defined as the identity on the topological space X and the q-th power map on the structure sheaf. If $\varphi \in Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ is the arithmetic Frobenius:

$$\varphi(\alpha) = \alpha^q \ \text{ for } \alpha \in \overline{\mathbb{F}_q}\ ,$$

and F^* is the map induced by $F \times id : \overline{X} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q} \to X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ on the cohomology, then is

$$F^* = \varphi^{-1} \text{ on } H^i(\overline{X}, A)\ .$$

Proof of Theorem 1.1:

(a) $\Rightarrow$ (b):

Lemma 1.2 (Bourbaki Algèbre IV 3, Exercise 3) Let $u(T) = \sum\limits_{n=0}^{\infty} a_n\, T^n$ be a formal power series over a field K. Then $u(T)$ lies in $K(T)$ (i.e., is the Taylor expansion of a rational function), if and only if there is an $N > 0$ that the Hankel-determinants

$$det(a_{i+j+M})_{0\le i,j\le N} = det \begin{pmatrix} a_M & a_{M+1} & \dots & a_{M+N} \\ a_{M+1} & a_{M+2} & & \\ \vdots & & & \\ a_{M+N} & \dots & & a_{M+2N} \end{pmatrix}$$

vanish for all $M >> 0$.

From (a) we first obtain that $Z(X,T)$ lies in $\mathbb{Q}_\ell(T)$. Thus the Hankel-determinants of the coefficients vanish as in Lemma 1.2. But the coefficients already lie in $\mathbb{Q}$, and with the same criterion $Z(X,T)$ then lies in $\mathbb{Q}(T)$ (this proof shows: $\mathbb{Q}[[T]] \cap \mathbb{Q}_\ell(T) = \mathbb{Q}(T)$) .

Remark 1.3 This proof does not show that the above $P_i(T)$ lie in $\mathbb{Q}[T]$.

(a) $\Longrightarrow$ (c): According to Poincaré-duality COH 4 and finiteness COH 5 one has an isomorphism of Galois modules

$$H^i(\overline{X}, \mathbb{Q}_\ell)^\vee = H^{2d-i}(\overline{X}, \mathbb{Q}_\ell)(d)$$

(Notation: $M^\vee = Hom_{\mathbb{Q}_\ell}(M, \mathbb{Q}_\ell)$ for a $\mathbb{Q}_\ell$-vector space M, $M(m) = M \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(m)$ for a $\mathbb{Z}_\ell$-Galois module M, where $\mathbb{Z}_\ell(m) = \lim\limits_{\leftarrow, n} \mathbb{Z}/\ell^n(m))$. Since the arithmetic Frobenius φ operates on $\mathbb{Q}_\ell(m)$ by multiplication with q^m, we have

$$\begin{array}{rl} & det(1 - F\frac{1}{q^d T} \mid H^i) \\ = & (q^dT)^{-b_i} det(F \mid H^i) \cdot (-1)^{b_i} det(d - F^{-1} q^d T \mid H^i) \\ = & (q^dT)^{-b_i} det(F \mid H^i) \cdot (-1)^{b_i} det(1 - FT \mid H^{2d-i}), \end{array}$$

where $H^i = H^i(\overline{X}, \mathbb{Q}_\ell)$ and $b_i = \dim_{\mathbb{Q}_\ell} H^i$. If $\alpha_{1,\dots,}\alpha_{b_i}$ are the eigenvalues of F on H^i, then $q^d\alpha_1^{-1}, \dots, q^d\alpha_{b_i}^{-1}$ are the eigenvalues on H^{2d-i} by Poincaré-duality. Thus one has

$$det(F \mid H^i) \cdot det(F \mid H^{2d-i}) = q^{b_i \cdot d} \text{ for } i \neq d\ .$$

Finally consider $i = d$. Let N_+ (resp. N_-) be the numbers of eigenvalues of F on H^d equal to $q^{\frac{d}{2}}$ (resp. $-q^{\frac{d}{2}}$). The remaining eigenvalues form pairs $\beta \neq q^d\beta^{-1}$, so that $b_d - N_+ - N_-$ is even.

Therefore we have

$$\begin{array}{rcl} det(F, H^d) & = & q^{d(b_d - N_+ - N_-)/2} q^{N_+ + N_-)d/d} (-1)^{N_-} \\ & = & q^{db_d/2} (-1)^{N_-}\ , \end{array}$$

where one should note that db_d is always even, since the Poincaré pairing alternating on H^d for odd d. This implies

$$\begin{array}{rcl} Z(X, \frac{1}{q^dT}) & = & \prod_{i=0}^{2d} det(1 - F\frac{1}{d^dT} \mid H^i)^{(-1)^{i+1}} \\ & = & (q^dT)^\chi\, q^{\frac{-\chi d}{2}} (-1)^{N_+} \prod_{i=0}^{2d} det(1 - FT \mid H^{2d-i})^{(-1)^{i+1}} \\ & = & (-1)^{N_+} q^{\frac{\chi d}{2}} T^\chi Z(X,T), \end{array}$$

where $\chi = \sum_{i=0}^{2d}(-1)^i b_i$ is the Euler-Poincaré-characteristic.

For the interpretation of χ as an intersection number, we need the following result.

Theorem 1.4 (Lefschetz formula, first version) Write $(\alpha \cdot \beta)$ for the image of $\alpha \otimes \beta$ under the Poincaré-pairing for $X \times X$

$$H^{2d-r}(X \times X)(d) \times H^{2d-r}(X \times X)(d) \to H^{4d}(X \times X)(2d) \stackrel{tr}{\to} \mathbb{Q}_\ell ,$$

where we write $H^i(-)$ for $H^i(-, \mathbb{Q}_\ell)$. Then one has

$$(\alpha \cdot {}^t\beta) = \sum_{i=0}^{2d}(-1)^i \; tr(\beta \circ \alpha \mid H^i(X)) ,$$

where $\beta \mapsto {}^t\beta$ is the transposition, which is induced by the changing of the factors of $X \times X$, and on the right side β and α are interpreted as endomorphisms of the cohomology, by the isomorphisms

$$\begin{aligned} & H^{2d+r}(X \times X)(d) & \\ \cong & \bigoplus_{i=0}^{2d} H^{2d-i}(X)(d) \otimes H^{i+r}(X) & \text{(Künneth formula)} \\ \cong & \bigoplus_{i=0}^{2d} H^i(X)^\vee \otimes H^{i+r}(X) & \text{(Poincaré duality)} \\ \cong & \bigoplus_{i=0}^{2d} Hom(H^i(X), H^{i+r}(X)) & \text{(linear algebra)} \quad . \end{aligned}$$

Proof Without restriction, let $\alpha \in H^{2d-i}(X)(d) \otimes H^j(X)$ and $\beta \in H^{2d-j}(X)(d) \otimes H^i(X)$, for instance $\alpha = \sum_\ell a'_\ell \otimes b_\ell$ and $\beta = \sum_\ell c_\ell \otimes a_\ell$ with $(a'_\ell \cdot a_m) = \delta_{\ell m}$. Then one has

$$(\alpha \cdot {}^t\beta) = \sum_\ell (b_\ell \cdot c_\ell) \cdot a_\ell + \sum_{\ell' \neq \ell}(b_\ell \cdot c_{\ell'}) a_{\ell'} ,$$

thus $Tr(\beta \circ \alpha \mid H^i(X)) = (-1)^i \sum_\ell (b_\ell \cdot c_\ell) = (-1)^i(\alpha \cdot {}^t\beta)$.

Therefore we now only need

COH 7: Cycle map: There are homomorphisms

$$cl : CH^j(X) \longrightarrow H^{2j}(\overline{X}, \mathbb{Q}_\ell)(j)$$

where $CH^j(X)$ is the group of algebraic cycles of codimension j on X modulo rational equivalence, such that the intersection product $(x \cdot y)$ corresponds with the intersection number $(cl(x) \cdot cl(y))$.

For the diagonal $\Delta \subseteq X \times X$, which induces the identity on $H^*(X)$, we then obtain

$$(\Delta \cdot \Delta) = \sum_{i=0}^{2d} (-1)^i \; tr(id \mid H^i(X)) = \chi(X, \mathbb{Q}_\ell) .$$

With the same methods we now obtain two proofs of 1.1 (a):

First proof of 1.1 (a): Via the intersection theory of algebraic cycles one shows

$$|X(\mathbb{F}_{q^n})| = (F^n \cdot \Delta)\ ,$$

where F^n also stands for the graph of F^n in $X \times X$. Together with the Lefschetz-formula 1.4 above we obtain

Theorem 1.5 (Lefschetz-formula, second version)

$$|X(\mathbb{F}_{q^n})| = \sum_{i=0}^{2d} (-1)^i\ tr(F^n \mid H^i(\overline{X}, \mathbb{Q}_\ell))\,. \tag{1.5.1}$$

Furthermore one has the well-known formula

$$exp(\sum_{n=1}^{\infty}\ tr(\alpha^n \mid V)\frac{T^n}{n}) = det(1 - \alpha T \mid V)^{-1} \tag{1.5.2}$$

for an endomorphism α on a vector space V over a field L of the characteristic 0 (by considering the eigenvalues of α one only needs to prove the formula for a number α in an algebraic closure L, so the proposition follows by the equality $\sum_{n=1}^{\infty}\ \alpha^n\frac{T^n}{n} = -log(1-\alpha T)$). Now 1.1(a) obviously follows from (1.5.1) and (1.5.2).

Second proof of 1.1 (a): One proves Theorem 1.5 by purely cohomological methods. Indeed, one obtains the more general fact

COH 8 = Theorem 1.6 (Lefschetz-formula, third version) Let X be a seperated scheme of finite type over $\mathbb{F}_q$ and let $\mathcal{F}$ be a constructible $\mathbb{Q}_\ell$-sheaf on X. Then one has

$$\sum_{x \in \overline{X}^{F^n}} tr(F^n_x, \mathcal{F}_{\overline{x}}) = \sum_{i=0}^{2\dim(X)} (-1)^i tr(F^n \mid H^i_c(\overline{X}, \mathcal{F}))\ .$$

In particular, for $\mathcal{F} = \mathbb{Q}_\ell$ we get

$$|X(\mathbb{F}_{q^n})| = |\overline{X}^{F^n}| = \sum_{i=0}^{2\dim(X)} (-1)^i tr(F^n \mid H^i_c(\overline{X}, \mathbb{Q}_\ell))\ ,$$

and thus by the formula (1.5.2) above

$$\begin{aligned} Z(X,T) &= \exp\left(\sum_{n=1}^{\infty} \mid X(\mathbb{F}_{q_n}) \mid \frac{T^n}{n}\right) = \exp\left(\sum_{n=1}^{\infty}\sum_{i=1}^{2\dim X} (-1)^i tr(F^n \mid H^i_c(\overline{X}, F))\right) \\ &= \prod_{i=0}^{2\dim(X)} det(1 - FT \mid H^i_c(\overline{X}, \mathbb{Q}_\ell))^{(-1)^{i+1}}\ . \end{aligned}$$

The notations are explained in the next chapters.

2 Constructible sheaves

In the following, let all sheaves be sheaves for the étale cohomology.

Reminder 2.1 (compare [Mi]; esp. I §5 and V §1) Let Z be a scheme.

(a) A geometric point of Z is a morphism $\overline{x} \to Z$, where $\overline{x} = \mathrm{Spec}(\Omega)$ for a separably closed field Ω. Equivalent is the specification of a point $x \in Z$ (the image of $\overline{x}$), and an embedding of the residue field $k(x)$ into Ω.

(b) An étale neighborhood of $\overline{x}$ is a commutative diagram

$$\begin{array}{ccc} & & U \\ & \nearrow & \Big\downarrow \\ \overline{x} & & \\ & \searrow & \Big\downarrow \\ & & Z\ , \end{array}$$

where $U \to Z$ is étale. The étale neighborhoods of $\overline{x}$ form a projective system, where those neighborhoods form a cofinal system, for which U is affine, connected, with $U \to Z$ of finite type.

(c) The strict henselization of Z in $\overline{x}$ is defined as

$$\mathcal{O}_{Z,\overline{x}} = \varinjlim \Gamma(U, \mathcal{O}_U)\ ,$$

where U runs over the étale neighborhoods of $\overline{x}$. Then $\mathcal{O}_{Z,\overline{x}}$ is a strictly henselian ring, i.e., local, henselian, with a separably closed residue field.

(d) If $\mathcal{F}$ is an étale sheaf on Z, then the stalk of $\mathcal{F}$ in $\overline{x}$ is defined as

$$\mathcal{F}_{\overline{x}} = \varinjlim \mathcal{F}(U)\ ,$$

where U runs over the étale neighborhoods of $\overline{x}$ (In particular, $\mathcal{O}_{Z,\overline{x}}$ is the stalk of the ring sheaf $\mathbb{G}_a$ in $\overline{x}$).

(e) Let Z be connected and $\overline{x}$ a geometric point of Z. Define the functor

$$\phi = \phi_{\overline{x}} : \begin{pmatrix} \text{finite étale morphisms} \\ Z' \to Z \end{pmatrix} \begin{array}{c} \to \\ \\ \end{array} \begin{array}{c} \text{(finite sets)} \\ \\ \end{array}$$
$$\qquad\qquad Z' \qquad \mapsto \quad Hom_Z(\overline{x}, Z')\,.$$

and the profinite group

$$\pi_1(Z, \overline{x}) = Aut(\phi) = \varprojlim\ Aut_Z(Z')\ ,$$

where the limit runs over the finite étale morphisms $Z' \to Z$, i.e., over the finite étale Z-schemes Z'. Then the induced functor

$$\phi : \begin{pmatrix} \text{finite étale} \\ Z\text{-schemes} \end{pmatrix} \longrightarrow \begin{pmatrix} \text{finite discrete} \\ \pi_1(Z,\overline{x})\text{-sets} \end{pmatrix}$$

is an equivalence of categories. For a profinite group G a finite discrete G-set is a finite set M with an operation of G such that the stabilizer of any element $m \in M$ is open in G.

Definition 2.2 A sheaf $\mathcal{F}$ on Z is called **locally constant**, if there is an étale covering $(U_i \to Z)$ such that $\mathcal{F} \mid U_i$ is constant for all i.

Remark 2.3 If $\mathcal{F}$ in addition is a sheaf of groups and has finite stalks and Z is quasi-compact, one can deduce from the descent theory that $\mathcal{F}$ is given by a finite étale group scheme H over Z (i.e., $\mathcal{F}$ is isomorphic to the functor $U \mapsto Hom_Z(U, H)$). Conversely, if H is a finite étale group scheme, then the sheaf $\mathcal{F}$ represented by H is locally constant with finite stalks:

Without restriction Z is connected. Then H is connected, because $H \to Z$ is closed (as a finite morphism) and open (as an étale morphism of finite type). Let $\overline{x}$ be a geometric point of Z. Then H corresponds to a connected finite $\pi_1(Z, \overline{x})$-set $M = \pi_1(Z, \overline{x})/U$, where $U \subseteq \pi_1(Z, \overline{x})$ is an open subgroup. There is an open normal subgroup $N \subseteq \pi_1(Z, \overline{x})$ with $N \subseteq U$, which corresponds to an étale cover

$$H' \to H \to Z\,.$$

If $\overline{y}$ is a geometric point of H' over $\overline{x}$, then $\pi_1(H', \overline{y}) \cong N$, and the restriction of $M' = \pi_1(H, \overline{x})$ to $\pi_1(H', \overline{x})$ is trivial. This shows that the pullback $H' \times_Z H' \to H'$ of the cover $H' \to Z$ is trivial, as well as the pullback $H \times_Z H' \to H'$. So the restriction of $Hom_Z(-, H)$ to H' is constant.

According to the Yoneda-Lemma one obtains an equivalence of categories

$$\begin{pmatrix} \text{finite étale (commmutative)} \\ \text{group schemes over } Z \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{locally constant (abelian) sheaves} \\ \text{with finite stalks on } Z \end{pmatrix}$$
$$H \mapsto Hom_Z(-, H)$$

We now calculate the stalk of $\mathcal{F} = Hom_Z(-, H)$ at a geometric point $\overline{x} = \mathrm{Spec}(\Omega)$: One has

$$\begin{aligned} \mathcal{F}_{\overline{x}} &= \varinjlim Hom_Z(U, H) \xrightarrow[\sim]{(1)} Hom_Z(\mathrm{Spec}(\mathcal{O}_{Z,\overline{x}}), H) \\ &= Hom_{\mathrm{Spec}(\mathcal{O}_{Z,\overline{x}})}(\mathrm{Spec}(\mathcal{O}_{Z,\overline{x}}), H \times_Z \mathrm{Spec}(\mathcal{O}_{Z,\overline{x}})) \\ &\xrightarrow[\sim]{(2)} Hom_{\mathrm{Spec}(k(\overline{x}))}(\mathrm{Spec}(k(\overline{x})), H \times_Z Spec(k(\overline{x}))) \\ &= Hom_Z(\mathrm{Spec}(k(\overline{x})), H) \xrightarrow[\sim]{(3)} Hom_Z(\overline{x}, H) = \phi_{\overline{x}}(H)\,. \end{aligned}$$

Here the limit without restrictions runs over affine étale neighborhoods of $\overline{x}$, hence (1) is a bijection ([Mi] II 3.3), $k(\overline{x})$ is the residue field of the henselian ring $\mathcal{O}_{Z,\overline{x}}$, hence (2) is a bijection ([Mi] I 4.4), and (3) is a bijection, because for a point $y \in H$ over the image point x of $\overline{x}$ the residue extension $k(y)/k(x)$ is separable and hence $Hom_{k(x)}(k(y), k(\overline{x})) = Hom_{k(x)}(k(y), \Omega)$ for the separably closed fields $k(\overline{x})$ and Ω.

Together with the equivalence of categories in 2.1 (e) we get an equivalence of categories for connected, quasi-compact Z with a geometric point $\overline{x}$:

$$\begin{pmatrix} \text{locally constant abelian sheaves} \\ \text{on } Z \text{ with finite stalks} \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{finite discrete} \\ \pi_1(Z, \overline{x})\text{-modules} \end{pmatrix}$$
$$\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}\,.$$

There is another characterization of locally constant sheafs.

Definition 2.4 (a) A geometric point $\overline{x}$ of a scheme Z is called specialization of another geometric point $\overline{y}$ of Z, if there is a ring homomorphism over Z

$$\varphi : \mathcal{O}_{Z,\overline{x}} \to \mathcal{O}_{Z,\overline{y}}$$

between the strict henselizations. We then call φ (or Spec(φ)) a specialization morphism.
(b) If $\mathcal{F}$ is an étale sheaf on Z, one obtains a an associated **specialization morphism**

$$\varphi_* : \mathcal{F}_{\overline{x}} \longrightarrow \mathcal{F}_{\overline{y}}$$

as follows. Since $\mathrm{Spec}(\mathcal{O}_{Z,\overline{x}}) = \varprojlim_U U$, where U runs over the étale neighborhoods of $\overline{x}$, φ corresponds to an element of

$$\varprojlim_U Hom_Z(\mathrm{Spec}(\mathcal{O}_{Z,\overline{y}}),\ U)\ .$$

Let U be of a finite type over Z without restriction. Since we also have $\mathrm{Spec}(\mathcal{O}_{Z,\overline{y}}) = \varprojlim_V V$, where V runs over the étale neighborhoods of $\overline{y}$, which can be assumed to be affine without restriction, one has

$$Hom_Z(\mathrm{Spec}(\mathcal{O}_{Z,\overline{y}}), U) = \varinjlim_V Hom_Z(V, U)$$

(compare [Mi] II 3.3). Therefore φ corresponds to an element in

$$\varprojlim_U \varinjlim_V Hom_Z(V, U)\ ,$$

hence a morphism between pro-objects $(V) \to (U)$. This then induces a homomorphism

$$\mathcal{F}_{\overline{x}} = \varinjlim \mathcal{F}(U) \longrightarrow \varinjlim \mathcal{F}(V) = \mathcal{F}_{\overline{y}}\ .$$

Remark 2.5 (a) By using pullbacks of sheaves in the commutative diagram

$$f = \mathrm{Spec}(\varphi) : \qquad \begin{array}{ccc} \mathrm{Spec}(\mathcal{O}_{Z,\overline{y}}) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{Z,\overline{x}}) \\ & {\scriptstyle \pi'} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & Z & \end{array}$$

one also obtains $\varphi_* = f^*$ as the following composition:

$$\mathcal{F}_{\overline{x}} = (\pi^*\mathcal{F})_{\overline{x}} \overset{(1)}{=} (\pi^*\mathcal{F})(\mathcal{O}_{Z,\overline{x}}) \overset{f^*}{\to} (\pi'^*\mathcal{F})(\mathcal{O}_{Z,\overline{y}}) \overset{(2)}{=} (\pi'^*\mathcal{F})_{\overline{y}} = \mathcal{F}_{\overline{y}}\ ,$$

where the isomorphism (1) follows from the fact, that $\overline{x}$ has only trivial étale neighborhoods for the strict henselian ring $\mathcal{O}_{Z,\overline{x}}$, by the same argument this holds for (2) and $\overline{y}$. The middle arrow is induced by the (adjunction) map

$$\pi^*\mathcal{F} \to f_* f^* \pi^* \mathcal{F} = f_* \pi'^* \mathcal{F}\ .$$

(b) If $\mathcal{F}$ is a constant sheaf, with stalk A, then obviously φ_* is an isomorphism, because all U and V above can be chosen as connected; then all groups are equal to A and the maps are identities.

(c) Finally we remark that $\overline{x}$ is a specialization of $\overline{y}$, if the image point $x \in Z$ is a specialization of the image point y of $\overline{y}$, i.e., is contained in $\overline{\{y\}}$. In fact, then y is contained in the image of U for every étale neighborhood U of $\overline{x}$, so that the set $Hom_Z(\overline{y}, U)$ is non-empty and finite. Furthermore this set coincides with $Hom_Z(Spec\ \mathcal{O}_{Z,\overline{y}}, U)$, and the projective limit

$$Hom_Z(Spec\ \mathcal{O}_{Z,\overline{y}}, Spec\ \mathcal{O}_{Z,\overline{x}}) = \varprojlim_U Hom_Z(Spec\ \mathcal{O}_{Z,\overline{y}}, U)$$

of finite sets in non-empty.

Lemma 2.6 Let Z be a locally noetherian scheme. Then an étale sheaf $\mathcal{F}$ on Z with finite stalks is locally constant if and only if all specialization morphisms are bijective.

Proof One direction follows from 2.5 (b). Conversely let all specialization morphisms be isomorphisms. The question is local, so we can assume that Z is noetherian. Let $\overline{x}$ be a geometric point of Z, and let $A = \mathcal{F}_{\overline{x}}$. Then

$$A = \bigoplus_{i=1}^{r} \mathbb{Z}/m_i\mathbb{Z} \cdot t_i$$

with $t_1, \ldots, t_r \in A$ and $m_1, \ldots, m_r \in \mathbb{N}$.

There are an étale neighborhood U of $\overline{x}$ and sections $s_1, \ldots, s_r \in \mathcal{F}(U)$, which are mapped to $t_s, \ldots, t_r$. Further we can assume (by passing to a 'smaller' étale neighborhood), that s_i is annihilated by m_i. We obtain a morphism of étale sheaves

$$\psi_U : \mathcal{G} = \left(\bigoplus_{i=1}^{r} \mathbb{Z}/m_i\mathbb{Z}t_i \right)_U \to \mathcal{F}|_U ,$$

which maps the basis element t_i to s_i. Here $\mathcal{G}$ is the sheaf on U associated to A. φ_U induces an isomorphism of the stalks

$$\psi_{\overline{x}} : \mathcal{G}_{\overline{x}} \xrightarrow{\sim} \mathcal{F}_{\overline{x}} .$$

Let $Z_1, \ldots, Z_k$ be the irreducible components of Z which contain the image point x of $\overline{x}$, let $Z_{k+1}, \ldots, Z_m$ be the the remaining components and let $V \subseteq U$ be an open subset, that is generated by removal of the inverse images of $Z_{k+1}, \ldots, Z_m$. If then $\overline{\eta}_1, \ldots, \overline{\eta}_k$ are geometric points of V over the generic points $\eta_1, \ldots, \eta_k$ of $Z_1, \ldots, Z_k$, we obtain commutative diagrams

$$\begin{array}{ccc} \mathcal{G}_{\overline{\eta}_i} & \xrightarrow{\psi_{\overline{\eta}_i}} & \mathcal{F}_{\overline{\eta}_i} \\ \wr\uparrow & & \uparrow\wr \\ \mathcal{G}_{\overline{x}} & \xrightarrow[\sim]{\psi_{\overline{x}}} & \mathcal{F}_{\overline{x}} , \end{array}$$

where we have vertical isomorphisms by specialization transformations (for $\mathcal{G}$ by 2.5 (b), and for $\mathcal{F}$ by assumption). Therefore the $\psi_{\overline{\eta}_i}$ are isomorphisms. If $\overline{y}$ is now a any geometric point of V, also regarded as a geometric point of Z, then $\overline{y}$ is a specialization of (at least) one $\overline{\eta}_i$, and we obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{G}_{\overline{\eta}_i} & \xrightarrow[\sim]{\psi_{\overline{\eta}_i}} & \mathcal{F}_{\overline{\eta}_i} \\ \uparrow\wr & & \uparrow\wr \\ \mathcal{G}_{\overline{y}} & \xrightarrow{\psi_{\overline{y}}} & \mathcal{F}_{\overline{y}} \,. \end{array}$$

Therefore $\psi_{\overline{y}}$ is an isomorphism and, since $\overline{y}$ was arbitrary, $\psi : \mathcal{G}|_V \to \mathcal{F}|_V$ is an isomorphism.

Definition 2.7 A sheaf $\mathcal{F}$ on Z is called **constructible**, if any closed subscheme $Y \subset Z$ contains an open, non-empty subscheme $U \subset Y$ such that $\mathcal{F}|_U$ is locally constant with finite stalks.

Remark 2.8 If Z is noetherian, it is equivalent that there is a stratification $Z = \dot{\cup} Z_i$ by finite number of locally closed subschemes Z_i, such that $\mathcal{F}\mid_{Z_i}$ is locally constant with finite stalks for all i.

Examples 2.9 (a) Let ℓ be prime and $\mu_{\ell^n} = \ker(\mathbb{G}_m \xrightarrow{\ell^n} \mathbb{G}_m)$, i.e., the étale sheaf on Z with

$$\mu_{\ell^n}(U) = \{\alpha \in \Gamma(U, \mathcal{O}) \mid \alpha^{\ell^n} = 1\}$$

for U étale over Z. Then μ_{ℓ^n} is represented by

$$\mu_{\ell^n, Z} = \mathrm{Spec}(\mathbb{Z}[T]/(T^{\ell^n} - 1)) \times_{Spec(\mathbb{Z})} Z \quad ,$$

since one has

$$\begin{aligned} Hom_Z(U, \mu_{\ell^n, Z}) &= Hom(U, \mathrm{Spec}(\mathbb{Z}[T]/(T^{\ell^n} - 1))) \\ &= Hom_{Ringe}(\mathbb{Z}[T]/(T^{\ell^n} - 1), \Gamma(U, \mathcal{O})) \xrightarrow{\sim} \mu_{\ell^n}(U) \,, \end{aligned}$$

where the last image represents a ring homomorphism φ on the element $\varphi(T)$. If ℓ is invertible, then $\mu_{\ell^n, Z}$ is finite and étale over Z: Since these properties are respected by base change, it suffices to show that

$$\mu_{\ell^n, Z[\frac{1}{\ell}]} = \mathrm{Spec}(\mathbb{Z}[\frac{1}{\ell}][T]/(T^{\ell^n} - 1))$$

is finite and étale over $\mathbb{Z}[\frac{1}{\ell}]$ (Note that $Z \to \mathrm{Spec}(\mathbb{Z})$ factors through $\mathrm{Spec}\mathbb{Z}[\frac{1}{\ell}]$ by assumption). The finiteness is obvious, and μ_{ℓ^n} is étale, since the ideal generated by $T^{\ell^n} - 1$ and its derivative $\ell^n T^{\ell^n - 1}$ contains 1 if ℓ is invertible (compare the criterion [Mi] I 3.4). If ℓ is invertible, then μ_{ℓ^n} is a local constant sheaf with finite stalks. If $\overline{x} = \mathrm{Spec}(\Omega)$ is a geometric point of $\overline{x}$, then the stalk is calculated as follows:

$$(\mu_{\ell^n})_{\overline{x}} = \varinjlim \mu_{\ell^n}(U) = \mu_{\ell^n}(\mathcal{O}_{Z,\overline{x}}) \underset{\sim}{\longrightarrow} \mu_{\ell^n}(k(\overline{x})) = \mu_{\ell^n}(\Omega) \,.$$

Here U runs over the étale neighborhoods of $\overline{x}$, $k(\overline{x})$ is the residue field of $\mathcal{O}_{Z,\overline{x}}$, and the next to last arrow is an isomorphisms by Hensel's lemma.

(b) If $j : U \hookrightarrow Z$ is an open immersion and $\mathcal{F}$ is a constructible sheaf on U, then $j_!\mathcal{F}$, the extension by zero, is constructible on Z. If $\mathcal{F}$ is locally constant, then $j_!\mathcal{F}$ is not in general locally constant again, e.g., not if Z is connected and $\emptyset \neq U \neq Z$.

Lemma 2.10 If

$$f : \mathcal{F} \longrightarrow \mathcal{F}'$$

is a morphism of constructible sheaves on any scheme Z, then $\ker f$, $\operatorname{im} f$ and $\operatorname{coker} f$ are constructible.

Proof It suffices to prove the analogous proposition, where "constructible" is replaced by "constant with finite stalks". Then the claim is clear.

Lemma 2.11 Let Z be locally noetherian.

(a) Quotients and subsheaves of constructible sheaves are constructible again.

(b) Extensions of constructible sheafs are constructible again, i.e., if

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is an exact sequence of sheaves, with $\mathcal{F}'$ and $\mathcal{F}''$ constructible, then $\mathcal{F}$ is constructible as well.

(c) Tensor products of constructible sheaves are constructible again.

(d) If Z is noetherian, then the following properties are equivalent for a sheaf $\mathcal{F}$ on Z:

(i) $\mathcal{F}$ is constructible.

(ii) $\mathcal{F}$ is a noetherian torsion sheaf (i.e., a noetherian object in the category of torsion sheaves).

(iii) There is an $m \in \mathbb{N}$ and $j : U \to Z$ étale of finite type such that $\mathcal{F}$ is a quotient of $j_!(\mathbb{Z}/m)$. Here $j_! : Sh(U_{\text{ét}}) \to Sh(Z_{\text{ét}})$ is the left adjoint functor to $j^* : Sh(Z_{\text{ét}}) \to Sh(U_{\text{ét}})$, where $Sh(X_{\text{ét}})$ denotes the category of étale sheaves on X (see [Mi] II Remark 3.18).

Proof We first show (d).

(i) $\Rightarrow$ (ii): Without restriction Z is irreducible. Let $U \subset Z$ be open, non-empty, such that $\mathcal{F}$ is locally constant on U, and let $\overline{\eta}$ be a geometric point over a generic point η of U. According to 2.6, for all geometric points $\overline{x}$ the specialization morphisms

$$\mathcal{F}_{\overline{x}} \longrightarrow \mathcal{F}_{\overline{\eta}}$$

are isomorphisms. Let now $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3 \subseteq \dots$ be an ascending chain of subsheaves. We have to show that the sequence becomes constant. Since $\mathcal{F}_{\overline{\eta}}$ is finite, the sequence of $\mathcal{F}_{i,\overline{\eta}}$ becomes constant, thus is constant without restriction. The specialization morphisms

$$\mathcal{F}_{i,\overline{x}} \longrightarrow \mathcal{F}_{i,\overline{\eta}}$$

are injective by the diagram

$$\begin{array}{ccc} \mathcal{F}_{i,\overline{x}} & \longrightarrow & \mathcal{F}_{i,\overline{\eta}} \\ \downarrow & & \downarrow \cong \\ \mathcal{F}_{\overline{x}} & \xrightarrow{\sim} & \mathcal{F}_{\overline{\eta}} . \end{array}$$

Let $s_1, \dots, s_r$ be generators of $\mathcal{F}_{1,\overline{\eta}}$, and let V be an étale neighborhood of $\overline{\eta}$ so that $s_1, \dots, s_r$ come from sections in $\mathcal{F}_1(V)$. Then for $\overline{x}$ over V the maps of stalks $\mathcal{F}_{1,\overline{x}} \longrightarrow \mathcal{F}_{i,\overline{x}}$ are bijective, by the commutative diagram

$$\begin{array}{ccccc} \mathcal{F}_1(V) & \twoheadrightarrow & \mathcal{F}_{1,\overline{\eta}} & \xrightarrow{\sim} & \mathcal{F}_{i,\overline{\eta}} \\ & \searrow & \uparrow & & \uparrow \\ & & \mathcal{F}_{1,\overline{x}} & \longrightarrow & \mathcal{F}_{i,\overline{x}} \, . \end{array}$$

Hence the sequence of the $\mathcal{F}_i$ is constant on the open image V' of V in Z. By noetherian induction one now can show that the sequence becomes constant on the closed complement $Z - V'$, and thereby the claim.

(ii) $\Rightarrow$ (iii): If $j: \; U \to Z$ is étale, $j_!$ is left adjoint to j^*; in particular one has

$$Hom_Z(j_!(\mathbb{Z}/m), \mathcal{F}) = Hom_U(\mathbb{Z}/m, j^*\mathcal{F}) = {}_m\mathcal{F}(U)\,,$$

where ${}_mA = \{a \in A \mid ma = 0\}$ for a abelian group A. If $\overline{x}$ is a geometric point of Z and $f \in \mathcal{F}_{\overline{x}}$, one hence obtains $m \in \mathbb{N}$ and U as above, such that f lies in the image of a morphism $j_!\mathbb{Z}/m \to \mathcal{F}$ (This means that the sheaves $j_!\mathbb{Z}/m$ are a family of generators in the category of torsion sheaves). If now $\mathcal{F}$ is noetherian, then there are finitely many $U_1, \dots, U_r$ and $m_1, \dots, m_r$ and a surjective morphism

$$\bigoplus_{i=1}^{r} (j_i)_!\mathbb{Z}/m_i \twoheadrightarrow \mathcal{F}\,.$$

The claim now follows with $U = \coprod U_i$ and $m = LCM(m_i)$.

(iii) $\Longrightarrow$ (i) Choose a surjection

$$\varphi : j_!\mathbb{Z}/m \twoheadrightarrow \mathcal{F}\ .$$

Since $j_!\mathbb{Z}/m$ is obviously constructible, hence noetherian, by the same arguments as above we get an epimorphism

$$j'_!\mathbb{Z}/m \twoheadrightarrow \ker\varphi$$

for an étale morphism of finite type $j' : U' \to Z$. By the claim shown at the beginning of the proof $\mathcal{F}$ is constructible, as cokernel of a morphism

$$j'_!\mathbb{Z}/m \to j_!\mathbb{Z}/m\ .$$

From this we easily obtain (a): The claim on quotients follows at once with the criterion (c) (iii); hence the subsheaf $\mathcal{F}'$ of a constructible sheaf can be constructed as the kernel of the morphism of constructible sheafs $\mathcal{F} \twoheadrightarrow \mathcal{F}/\mathcal{F}'$.

(b): It suffices to show the corresponding proposition for locally constant sheaves. Let

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

be an exact sequence of sheaves. If $\mathcal{F}'$ and $\mathcal{F}''$ have finite stalks, this obviously holds for $\mathcal{F}$. Let $\mathcal{F}'$ and $\mathcal{F}''$ be locally constant. If $\overline{x}$ and $\overline{y}$ are geometric points of Z, and if $\varphi : \mathcal{O}_{Z,\overline{x}} \to \mathcal{O}_{Z,\overline{y}}$ is a specialization morphism, we obtain a commutative diagram with exact lines

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}'_{\overline{x}} & \longrightarrow & \mathcal{F}_{\overline{x}} & \longrightarrow & \mathcal{F}''_{\overline{x}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\varphi_*} & & \downarrow{\scriptstyle\varphi_*} & & \downarrow{\scriptstyle\varphi_*} & & \\ 0 & \longrightarrow & \mathcal{F}'_{\overline{y}} & \longrightarrow & \mathcal{F}_{\overline{y}} & \longrightarrow & \mathcal{F}''_{\overline{y}} & \longrightarrow & 0\,. \end{array}$$

If the vertical specialization morphisms φ_* are isomorphisms for $\mathcal{F}'$ and $\mathcal{F}''$, then this holds for $\mathcal{F}$ as well by the five lemma. Thus the claim follows with Lemma 2.6.

(c): It suffices to show this for locally constant and then for constant sheaves, and the proposition follows.

3 Constructible $\mathbb{Z}_\ell$-sheaves

Definition 3.1 (see SGA 5 VI) (a) A $\mathbb{Z}_\ell$-sheaf $\mathcal{F}$ on a scheme Z is a projective system

$$\ldots \to \mathcal{F}_{n+1} \to \mathcal{F}_n \to \ldots \to \mathcal{F}_1$$

of sheaves on Z such that the following holds:

(i) $\mathcal{F}_n$ is annihilated by ℓ^n and hence is a $\mathbb{Z}/\ell^n$-sheaf,

(ii) $\mathcal{F}_{n+1}/\ell^n\mathcal{F}_{n+1} \xrightarrow{\sim} \mathcal{F}_n$ is an isomorphism. Below we write $\mathcal{F} = (\mathcal{F}_n)$.

(b) Morphisms of $\mathbb{Z}_\ell$-sheaves are morphisms of projective systems, therefore commutative diagrams

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & \mathcal{F}_{n+1} & \longrightarrow & \mathcal{F}_n & \longrightarrow & \ldots & \longrightarrow & \mathcal{F}_1 \\ & & \downarrow & & \downarrow & & & & \downarrow \\ \ldots & \longrightarrow & \mathcal{G}_{n+1} & \longrightarrow & \mathcal{G}_n & \longrightarrow & \ldots & \longrightarrow & \mathcal{G}_1 \end{array}$$

(Because of (a) (ii) we have:

$$Hom((\mathcal{F}_n), (\mathcal{G}_n)) = \varprojlim_n \; Hom(\mathcal{F}_n, \mathcal{G}_n) \quad ,$$

where the transition maps are given by

$$Hom(\mathcal{F}_{n+1}, \mathcal{G}_{n+1}) \longrightarrow Hom(\mathcal{F}_{n+1}, \mathcal{G}_n) \underset{\text{(ii)}}{\xrightarrow{\sim}} Hom(\mathcal{F}_n, \mathcal{G}_n)) \, .$$

(c) As objects, $\mathbb{Q}_\ell$-sheaves are the same as $\mathbb{Z}_\ell$-sheaves, only the sets of morphisms are tensored by $\mathbb{Q}_\ell$.

(d) (naive definition) The cohomology of a $\mathbb{Z}_\ell$-sheaf $\mathcal{F} = (\mathcal{F}_n)$ is defined as

$$H^i(Z, \mathcal{F}) = \varprojlim_n H^i(X, \mathcal{F}_n) \quad .$$

The stalk at a geometric point $\overline{x}$ of Z is defined as

$$\mathcal{F}_{\overline{x}} = \varprojlim_n (\mathcal{F}_n)_{\overline{x}} \quad .$$

For $\mathbb{Q}_\ell$-sheaves one tensorizes these groups with $\mathbb{Q}_\ell$ over $\mathbb{Z}_\ell$.

(e) A $\mathbb{Z}_\ell$- or $\mathbb{Q}_\ell$-sheaf $\mathcal{F}$ is called **twisted constant**, if the components $\mathcal{F}_n$ are locally constant sheaves. $\mathcal{F}$ is called **constructible**, if the components are constructible. Constructible twisted constant sheaves are also called **smooth**.

(f) The tensor product of two $\mathbb{Z}_\ell$- (or $\mathbb{Q}_\ell$-)sheaves $\mathcal{F}$ and $\mathcal{G}$ is defined by $\mathcal{F} \otimes \mathcal{G} = (\mathcal{F}_n \otimes \mathcal{G}_n)$, with the tensor product of the transition maps. The dual is defined by $\mathcal{F}^\vee = (\mathcal{F}_n^\vee)$, with the $\mathbb{Z}/\ell^n$-dual $\mathcal{F}_n^\vee = \underline{Hom}(\mathcal{F}_n, \mathbb{Z}/\ell^n)$. Here $\underline{\text{Hom}}$ is the sheaf-Hom, and the transition maps are formed similarly as in (b).

(g) A sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

of $\mathbb{Z}_\ell$-sheaves (resp. $\mathbb{Q}_\ell$-sheaves) is exact, if the associated sequence of stalks is exact for all geometric points $\overline{x}$ of Z.

Example 3.2 For ℓ invertible on Z and $m \in \mathbb{Z}$ set $\mathbb{Z}_\ell := (\mathbb{Z}/\ell^n(m))$ with the obvious transition maps, where

$$\mathbb{Z}/\ell^n(m) = \begin{cases} \mu_{\ell^n}^{\otimes m} & m \geq 0, \\ (\mu_{\ell^n}^{\otimes |m|})^\vee & m < 0 \end{cases} .$$

Then $\mathbb{Z}_\ell = \mathbb{Z}_\ell(0)$ is constant and $\mathbb{Z}_\ell(m)$ is a smooth $\mathbb{Z}_\ell$-sheaf for all $m \in \mathbb{Z}$ by example 2.9 (a). Obviously we have $\mathbb{Z}_\ell(m)^\vee = \mathbb{Z}_\ell(-m)$ and $\mathbb{Z}_\ell(m) \otimes \mathbb{Z}_\ell(n) = \mathbb{Z}_\ell(m+n)$.

Proposition 3.3 Let Z be locally noetherian.

(a) A $\mathbb{Z}_\ell$-sheaf $\mathcal{F}$ on Z is constructible if and only if $\mathcal{F}_1$ is constructible.

(b) A $\mathbb{Z}_\ell$-sheaf $\mathcal{F}$ on Z is constructible if and only if for every closed subscheme $Y \subset Z$ there is an open non-zero subscheme $U \subset Y$ such that $\mathcal{F}\mid_U$ is smooth.

(c) If Z is noetherian and connected, and $\overline{x} \to X$ is a geometric point, then there is an equivalence of categories

$$\begin{array}{ccc} (\text{smooth } \mathbb{Z}_\ell\text{-sheaves on } Z) & \leftrightarrow & \begin{pmatrix} \text{finitely generated } \mathbb{Z}_\ell\text{-modules} \\ \text{with continuous operation of } \pi_1(Z,\overline{x}) \end{pmatrix} \\ \mathcal{F} & \mapsto & \mathcal{F}_{\overline{x}}\,. \end{array}$$

The same holds, if $\mathbb{Z}_\ell$ is replaced by $\mathbb{Q}_\ell$.

Proof (a): The condition is local, so Z is noetherian without restriction. Let $\mathcal{F}_1$ be constructible. We show by induction over n, that all $\mathcal{F}_n$ are constructible. If this is already proved for n, consider the exact sequences

$$(3.3.1) \qquad 0 \to \ell^n\mathcal{F}_{n+1} \to \mathcal{F}_{n+1} \to \mathcal{F}_{n+1}/\ell^n\mathcal{F}_{n+1} \cong \mathcal{F}_n \to 0$$

$$(3.3.2) \qquad \mathcal{F}_1 \cong \mathcal{F}_{n+1}/\ell\,\mathcal{F}_{n+1} \xrightarrow{\ell^n} \ell^n\mathcal{F}_{n+1} \to 0\,,$$

where the first isomorphism in (3.3.2) follows by iteration from 3.1 (ii):

$$\mathcal{F}_{n+1}/\ell\mathcal{F}_{n+1} \tilde{\to} \mathcal{F}_n/\ell\mathcal{F}_n \tilde{\to} \ldots \tilde{\to} \mathcal{F}_2/\ell\mathcal{F}_2 \tilde{\to} \mathcal{F}_1\,.$$

Then Lemma 2.10 implies that $\ell^n\mathcal{F}_{n+1}$ and $\mathcal{F}_{n+1}$ are constructible.

(b): For a $\mathbb{Z}_\ell$-sheaf $\mathcal{F}$ on Z let

$$gr^r\mathcal{F} := Ker(\mathcal{F}_r \longrightarrow \mathcal{F}_{r-1}) = \ell^{r-1}\mathcal{F}_r$$

for $r \in \mathbb{N}$ (where $\mathcal{F}_0 := 0$). Then

$$gr\mathcal{F} := \bigoplus_{r\geq 1} gr^r\mathcal{F}$$

is a graded $\mathbb{F}_\ell[T]$-module as follows: define

$$T^s: \qquad gr^r\mathcal{F} \longrightarrow gr^{r+s}\mathcal{F}$$

as the composition of the morphisms

$$\ell^{r-1}\mathcal{F}_r \cong \ell^{r-1}(\mathcal{F}_{r+s}/\ell^r\mathcal{F}_{r+s}) \xrightarrow{\cdot \ell^s} \ell^{r+s-1}\mathcal{F}_{r+s} \quad ,$$

and extend the operation to an $\mathbb{F}_\ell$-linear operation of $\mathbb{F}_\ell[T]$. (Explanation: if $\mathcal{F}$ really came from an object with $\mathbb{Z}_\ell$-operation, then we had $\mathcal{F}_r = \mathcal{F}/\ell^r\mathcal{F}$, and the usual operation of $gr\ \mathbb{Z}_p = \bigoplus_{r\geq 0} \ell^r\mathbb{Z}_p/\ell^{r+1}\mathbb{Z}_p \cong \mathbb{F}_\ell[T]$ on $gr\ \mathcal{F} = \bigoplus_{r\geq 0} \ell^r\mathcal{F}/\ell^{r+1}\mathcal{F}$). The surjections constructed in the proof of (a) $\mathcal{F}_1 \longrightarrow gr^p\mathcal{F}$ define a surjection of graded $\mathbb{F}_\ell[T]$-sheaves

$$\varphi : \mathbb{F}_\ell[T] \otimes \mathcal{F}_1 \twoheadrightarrow gr\ \mathcal{F} \quad .$$

Now we apply the well-known

Hilbert-Lemma 3.4 If $\mathcal{F}_1$ is a noetherian sheaf, then $\mathbb{F}_\ell[T] \otimes \mathcal{F}_1$ is noetherian as a graded $\mathbb{F}_\ell[T]$-sheaf (this is more generally true for an object in an abelian category).

The proof of this claim is easily obtained by examination of a double filtration in $\mathcal{F}_1$, compare SGA 5 V 5.1.4.

As in the proof of Lemma 2.10 (c), one obtains a surjection of graded $\mathbb{F}_\ell[T]$-sheaves

$$\mathbb{F}_\ell[T] \otimes \mathcal{G} \twoheadrightarrow Ker\ \varphi \quad ,$$

with a constructible, graduated sheaf $\mathcal{G}$ on Z. Then $gr\mathcal{F}$ is the cokern of

$$\mathbb{F}_\ell[T] \otimes \mathcal{G} \longrightarrow \mathcal{F}_q[T] \otimes \mathcal{F}_1 \quad .$$

As $\mathcal{F}_1$ and $\mathcal{G}$ are constructible, it follows that $gr\mathcal{F}$ is constructible, in the sense that there exists an open U in Y for every closed Y in Z, so that $gr\mathcal{F}$ restricted to U is locally constant, i.e., this holds true for all $gr^p\mathcal{F}$. Since locally constant sheaves are closed under extensions (see the proof of Lemma 2.10 (b)), the claim follows: all $\mathcal{F}_n$ are locally constant on U.

The conclusion (c) of Proposition 3.3 is clear; one notes that one has the following equivalence of categories:

$$\begin{pmatrix} \text{finitely generated} \\ \mathbb{Z}_\ell\text{-modules with} \\ \text{continuous operation of } \pi_1(Z,\overline{x}) \end{pmatrix} \leftrightarrow \begin{pmatrix} \ell\text{-adic projective} \\ \text{systems of finite} \\ \text{discrete } \pi_1(Z,\overline{x})\text{-modules} \end{pmatrix}$$

$$\begin{array}{ccc} M & \mapsto & (M/\ell^n M) \\ \lim\limits_{\longleftarrow, n}(M_n) & \leftarrow\!\shortmid & (M_n)\,. \end{array}$$

Here an ℓ-adic projective system in an abelian category is a projective system

$$\cdots \to A_{n+1} \to A_n \to \cdots \qquad \to A_1$$

with (i) $\ell^n A_n = 0$ and (ii) $A_{n+1}/\ell^n A_{n+1} \xrightarrow{\sim} A_n$. Finally, from the category and the left hand side one obtains the category of the $\mathbb{Q}_\ell$-representations of $\pi_1(Z,\overline{x})$ of the left category, i.e., the finite dimensional $\mathbb{Q}_\ell$-vector space with continuous operation of $\pi_1(Z,\overline{x})$, if one tensors the sets of homomorphisms over $\mathbb{Z}_\ell$ with $\mathbb{Q}_\ell$.

4 Cohomology with compact support

We record further definitions and properties of étale cohomology, which we need for understanding the Lefschetz formula COH 8/Theorem 1.6.

COH 9 Cohomology with compact support: Let X be separated of finite type over a field k. By Nagata, there is an open immersion $\mu : X \hookrightarrow X_1$ into a proper k-scheme X_1, and for a torsion sheaf $\mathcal{F}$ on X one defines the cohomology with compact support by

$$H_c^i(X, \mathcal{F}) := H^i(X_1, \mu_! \mathcal{F}) ,$$

where $\mu_! \mathcal{F}$ is the extension by zero of $\mathcal{F}$ to X_1: $\mu_! \mathcal{F}$ is associated to the presheaf $\mu_!^P \mathcal{F}$:

$$\mu_!^P \mathcal{F}(V) = \begin{cases} \mathcal{F}(V) & V \to X_1 \text{ factorizes over } X, \\ 0 & \text{otherwise} . \end{cases}$$

Lemma 4.1 (a) $H_c^i(X, \mathcal{F})$ does not depend on the choice of the "compactification" $\mu : X \hookrightarrow X_1$.

(b) $\mathcal{F} \mapsto H_c^i(X, \mathcal{F})$ is an exact δ-functor.

(c) If $i : Z \hookrightarrow X$ is closed with open complement $j : U \hookrightarrow X$, one has a long exact sequence

$$\ldots \to H_c^{i-1}(Z, \mathcal{F}) \to H_c^i(U, \mathcal{F}) \to H_c^i(X, \mathcal{F}) \to H_c^i(Z, \mathcal{F}) \to H_c^{i+1}(U, \mathcal{F}) \to \ldots ,$$

where the restrictions $j^* \mathcal{F}$, $i^* \mathcal{F}$ are again denoted by $\mathcal{F}$.

To prove (a), one needs

COH 10 Proper base change: Let $f : X \to Y$ be proper.

(a) If $\mathcal{F}$ is constructible on X, then $R^i f_* \mathcal{F}$ is constructible for all $i \geq 0$.

(b) For a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

and a torsion sheaf $\mathcal{F}$ on X, the base change morphism

$$g^* R^i f_* \mathcal{F} \longrightarrow R^i f'_* g'^* \mathcal{F}$$

is an isomorphism for all $i \geq 0$.

Remarks 4.2 (a) The complicated part is (a); part (b) follows easily, compare [Mi] VI §2.

(b) For $i = 0$ the base change morphism is defined as follows: since g^* is left adjoint to g_* it suffices to define a morphism

$$f_* \mathcal{F} \longrightarrow g_* f'_* g'^* \mathcal{F} = f_* g'_* g'^* \mathcal{F} ;$$

and we define it by applying f_* to the adjunction morphism

$$\mathcal{F} \longrightarrow g'_* g'^* \mathcal{F} .$$

For $i \geq 0$ one obtains the morphism by considering an injective resolution of $\mathcal{F}$.

(c) Let X be a quasi-compact scheme, let k be a field with separable closure k_s and let $f : X \to \mathrm{Spec}(k)$ be a morphism. If $\mathcal{F}$ is a sheaf on X and $\overline{x}$ denotes the geometric point $\mathrm{Spec}(k_s) \to \mathrm{Spec}(k)$, one has a canonical isomorphism

$$(R^i f_* \mathcal{F})_{\overline{x}} \cong H^i(X \times_k k_s, \overline{x}^* \mathcal{F}) ,$$

where $\overline{x}$ also stands for the base change $X \times_k k_s \to X$ of $\overline{x}$. Since $R^i f_* \mathcal{F}$ is the associated sheaf to the presheaf $(g : U \to \mathrm{Spec}(k)) \longmapsto H^i(X \times_k U, g^* \mathcal{F})$, and this presheaf has the same stalks, the claim follows from the compatibility of cohomology with limits (compare [Mi]III 1.16), namely from the equalities

$$\varinjlim_K H^i(X \times_k K, h_K^* \mathcal{F}) = H^i(X \times_k k_s, \overline{x}^* \mathcal{F}) ,$$

where $h_K : \mathrm{Spec}(K) \to \mathrm{Spec}(k)$ runs through the connected étale neighborhoods of $\overline{x}$, hence the finite separable extensions K of k inside k_s.

In particular, it follows from COH 10 (a) that for proper f and constructible $\mathcal{F}$ on X the group $H^i(X \times_k k_s, \overline{x}^* \mathcal{F})$ is finite. As a special case, viz $k = k_s, \mathcal{F} = \mathbb{Z}/\ell^n$, we obtain COH 5.

(d) If $f : X \to Y$ is proper, $\overline{y} \to Y$ is a geometric point, and if

$$\begin{array}{ccc} X_{\overline{y}} & \xrightarrow{\pi'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \overline{y} & \xrightarrow{\pi} & Y \end{array}$$

is a cartesian diagram, then it follows from (c) and COH 10 (b) that for torsion sheaves $\mathcal{F}$ on X there are canonical isomorphisms

$$(R^i f_* \mathcal{F})_{\overline{y}} \quad \cong \quad H^i(X_{\overline{y}}, (\pi')^* \mathcal{F}) ,$$

since the left hand side is isomorphic to $(\pi^* R^i f_* \mathcal{F})_{\overline{y}}$ and the right hand side is isomorphic to $(R^i f'_* \pi'^* \mathcal{F})_{\overline{y}}$.

Proof of 4.1: We only prove (a); (b) and (c) follow easily from the exactness of $\mu_!$, see [Mi] III 1.29.

Let $\nu : X \hookrightarrow X_2$ be another compactification of X. By considering the closure of X in $X_1 \times X_2$, one can assume without restriction that there is a morphism $g : X_1 \to X_2$ with $g\mu = \nu$. Then the claim follows from the Leray-spectral sequence for $\mu_! \mathcal{F}$,

$$E_2^{p,q} = H^p(X_2, g_* \mu_! \mathcal{F}) \Rightarrow H^{p+q}(X_1, \mu_! \mathcal{F}) ,$$

if one shows

$$R^q g_* \mu_! \mathcal{F} = \begin{cases} \nu_! \mathcal{F} & q = 0, \\ 0 & q > 0 \end{cases} .$$

It suffices to show this on the stalks in a geometric point $\overline{x}$ of X_2 (for $q = 0$ note that $g_*\mu_!\mathcal{F}$ and $\nu_!\mathcal{F}$ are subsheaves of $g_*\mu_*\mathcal{F} = \nu_*\mathcal{F}$). But by proper base change (see 4.2 (d)) we get

$$(R^q g_*\mu_!\mathcal{F})_{\overline{x}} = H^q(X_{1,\overline{x}}, \mu_!\mathcal{F}|X_{1,\overline{x}}) = \begin{cases} \mathcal{F}_{\overline{x}} & q = 0 \quad \text{and} \quad x \in X\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

since $\mu_!\mathcal{F}|X_{1,\overline{x}} = 0$, if the image x of $\overline{x}$ in X_2 is not in X, and since $X_{1,\overline{x}}$ consists only of the $\overline{x}$, if x is in X.

Everything carries over to $\mathbb{Z}_\ell$- and $\mathbb{Q}_\ell$-sheaves. In particular, for a proper scheme X_1 of finite type over a separably closed field L and a constructible $\mathbb{Z}_\ell$-(resp. $\mathbb{Q}_\ell$-)sheaf $\mathcal{F}$ on X, the cohomology $H^q(X_1, \mathcal{F})$ is a finitely generated $\mathbb{Z}_\ell$-(resp. $\mathbb{Q}_\ell$-)module. If $j : X \hookrightarrow X_1$ is an open immersion and $\mathcal{F}$ a (constructible) $\mathbb{Z}_\ell$-(resp. $\mathbb{Q}_\ell$)-sheaf on X, then this also holds for $j_!\mathcal{F}$ on X_1. It follows that $H^q_c(X, \mathcal{F}) = H^q(X_1, j_!\mathcal{F})$ is a finitely generated $\mathbb{Z}_\ell$-(bzw. $\mathbb{Q}_\ell$-)module.

5 The Frobenius-endomorphism

To explain the last notations of 1.6, we need to consider the following functoriality.

For every morphism $f : X' \to X$ of schemes, one has a homomorphism

(5.1.1) $$H^i(X, \mathcal{F}) \longrightarrow H^i(X, f^*\mathcal{F}),$$

defined by the composition

$$H^i(X, \mathcal{F}) \xrightarrow{\alpha} H^i(X, f_*f^*\mathcal{F}) \xrightarrow{\beta} H^i(X', f^*\mathcal{F}),$$

where α is induced by the adjunction morphism $\mathcal{F} \to f_*f^*\mathcal{F}$ and β is the edge morphism for the Leray-spectral sequence

$$E_2^{p,q} = H^p(X, R^q f_*\mathcal{G}) \Rightarrow H^{p+q}(X', \mathcal{G}),$$

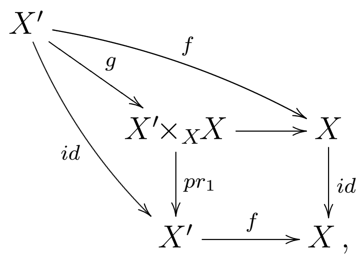
for $\mathcal{G} = f^*\mathcal{F}$. Alternatively,

$$\beta : H^0(X, f_*\mathcal{G}) = H^0(X' \times_X X, \mathcal{G}) \xrightarrow{\sim} H^0(X', \mathcal{G})$$

is defined by the canonical identification $X' \times_X X = X'$ and extended to higher cohomology groups by looking at injective resolutions of $\mathcal{G}$ and $f_*\mathcal{G}$.

By the commutative diagram

(5.1.2)



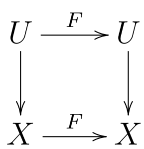
one can identify $X' \times_X X$ with X' by the morphisms pr_1 and g (which are inverse to each other).

Now we consider the case that X is a scheme over $\mathbb{F}_q$, $\mathcal{F}$ is a (usual or $\mathbb{Z}_\ell$- or $\mathbb{Q}_\ell$-) sheaf on X, and $f = F : X \to X$ is the q-Frobenius.

Lemma 5.1 There is a canonical isomorphism

$$F^*_{/X} : \mathcal{F} \xrightarrow{\sim} F_*\mathcal{F}.$$

Proof: Let U be étale over X, then the commutative diagram



induces a morphism of X-schemes

$$F_{U/X} : U \longrightarrow X_F {\times}_X U\,.$$

Since $U \to X$ and therefore $X_F {\times}_X U \to X$ are étale, $F_{U/X}$ is étale ([Mi] I 3.6), and since F is entire and radicial, one can easily see that $F_{U/X}$ is a isomorphism (compare SGA 5 XV). We obtain an isomorphism

$$F^*_{U/X} : (\mathcal{F})(U) = \mathcal{F}(X_F {\times}_X U) \xrightarrow{\sim} \mathcal{F}(U)\,,$$

which is functorial in U, and therefore the wanted isomorphism.

By adjunction, $(F^*_{/X})^{-1}$ gives a morphism

$$F^* : F^*\mathcal{F} \longrightarrow \mathcal{F}\,.$$

Lemma 5.2 The induced homomorphism in the cohomology

$$H^i(X,\mathcal{F}) \longrightarrow H^i(X,F^*\mathcal{F}) \xrightarrow{F^*} H^i(X,\mathcal{F})$$

is the identity.

Proof For $i = 0$, we obtain this as follows. Let

$$Ad : \mathcal{F} \to F_*F^*\mathcal{F} \quad \text{and} \quad ad : F^*F_*\mathcal{F} \to \mathcal{F}$$

be the adjunction morphisms. By definition, F^* is given by the composition

$$F^*\mathcal{F} \xrightarrow{F^*((F^*{/x})^{-1})} F^*F : *\mathcal{F} \xrightarrow{ad} \mathcal{F}\ .$$

The claim now follows from the commutative diagram

$$\begin{array}{ccccc}
H^0(X,\mathcal{F}) & \xrightarrow{Ad} & H^0(X,F_*F^*\mathcal{F}) & \xrightarrow[\sim]{\beta} & H^0(X,F^*\mathcal{F}) \\
\downarrow{\scriptstyle (F^*_{/x})^{-1}} & (1) & \downarrow{\scriptstyle F_*F^*(F^*_{/x})^{-1}} & (2) & \downarrow{\scriptstyle F^*(F^*_{/x})^{-1}} \\
H^0(X,F_*\mathcal{F}) & \xrightarrow{AdF_*} & H^0(X,F_*F^*F_*\mathcal{F}) & \xrightarrow[\sim]{\beta} & H^0(X,F^*F_*\mathcal{F}) \\
& \searrow\!\!\!= \ (3) & \downarrow{\scriptstyle F_*ad} & (4) & \downarrow{\scriptstyle ad} \\
& & H^0(X,F_*\mathcal{F}) & \xrightarrow[\sim]{\beta} & H^0(X,\mathcal{F})\,.
\end{array}$$

Here (1) and (4) are commutative, since Ad and ad are natural transformations, (3) is commutative by definition of the adjunction morphisms, and (2) is commutative, since β is functorial. Finally, $(F^*_{/X})^{-1}$ and β are inverse to each other as noted in (5.1.2).

For $i > 0$ the claim follows by considering injective resolutions, since the functorial isomorphism $\mathcal{F} \cong F_*\mathcal{F}$ also shows that F_* is exact.

Let $\overline{X} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$, and let $\pi : \overline{X} \to X$ be the projection. Then we have a commutative diagram

$$\begin{array}{ccc}
\overline{X} & \xrightarrow{F\times id} & \overline{X} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
X & \xrightarrow{F} & X
\end{array}$$

and an induced homomorphism

$$(F \times id)^* \pi^* \mathcal{F} = \pi^* F^* \mathcal{F} \overset{\pi^*(F^*)}{\longrightarrow} \pi^* \mathcal{F}\,.$$

We obtain a homomorphism in the cohomology by composition

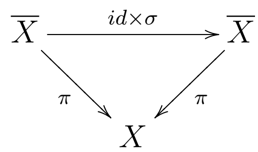
$$F = F^* : H^i(\overline{X}, \pi^*\mathcal{F}) \to H^i(\overline{X}, (F \times id)^* \pi^* \mathcal{F}) \to H^i(\overline{X}, \pi^* \mathcal{F})\,.$$

Remark 5.3 This can be extended to the cohomology with compact support for a separated $\mathbb{F}_q$-scheme of finite type, since F is finite and thus induces a map

$$H^i_c(\overline{X}, \pi^*\mathcal{F}) \longrightarrow H^i_c(\overline{X}, (F \times id)^* \pi^* \mathcal{F})\,,$$

since for a compactification $\mu : X \hookrightarrow X_1$, we have $\mu_! F_* = F_* \mu_!$. This gives the Frobenius-endomorphism in 1.6.

On the other hand, let $\sigma : \operatorname{Spec} \overline{\mathbb{F}}_q \to \operatorname{Spec} \overline{\mathbb{F}}_q$ be the q-Frobenius (i.e., the arithmetic Frobenius). Then we have the commutative diagram



and hence an equality $(id \times \sigma)^* \pi^* \mathcal{F} = \pi^* \mathcal{F}$. With this we get an induced map

$$\sigma : H^i(\overline{X}, \pi^*\mathcal{F}) \longrightarrow H^i(\overline{X}, (id \times \sigma)^* \mathcal{F}) = H^i(\overline{X}, \pi^* \mathcal{F})\,,$$

which gives an operation of the Galois group.

Obviously, $Fr \times \sigma = (Fr \times id) \circ (id \times \sigma) = (id \times \sigma) \circ (Fr \times id)$ is the q-Frobenius of $\overline{X}$. Now Lemma 5.2 implies:

Theorem 5.4 We have $F = \sigma^{-1}$.

In particular, this implies **COH 6**. First of all, there are canonical bijections

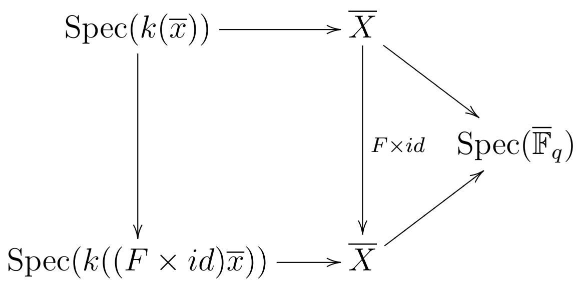
Finally we explain the Frobenius-operation on stalks. Initially we need canonical bijections

(5.5.1) $$\overline{X}_0 \xrightarrow{\sim} \overline{X}(\overline{\mathbb{F}}_q) = \operatorname{Hom}_{\overline{\mathbb{F}}_q}(\operatorname{Spec} \overline{\mathbb{F}}_q, \overline{X}) \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{F}_q}(\operatorname{Spec} \overline{\mathbb{F}}_q, X) = X(\overline{\mathbb{F}}_q)\,,$$

(where $\overline{X}_0$ denotes the set of the closed points of $\overline{X}$), which are defined as follows: If $\overline{x} \in \overline{X}_0$, then the composition

$$p_{\overline{x}} : \operatorname{Spec} k(\overline{x}) \xrightarrow{i_{\overline{x}}} \overline{X} \xrightarrow{p} \operatorname{Spec}(\overline{\mathbb{F}}_q)$$

is necessarily an isomorphism, and we assign to $\overline{x}$ the morphism $\varrho_{\overline{x}} = i_{\overline{x}} p_{\overline{x}}^{-1}$. The second map in (5.5.1) is obtained by composing with the projection $\pi : \overline{X} \to X$. The commutative diagram



shows that the operation of $F \times id$ on $\overline{X}$ corresponds to the operation of $F \times id$ on $\overline{X}(\overline{\mathbb{F}}_q)$ (operation by composition). Under the second bijection this corresponds to the Frobenius operation on $X(\overline{F}_q)$, which, because of the commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} \overline{\mathbb{F}}_q & \xrightarrow{f} & X \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ \operatorname{Spec} \overline{\mathbb{F}}_q & \xrightarrow{f} & X \end{array}$$

can be defined by composing f by F from the left or the right.

In particular, by considering the powers of F, we obtain:

Lemma 5.5 There is a bijection of the fixed modules

$$\overline{X}^{(F^n \times id)} \xrightarrow{\sim} X(\mathbb{F}_{q^n})\,.$$

Now let $\mathcal{F}$ be a sheaf on X. For every $\overline{x} \in \overline{X}_0$, regarded as geometric point of X, the morphism

$$F^* : \ F^*\mathcal{F} \longrightarrow \mathcal{F}$$

induces a homomorphism of stalks

$$F_{\overline{x}} = F^*_{\overline{x}} : \mathcal{F}_{F\overline{x}} \cong (F^*\mathcal{F})_{\overline{x}} \longrightarrow \mathcal{F}_{\overline{x}}\,.$$

For any $n \in \mathbb{N}$ with $\overline{x} \in \overline{X}^{(F^n \times id)} = X(\overline{F}_q)^{F^n}$ one then obtains a endomorphism by iteration

$$F^n_{\overline{x}} : \mathcal{F}_{\overline{x}} = \mathcal{F}_{F^n\overline{x}} \longrightarrow \mathcal{F}_{\overline{x}}\,.$$

If x is the image of $\overline{x}$ in X, one can choose $n = \deg(x) = [k(x) : \mathbb{F}_q]$ and let

$$F_x = F^{\deg(x)}_{\overline{x}} : \mathcal{F}_{\overline{x}} \longrightarrow \mathcal{F}_{\overline{x}}\,.$$

Up to isomorphism, the pair $(F_x, \mathcal{F}_{\overline{x}})$ is independent of the choice of (the $\deg(x)$ many) $\overline{x}$ over x, and thus the term

$$\det(1 - F_x T \mid \mathcal{F}_{\overline{x}})$$

is well defined, i.e., only depends on x; in particular, this holds for

$$tr(F_x \mid \mathcal{F}_{\overline{x}})\,.$$

Remarks 5.6 (a) The action on the stalks can also be described by the action of $F \times id$ on $\overline{X} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$: In fact, $F_{\overline{x}}$ can be identified with the homomorphism

$$F_{\overline{x}} = (F \times id)^*_x : (\pi^*\mathcal{F})_{(F\times id)\overline{x}} = ((F \times id)^*\pi^*\mathcal{F})_{\overline{x}} \longrightarrow (\pi^*\mathcal{F})_{\overline{x}}\,,$$

which is induced by the homomorphism $(F \times id)^* : (F \times id)^*\pi^*\mathcal{F} \longrightarrow \pi^*\mathcal{F}$ in the geometric point $\overline{x}$ of $\overline{X}$ (by the canonical isomorphism $(\pi^*\mathcal{F})_{\overline{x}} = \mathcal{F}_{\overline{x}}$, where $\overline{x}$ denotes $\rho_{\overline{x}}$ on the left and $\pi_{\overline{x}} := \pi\rho_{\overline{x}}$ on the right). This reduces everything to objects (sheaves, endomorphisms etc.), which are explained for $\overline{X}/\overline{F}_q$.

(b) On the other hand, in the situation above, where everything originates from $X/\mathbb{F}_q$, the operation on the stalks can also be explained by Galois theory. For this note that the Galois group $Gal(\overline{\mathbb{F}}_q/k(x))$ always operates on the stalk $\mathcal{F}_{\overline{x}}$ in the geometric point $\overline{x}$ of X (x is the image of $\overline{x}$ in X). If $\varphi_x : a \mapsto a^{q \cdot \deg(x)}$ is the arithmetic Frobenius of $\overline{\mathbb{F}}_q$ over $k(x)$, then we have

$$F_x = \varphi_x^{-1} \text{ on } \mathcal{F}_{\overline{x}}\,.$$

To see this, one notes that one has to show this only for $\deg(x) = 1$ (by base change to $k(x) = \mathbb{F}_{q^{\deg(x)}}$), since $F^{\deg(x)}$ is the $q^{\deg(x)}$-Frobenius.
Then $\mathcal{F}_{\overline{x}}$ identifies with the Galois module $H^0(\overline{\mathbb{F}}_q, \pi_{\overline{x}}^*\mathcal{F}) = H^0(\mathrm{Spec}(k(x)) \times_{\overline{\mathbb{F}}_q} \overline{\mathbb{F}}_q, \pi^*(i_x^*\mathcal{F}))$ (where $i_x : \mathrm{Spec}(k(x)) \to X$, $\pi : \mathrm{Spec}(k(x)) \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q \to \mathrm{Spec}(k(x))$ are the canonical morphisms), and the claim follows from Theorem 5.4 for $X = Spec(k(x))$ and the sheaf $i_x^*\mathcal{F}$.

6 Deligne's theorem: Formulation and first reductions

Together with Grothendieck's formula one obviously obtains the Weil conjectures I - III from the following result.

Theorem 6.1 (Deligne) Let X be a smooth projective variety over $\mathbb{F}_q$. For every $i \geq 0$, the characteristic polynomial

$$\det(1 - FT \mid H^i(\overline{X}, \mathbb{Q}_\ell)) \quad (\ell \neq p)$$

has integral coefficients, which are independent of ℓ. If we write

$$\det(1 - FT \mid H^i(\overline{X}, \mathbb{Q}_\ell)) = \prod_\alpha (1 - \alpha T) \quad \text{with}\, \alpha \in \overline{\mathbb{Q}} \subset \mathbb{C}\,,$$

then we have

$$\mid \alpha \mid = q^{\frac{i}{2}} \quad \text{for all } \alpha\,.$$

Note: the α are exactly the eigenvalues of F on $H^i(\overline{X}, \mathbb{Q}_\ell)$.

Reduction 1 It suffices to show:

W(X,i): For every $i \geq 0$ and every $\ell \neq p$, the eigenvalues of F on $H^i(\overline{X}, \mathbb{Q}_\ell)$ are algebraic numbers, whose complex conjugates α all have the absolute value

$$\mid \alpha \mid = q^{\frac{i}{2}}\,.$$

Proof Let ℓ be fixed, let $P_i(T) = \det(1 - FT \mid H^i(\overline{X}, \mathbb{Q}_\ell))$ and let M_i be the set of zeros of $P_i(T)$. Let

$$\begin{aligned} P(T) &= \prod_{i\,\text{uneven}} P_i(T)\,, \\ Q(T) &= \prod_{i\,\text{even}} P_i(T)\,, \end{aligned}$$

so that

$$Z(X,T) = \frac{P(T)}{Q(T)}\,.$$

Let K be a Galois number field which contains all zeros. Then the last equation can be regarded as equation in $K[[T]]$, and the Galois group $Gal(K/\mathbb{Q})$ operates on this ring by operating on the coefficient of the power series. For $\sigma \in Gal(K/\mathbb{Q})$, $\sigma Z(X,T) = Z(X,T)$, since $Z(X,T) \in \mathbb{Z}[[T]]$. On the other side , by $W(X,i)$ for all i, the polynomial $\sigma P_i(T)$ is prime to $P_j(T)$ for $j \neq i$, since $\sigma(M_i)$ is disjoint to M_j for $i \neq j$. Since all $P_i(T)$ have constant coefficient 1, we have $\sigma P_i(T) = P_i(T)$, therefore $P_i(T) \in \mathbb{Q}[T]$, since this holds for all σ. The following lemma shows that P and Q even lie in $\mathbb{Z}[T]$.

Lemma 6.2 Let $P, Q \in \mathbb{Q}[T]$ be prime to each other with constant coefficient 1 and $P/Q = Z \in \mathbb{Z}[[T]]$. Then we have $P, Q \in \mathbb{Z}[T]$.

Proof Let p be a prime number and let $\lambda \in \overline{\mathbb{Q}_p}$ be a zero of $Q(T)$. We claim that λ^{-1} is p-integral. If this is not the case, then λ is p-integral, thus $\mid \lambda \mid_p < 1$ for the p-adic absolute value of $\mathbb{Q}_p$, normalized by $| p |_p = \frac{1}{p}$. Since Z has integral coefficients, $Z(x)$ converges for all $x \in \mathbb{Q}_p$ with $| x |_p < 1$, and we get

$$P(\lambda) = Q(\lambda) \cdot Z(\lambda) = 0\,,$$

in contradiction to the claim that P and Q are prime to each other. Since this holds for all p, the inverse zeros of $Q(T)$ are integral. Since Q has a constant coefficient 1, the rational coefficients are whole. Therefore we have $P(T) = Q(T) \cdot Z(T) \in \mathbb{Z}[T]$.

Furthermore we note that the $P_i(T)$ have integral coefficients, since they have constant coefficient 1 and their reciprocal zeros are integral as reciprocal zeros of P or Q (one can use the lemma of Gauss as well). We get the independence of ℓ of the coefficients as follows: The reciprocal zeroes of $P_i(T)$ are the reciprocal zeroes of $P(T)$ or $Q(T)$, whose complex conjugates all have the value $q^{\frac{i}{2}}$. Since the zeroes of $P(T)$ and $Q(T)$ are determined by $Z(X, T)$, the description is independent of ℓ.

Remark 6.3 (a) The proof above comes from [Fr-K] (see p. 258). It gives the rationality of $Z(X, T)$ independently, without using the Hankel-determinants, as did Deligne ([D1] p. 276). Lemma 6.2 seems to be the lemma of Fatou that is cited by Deligne.

(b) Deligne proved generally ([D2]3.3.4) that, for a separated scheme X of finite type over $\mathbb{F}_q$, the Frobenius eigenvalues on $H^i_c(\overline{X}, \mathbb{Q}_\ell)$ are algebraic numbers for every $i \geq 0$ and $\ell \neq p = char(\mathbb{F}_q)$. It is still unknown if these numbers are independent of ℓ ($\neq p$); one does not even know if $\dim_{\mathbb{Q}_\ell} H^i_c(\overline{X}, \mathbb{Q}_\ell)$ is independent of ℓ.

Reduction 2 It suffices to show $W(X, i)$ after passing to a finite extension $\mathbb{F}_{q^n}$ of $\mathbb{F}_q$. More precisely: $W(X, i) \Leftrightarrow W(X \times_{\mathbb{F}_q} \mathbb{F}_{q^n}, i)$, since under base change to $\mathbb{F}_{q^n}$, the eigenvalues α turn to α^n and q turns to q^n.

Reduction 3 It suffices to consider a geometrically irreducible X over any $\mathbb{F}_q$. This follows from reduction 2 and the following obvious fact: If $X = \amalg X_j$, then we have $W(X, i) \Leftrightarrow W(X_j, i)$ for all j.

Below, let X be geometrically irreducible of dimension d.

Reduction 4 It suffices to show $W(X, i)$ for $i \leq d$. In fact, by Poincaré duality we have $W(X, i) \Leftrightarrow W(X, d - i)$: if $\{\alpha\}$ are the eigenvalues of F on $H^i(\overline{X}, \mathbb{Q}_\ell)$, then $\{q^d \alpha^{-1}\}$ are the eigenvalues on $H^{2d-i}(\overline{X}, \mathbb{Q}_\ell)$.

Reduction 5 It suffices to show $W(X, d)$ (for all X as above). In fact, we have

COH 11 Weak Lefschetz: If X is smooth projective of the dimension d and $Y \subseteq X$ is a smooth hyperplane section, then the restriction map

$$H^i(\overline{X}, \mathbb{Q}_\ell) \longrightarrow H^i(\overline{Y}, \mathbb{Q}_\ell)$$

is bijective for $0 \leq i \leq d - 2$ and injective for $i = d - 1$.

The reduction follows by induction over the dimension of X: If, in the situation of COH 11, one knows $W(Y, i)$ for all $i \leq \dim(Y) = d - 1$, then, by injection, we also get $W(X, i)$ for all $i \leq d-1$. Note: by Bertini, there is always a smooth hyperplane section of X, which is defined over a finite extension of $\mathbb{F}_q$.

Before we continue with the reductions, we show how to derive COH 11 from the following fundamental properties of the étale cohomology.

COH 12 Weak Lefschetz (second version): If X is affine and of finite type over a separably closed field L, then for the cohomological dimension $cd(X)$ of X we have

$$cd(X) = \dim(X)\,,$$

i.e., for all étale torsion sheaves $\mathcal{F}$ on X, $H^i(X, \mathcal{F}) = 0$ for $i > \dim(X)$ (see for example [Mi] VI 7.2).

COH 13 Poincaré duality (second version): Let X be a smooth separated scheme of pure dimension d over a field k with separable ending k_s, and let ℓ be prime to the characteristic of k.

(a) There are canonical Galois equivalent homomorphisms (where $\overline{X} = X \times_k k_s$)

$$tr : H_c^{2d}(\overline{X}, \mathbb{Z}/\ell^n)(d) \longrightarrow \mathbb{Z}/\ell^n\,,$$

which are compatible with the projections $\mathbb{Z}/\ell^{n+1} \longrightarrow \mathbb{Z}/\ell^n$.

(b) If $\mathcal{F}$ is a constructible locally constant $\mathbb{Z}/\ell^n$-sheaf on X, then the composition of the cupproduct and tr

$$H_c^i(\overline{X}, \mathcal{F}) \times H^{2d-i}(\overline{X}, \mathcal{F}^\vee)(d) \longrightarrow H^{2d}(\overline{X}, \mathbb{Z}/\ell^n)(d) \xrightarrow{tr} \mathbb{Z}/\ell^n$$

is a perfect duality (the cohomology groups are finite by 2.9 (c)). (See for example [Mi] VI 11.2).

Lemma 6.4 COH 11 follows from COH 12 and COH 13.

Proof If X is smooth, projective, geometric irreducible over a field k and if $Y \subseteq X$ is a smooth hyperplane section, then the complement $U = X - Y$ is affine (for a hyperplane $H \subseteq \mathbb{P}^N$, $\mathbb{P}^N - H \cong \mathbb{A}^N$ is affine, and a closed immersion $X \hookrightarrow \mathbb{P}^N$ is affine). By COH 12 and COH 13 we have

$$H_c^i(\overline{U}, \mathcal{F}) = 0 \,\text{for}\; i < d = \dim X$$

for every local-constant $\mathbb{Z}/\ell^n$-sheaf $\mathcal{F}$ with finite stalks on U ($\ell \neq char(k)$ and $\overline{U} = U \times_k k_s$ as above). For such a sheaf $\mathcal{F}$ on X, the restriction map

$$H^i(\overline{X}, \mathcal{F}) \longrightarrow H^i(\overline{Y}, \mathcal{F})$$

is bijective for $i < d-1$ and injective for $i = d-1$ by the long exact cohomology sequence in 2.8 (c) (note that $H^i = H_c^i$ for X and Y). The claim now follows from this for $\mathcal{F} = \mathbb{Z}/\ell^n$ by passing to the limit.

Reduction 6 ("Rankin's trick") It suffices to show the following: For every q there exists a $N \geq 0$, so that for all geometric irreducible smooth projective varieties of dimension d over $\mathbb{F}_q$ we have:

$\mathbf{W(X, d; N)}$: The eigenvalues of F on $H^d(\overline{X}, \mathbb{Q}_\ell)$ are algebraic numbers, whose complex conjugates α all have the value

$$\mid \alpha \mid \leq q^{\frac{d}{2}+\frac{N}{2}} .$$

Furthermore one can limit it to the dimensions d, which are divisible by a fixed natural number M.

Proof By the Künneth formula, α^k is an eigenvalue of F on $H^{kd}(\overline{X}^k, \mathbb{Q}_\ell)$ for every $k \in \mathbb{N}$. By $W(X^k, kd; N)$ we have

$$\mid \alpha^k \mid \leq q^{\frac{dk}{2}+\frac{N}{2}} ,$$

therefore

$$\mid \alpha \mid \leq q^{\frac{d}{2}+\frac{N}{2k}} .$$

Since this holds for all k, we get

$$\mid \alpha \mid \leq q^{\frac{d}{2}} .$$

By the Poincaré duality, $q^d\alpha^{-1}$ is an eigenvalue, therefore also $\mid q^d\alpha^{-1} \mid \leq q^{\frac{d}{2}}$, i.e.,

$$\mid \alpha \mid \geq q^{\frac{d}{2}} ,$$

where we have equalities and therefore $W(X, d)$. Finally, we can restrict ourselves on k, which are divisible by M.

Remark 6.5 The trick to consider higher powers, either of X or of sheaves on X, appears in several places in Deligne's proof. Deligne writes ([D 1] S. 283) that he was inspired by Rankin's work [Ran], where Rankin obtains his estimation for the Ramanujan function (compare §0 Application 1!), by considering the Dirichlet series

$$\sum \tau(n)^2 n^{-s}$$

instead of $\sum \tau(n) n^{-s}$.

Reduction 7 In the statements above, one can replace the terms "the eigenvalues of Frobenius are algebraic numbers, whose complex conjugates α have the value $\mid \alpha \mid \leq r \in \mathbb{R}$ ", by the term "the eigenvalues $\alpha \in \overline{\mathbb{Q}}_\ell$ of Frobenius have the property that $\mid \iota\alpha \mid \leq r$ for every embedding $\iota : \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$". In fact, from the last property it follows automatically that α is algebraic: if α is transcendental over $\mathbb{Q}$, then for every transcendental number $\beta \in \mathbb{C}$ there is an embedding: $\iota : \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$ with $\iota(\alpha) = \beta$; since obviously there is such an embedding for $\overline{\mathbb{Q}}(\alpha)$, and this can be extended on $\overline{\mathbb{Q}}_\ell$. With this, $\mid \iota\alpha \mid$ can be large.

Remark 6.6 For the extension of the embedding $\mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$ to $\overline{\mathbb{Q}}_\ell$, one needs the axiom of choice. But it is always possible to choose a finitely generated field $K \subseteq \overline{\mathbb{Q}}_\ell$, where all the considered eigenvalues lie, so that one only needs the unproblematic embeddings of K. The embeddings of $\overline{\mathbb{Q}}_\ell$ are more of a comfortable way of speaking.

7 Weights and determinant weights

The previous considerations suggest the following definitions.

Definition 7.1 Let q be a prime power and $n \in \mathbb{Z}$. An element α in a field of characteristic zero is called pure of weight n with respect to q, if it is algebraic and all its complex conjugates have the absolute value $q^{\frac{n}{2}}$.

Definition 7.2 Let X be a scheme of finite type over $\mathbb{Z}$ and let $\mathcal{F}$ be a constructible $\mathbb{Q}_\ell$-sheaf on X.

(a) $\mathcal{F}$ is called **pure** of weight $n \in \mathbb{Z}$, if for all closed points x of X the eigenvalues of F_x on $\mathcal{F}_{\overline{x}}$ are pure of weight n with respect to $N(x)$. Here $\overline{x} : \mathrm{Spec}(\overline{k(x)}) \longrightarrow X$ is a geometric point over x, $F_x \in Gal(\overline{k(x)}/k(x))$ is the geometric Frobenius which operates on $\mathcal{F}_{\overline{x}}$, and $N(x) = |k(x)|$.

(b) $\mathcal{F}$ is called **mixed**, if it has a finite filtration $\ldots \subseteq \mathcal{F}_{n-1} \subseteq \mathcal{F}_n \subseteq \ldots \mathcal{F}$ by constructible sheaves such that the successive quotients $\mathcal{F}_n/\mathcal{F}_{n-1}$ are pure. The weights of the non-trivial quotients are called the weights of $\mathcal{F}$.

Examples 7.3 (a) $\mathbb{Q}_\ell(m)$ is pure of weight $-2m$ (F_x operates by multiplication with $N(x)^{-m}$).

(b) Let X be smooth and projective over $\mathbb{F}_q$. If the $Gal(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-representation $H^i(\overline{X}, \mathbb{Q}_\ell)$ is regarded as $\mathbb{Q}_\ell$-sheaf on $\mathrm{Spec}(\mathbb{F}_q)$, then $W(X, i)$ means that $H^i(\overline{X}, \mathbb{Q}_\ell)$ is pure of weight i.

Lemma 7.4 (a) The category of sheaves which are pure of weight n is closed under forming quotients, subsheaves, extensions, inverse images, and direct images under finite morphisms.

(b) If $\mathcal{F}$ and $\mathcal{G}$ are pure of weight m and n, respectively, then $\mathcal{F}^\vee$ is pure of weight $-m$ and $\mathcal{F} \otimes \mathcal{G}$ is pure of weight $m+n$.

(c) The category of mixed sheaves is closed under the operations in (a) as well as by forming tensor products and duals.

The claims follow immediately from the compatibility of the operations with forming stalks. For a finite morphism $f : X \to Y$ and $y \in Y$ note that one has a Galois equivariant isomorphism

$$(f_*\mathcal{F})_{\overline{y}} \cong \bigoplus_{f(x)=y} \mathcal{F}_{\overline{x}}\,.$$

Furthermore, the tensor product is exact on the category of the $\mathbb{Q}_\ell$-sheaves.

The last reduction in §6 motivates the following

Definition 7.5 Let $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ be an embedding.

(a) For a prime power q and a number $\alpha \in \overline{\mathbb{Q}}_\ell^\times$, $\iota\text{-}w_q(\alpha) := 2\ log_q|\iota\alpha| \in \mathbb{R}$ is called the ι-weight of α, with respect to q, . (Hence $|\iota\alpha| = q^{\frac{\iota - w_q(\alpha)}{2}}$).

(b) Let X be of finite type over $\mathbb{Z}$ and let $\mathcal{F}$ be a constructible $\mathbb{Q}_\ell$-sheaf on X. $\mathcal{F}$ is called ι-pure of weight $\beta \in \mathbb{R}$, if, for all $x \in |X|$ and all eigenvalues α of F_x on $\mathcal{F}_{\overline{x}}$, we have: $\iota\text{-}w_{N(x)}(\alpha) = \beta$, i.e., $|\iota\alpha| = N(x)^{\frac{\beta}{2}}$.

(c) The sheaf $\mathcal{F}$ is called ι-mixed, if it has a finite filtration with successive ι-pure quotients.

The obvious analogs of 7.4 apply. The first non trivial claims about weights are obtained by the so-called determinant weights. Let X be a normal geometric connected scheme of finite type over $\mathbb{F}_q$ and let $\overline{y}$ be a geometric point. One has an exact sequence of the fundamental groups

$$(7.6.1) \qquad 1 \to \pi_1(\overline{X}, \overline{y}) \to \pi_1(X, \overline{y}) \to Gal(\overline{\mathbb{F}}_q/\mathbb{F}_q) \to 1\,,$$

where $\overline{y}$ also denotes a geometric point of $\overline{X}$, lying over $\overline{y}$: For normal X, this follows from the Galois theory of fields: without restriction, $\overline{y}$ lies over the generic point of X, then $\pi_1(\overline{X}, \overline{y})$ and $\pi_1(X, \overline{y})$ are the Galois groups of the maximal extensions of the function fields $\mathbb{F}_q(\overline{X})$ or $\mathbb{F}_q(X)$, respectively, which are unramified over $\overline{X}$ or X, respectively, and $Gal(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ is isomorph to the Galois group of the unramified extension $\mathbb{F}_q(X) \cdot \overline{\mathbb{F}}_q/\mathbb{F}_q(X)$.

Definition 7.6 The Weil group $W(X, \overline{y})$ is the full inverse image in $\pi_1(X, \overline{y})$ of the subgroup $\{F^n \mid n \in \mathbb{Z}\} \cong \mathbb{Z} \subseteq Gal(\overline{\mathbb{F}}_q/\mathbb{F}_q) \cong \hat{\mathbb{Z}}$.

We thus have an exact sequence

$$(7.6.2) \qquad 1 \to \pi_1(\overline{X}, \overline{y}) \to W(X, \overline{y}) \to \mathbb{Z} \to 0\,,$$

where we denote the homomorphism $W(X, \overline{y}) \to \mathbb{Z}$ by deg and call it the degree map.

In the following we consider smooth $\mathbb{Q}_\ell$-sheaves on X. These correspond to continuous $\mathbb{Q}_\ell$-representations of $\pi_1(X, \overline{y})$, but for the following conclusions it is useful to work with $\overline{\mathbb{Q}_\ell}$-coefficients. Therefore we define

Definition 7.7 A smooth $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{F}$ on X is a continuous finite-dimensional $\overline{\mathbb{Q}_\ell}$-representation of $\pi_1(X, \overline{y})$.

Since $\pi_1(X, \overline{y})$ is compact, every $\overline{\mathbb{Q}_\ell}$-representation comes by tensorizing with $\overline{\mathbb{Q}_\ell}$ from an E-representation for a finite extension E of $\mathbb{Q}_\ell$. Conversely, every smooth $\mathbb{Q}_\ell$-sheaf or E-sheaf gives a smooth $\overline{\mathbb{Q}_\ell}$-sheaf by tensorizing with $\overline{\mathbb{Q}_\ell}$.

For another geometric point $\overline{x}$ of X, $\pi_1(X, \overline{x})$ is isomorph to $\pi_1(X, \overline{y})$, and such an isomorphism is unique up to an inner automorphisms. Therefore, for every $\overline{x} \in X(\overline{\mathbb{F}_q})$, with image x in X, one obtains a homomorphism

$$Gal(\overline{\mathbb{F}_q}/k(x)) = \pi_1(\{x\}, \overline{x}) \to \pi_1(X, \overline{x}) \xrightarrow{\sim} \pi_1(X, \overline{y})\,,$$

which is well-defined up to conjugation in $\pi_1(X, \overline{y})$. The stalk of a $\overline{\mathbb{Q}_\ell}$-sheaf in $\overline{x}$ is the $\overline{\mathbb{Q}_\ell}$-representation of $Gal(\overline{\mathbb{F}_q}/k(x))$, which one obtains by restriction via the homomorphism above. In particular, the eigenvalues of F_x are defined, and one can transfer the notions of pureness, ι-weight etc.

Proposition 7.8 A smooth $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{F}$ of rank 1 is ι-pure for every $\iota : \overline{\mathbb{Q}_\ell} \hookrightarrow \mathbb{C}$. More precisely, the following holds:

(a) Let $\chi : W(X, \overline{y}) \longrightarrow \overline{\mathbb{Q}_\ell}^\times$ be the character induced by $\mathcal{F}$. Then χ is the product of a finite character and a character of the form

$$w \mapsto c^{\deg(w)}$$

for a $c \in \overline{\mathbb{Q}_\ell}^\times$.

(b) $\mathcal{F}$ is ι-pure of weight ι-$w_q(c)$.

Proof Obviously (b) follows from (a), since

$$|\ \iota\chi(F_x)\ |=|\ (\iota c)^{\deg(x)}\ |=|\ \iota c\ |^{\deg(x)}= N(x)^{\frac{\iota - w(c)}{2}}$$

for all $x \in |X_0|$. For (a) is suffices to show that the restriction of χ to $\pi_1(\overline{X}, \overline{y})$ has finite order. In fact, then χ^n is of the form $w \mapsto b^{\deg(b)}$ for n big enough, and the claim follows with an n-th root c of b. But $\chi(\pi_1(\overline{X}, \overline{y}))$ is a compact subgroup of $E^\times$ for a finite extension $E/\mathbb{Q}_\ell$, and hence a product of a finite group and a pro-ℓ-group. On the other hand, one can show (note that $\ell \neq p$):

Theorem 7.9 The image of $\pi_1(\overline{X}, \overline{y})$ in the maximal abelian quotient $W(X, \overline{y})^{ab}$ of $W(X, \overline{y})$ ($= W(X, \overline{y})$ modulo of the closure of the commutator group) is a product of a finite group and a pro-p-group.

Proof We just show this for a smooth curve X, since we only need this case later. Let X_1 be the smooth compactification of X and let $S = X_1 - X$.

First Proof, by class field theory: By this we have a canonical isomorphism

$$W(X, \overline{y})^{ab} \cong K^\times \setminus A^\times / \prod_{x \in X_0} O_x^\times\ ,$$

where K is the function field of X, $A^\times$ is the idele group of K and O_x is the completion of the local ring of X at x. The image of $\pi_1(\overline{X}, \overline{y})$ is the kernel of the degree map on the above group. But the kernel of the map

$$W(X, \overline{y})^{ab} \longrightarrow W(X_1, \overline{y})^{ab} = K^\times \setminus A^\times / \prod_{x \in (X_1)_0} O_x^\times$$

is a product of a finite and of a pro-p-group, as a quotient of $\prod_{x \in S} O_x^\times$, and the kernel of the degree map on $W(X_1, \overline{y})^{ab}$ is the finite class group $Pic^0(X_1)$ of X_1.

Second Proof, geometrical: It suffices to show that for $\ell \neq p$ the order of the fixed modules under the Frobenius F

$$Hom(\pi_1(\overline{X}, \overline{y}), \mathbb{Z}/\ell^n)^F = H^1(\overline{X}, \mathbb{Z}/\ell^n)^F$$

is bounded, independently of ℓ and n. By Poincaré duality, this group is dual to $H_c^1(\overline{X}, \mu_{\ell^n})_F$, the cofixed module for F. Because of the exact sequence

$$H^0(\overline{S}, \mu_{\ell^n}) \longrightarrow H_c^1(\overline{X}, \mu_{\ell^n}) \longrightarrow H^1(\overline{X}_1, \mu_{\ell^n}) \longrightarrow 0$$

it suffices to bound the orders of $H^0(\overline{S}, \mu_{\ell^n})_F$ and $H^1(\overline{X}_1, \mu_{\ell^n})_F$, and because of the exact sequence

$$0 \longrightarrow A^F \longrightarrow A \xrightarrow{F-1} A \longrightarrow A_F \longrightarrow 0\,, \tag{7.9.1}$$

one can consider the order of the fixed modules for an F-module A, since for finite A it follows from (7.9.1) that A^F and A_F have the same order. But

$$H^0(\overline{S},\mu_{\ell^n})^F = H^0(S,\mu_{\ell^n}) = \bigoplus_{x\in S} \mu_{\ell^n}(k(x)) \subseteq \bigoplus_{x\in S} k(x)^\times$$

and

$$H^1(\overline{X}_1,\mu_{\ell^n})^F \overset{(1)}{=} {}_{\ell^n}Pic(\overline{X}_1)^F = {}_{\ell^n}Pic^0(\overline{X}_1)^F \overset{(2)}{\subseteq} Pic^0(X_1)$$

are contained in finite groups, since we are over a finite field. The equality (1) follows from the isomorphism $Pic(\overline{X}_1) \cong H^1(\overline{X}_1,\mathbb{G}_m)$ and the cohomology sequence to the Kummer sequence

$$0 \longrightarrow \mu_{\ell^n} \longrightarrow \mathbb{G}_m \overset{\ell^n}{\longrightarrow} \mathbb{G}_m \longrightarrow 0\,,$$

since $H^0(\overline{X}_1,\mathbb{G}_m) = \overline{\mathbb{F}}_q^\times$ is ℓ-divisible. The inclusion (2) follows from the Hochschild-Serre spectral sequence, since $H^1(\mathbb{F}_q,\overline{\mathbb{F}}_q^\times) = 0$ (Hilbert 90) and $H^2(\mathbb{F}_q,\overline{\mathbb{F}}_q^\times) = 0$ ($cd(\mathbb{F}_q) = 1$). A geometric proof for the finiteness of $Pic^0(X_1)$ follows for example from the fact that this is the set of the $\mathbb{F}_q$-rational points of an abelian variety over $\mathbb{F}_q$, namely, the Jacobi variety of X_1.

By 7.8, the following definition makes sense.

Definition 7.10 Let $\mathcal{F}$ be a smooth $\overline{\mathbb{Q}_\ell}$-sheaf on X and let $\iota : \overline{\mathbb{Q}_\ell} \hookrightarrow \mathbb{C}$ be an embedding. The ι-determinant weights of $\mathcal{F}$ are the numbers $\frac{1}{g}\cdot$ (ι-weight of $\Lambda^g\mathcal{G}$), where $\mathcal{G}$ is a composition factor (= irreducible subquotient) of $\mathcal{F}$ and $g = \dim\mathcal{G}$.

Non-trivial claims about determinant weights follow from the theory of algebraic (monodromy) groups.

Definition 7.11 Let $\mathcal{F}$ be a smooth $\overline{\mathbb{Q}_\ell}$-sheaf on X. Let G_1 be the Zariski closure of the image of $\pi_1(X,\overline{y})$ in $GL(\mathcal{F}_{\overline{y}})$ and let G be the semi-direct product of $\mathbb{Z}$ with G_1, which makes the diagram with exact rows

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(\overline{X},\overline{y}) & \longrightarrow & W(X,\overline{y}) & \overset{\deg}{\longrightarrow} & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & G_1 & \longrightarrow & G & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & & & \downarrow & & & & \\
 & & & & GL(\mathcal{F}_{\overline{y}}) & & & &
\end{array}$$

commutative (if $F \in \deg^{-1}(1)$, then F normalizes the normal subgroup $\pi_1(\overline{X},\overline{y})$, as well as G_1, the operation is algebraic, and G is a semi-direct product of $< F >$ and G_1).

Theorem 7.12 (Grothendieck) Let G_1^0 be the connected component of the unit in G_1. Then the radical of G_1^0 is unipotent.

For the proof we use:

Lemma 7.13 If $\mathcal{F}$ is semi-simple, regarded as a representation of $\pi_1(X, \overline{y})$, then G_1^0 is semi-simple. (A representation is called semi-simple, if it is a direct sum of irreducible representation. Analogously, semi-simple objects are defined in an abelian category).

Proof If $\mathcal{F}$ is semi-simple, then the restriction to the normal subgroup $\pi_1(\overline{X}, \overline{y})$ is semi-simple: if $W \subset V = \mathcal{F}_{\overline{y}}$ is a simple $\pi_1(\overline{X}, \overline{y})$-module, then the sum W' of its $W(X, \overline{y})$-conjugates is a semi-simple $\pi_1(\overline{X}, \overline{y})$-module and has a complement in V (for $W(X, \overline{y})$ and thus for $\pi_1(\overline{X}, \overline{y})$). Then G_1^0 is reductive, i.e., the unipotent radical is trivial (for this, V is simple without restriction; one uses that a unipotent group always has a fixed vector $\neq 0$, therefore the unipotent radical would have a fixed module $0 \neq V' \neq V$). One has to show that the maximal central torus T_1 is trivial.

$W(X, \overline{y})$ operates on T_1 by conjugation and thus also acts on the character module $X(T_1) = Hom(T_1, \mathbb{G}_m)$, respecting the finite set E of the characters by which T_1 operates on V. The set E generates $X(T_1)$, since, by assumption, T_1 operates faithfully on V. The operation of $W(X, \overline{y})$ factorizes over a finite quotient of $\mathbb{Z}$, and we can consider the kernel of the operation without restriction, which corresponds to a base change to a finite extension of $\mathbb{F}_q$. Then the operation on T_1 is trivial. But there are only finitely many outer automorphisms which are trivial on T_1, by another base change the operation on G_1^0 is trivial. By passing to an open subgroup of $\pi_1(X, \overline{y})$, i.e., a finite covering of X, we get $G_1 = G_1^0$ without restriction.

Hence we can also assume that $G = G_1^0 \times \mathbb{Z}$. Let T be the maximal torus quotient of G_1^0. This is isogenic to T_1, hence we have to show that T_1 is trivial. The map $W(X, \overline{y}) \longrightarrow G \longrightarrow G_1^0 \longrightarrow T$ has the property that the image of $\pi_1(\overline{X}, \overline{y})$ is Zariski dense. Since T is commutative, this map is finite by Theorem 7.9, therefore $T = \{1\}$.

Proof of Theorem 7.12 Let $F^{\cdot}$ be a Jordan-Hoelder filtration of V, let P be the subgroup of $GL(V)$, which respects the filtration $F^{\cdot}$ and let $N \subseteq P$ be the subgroup, which operates trivially on the quotients of $F^{\cdot}$, and $L = P/N$. Then $G_1 \subseteq P$, its image G_2 in L is the Zariski closure of $\pi_1(\overline{X}, \overline{y})$ in $GL(Gr_F^{\cdot} V)$ and the kernel of $G_1 \twoheadrightarrow G_2$ is a unipotent normal subgroup of G_1 (since N is a unipotent normal subgroup of P). By 7.13, G_2^0 is reductive, and the claim follows.

Corollary 7.14 Let $\mathcal{F}$ be semi-simple and let Z be the center of G. Then the kernel and cokernel of $\deg : Z \to \mathbb{Z}$ are finite.

Proof $Z \cap G_1$ is in the center of G_1 and thus finite. Furthermore, in the proof of 7.13 we showed that there exists an element g in G with $\deg(g) = n \neq 0$, which commutes with G_1^0. Then a suitable power commutes with G, i.e., is in Z:

In fact, first we can assume that g operates trivial on G_1/G_1^0 by conjugation. Then, for $h \in G_1$, let the element $x_h \in G_1^0$ be defined by

$$ghg^{-1} = x_h \cdot h \,.$$

Then $x_{hh'} = x_h$ and $x_{h'h} = h'x_h(h')^{-1}$ for $h' \in G_1^0$. Since G_1^0 is a normal subgroup in G_1, we get $h'x_h(h')^{-1} = x_h$, i.e., x_h is in the center of G_1^0. Since this is finite, there is a $m \neq 0$ with

$$g^m h g^{-m} = x_h^m \cdot h = h$$

for all $h \in G_1$.

Corollary 7.15 Let $\mathcal{F}$ be semi-simple and let g be a central element in G with $\deg(g) = n \neq 0$. Let $\mathcal{F}'$ be a smooth $\overline{\mathbb{Q}_\ell}$-sheaf on X, which is induced by a representation V' of G (see 7.11). Then $\beta \in \mathbb{R}$ is a ι-determinant weight on $\mathcal{F}$ if and only if there is a eigenvalue α of g on V' with $|\iota\alpha| = q^{\frac{n\beta}{2}}$.

Proof Without restriction, V' is simple. Then g is scalar (here, one needs $\overline{\mathbb{Q}_\ell}$-coefficients, i.e., the Lemma of Schur!), say equal to the multiplication with α, and the eigenvalue on $\det V'$ is equal to α^r, $r = \dim V'$. By Proposition 7.8, the determinant weight β is equal to $\frac{1}{n} \cdot \iota\text{-}w(\alpha)$: If χ is the character to $\det V'$, we have $|\iota\chi(w)| = q^{\frac{\deg(w)\cdot\beta\cdot r}{2}}$; if one chooses w with $\deg(w) = n$, then one has $|\iota\chi(w)| = |\iota\alpha|$).

Theorem 7.16 (a) For $\beta \in \mathbb{R}$, let $n(\beta)$ be the sum of the ranks of the composition factors with ι-determinant weights β. Then the determinant weights of $\Lambda^a\,\mathcal{F}$ are the sums

$$\sum m(\beta)\beta$$

with $m(\beta) \in \mathbb{Z}$, $\sum\ m(\beta) = a$ and $0 \leq m(\beta) \leq n(\beta)$.

(b) If the smooth $\overline{\mathbb{Q}_\ell}$-sheaves $\mathcal{F}$ and $\mathcal{F}'$ are of pure ι-determinant weight β and β', then $\mathcal{F} \otimes \mathcal{F}'$ is of pure ι-determinant weight $\beta + \beta'$.

(c) Let $f : X' \to X$ be a dominant morphism of normal connected schemes, which are of finite type over $\mathbb{F}_q$. A smooth $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{F}$ on X is of pure ι-determinant weight β, if and only if this holds for $f^*\mathcal{F}$.

Proof (a) The eigenvalues of Z on $\Lambda^a\mathcal{F}$ are products of a eigenvalues on $\mathcal{F}$, which belong to different eigenvectors in $\mathcal{F}$. By forming ι-absolute valves and their logarithms, one sees that one just obtains all sums of a determinant weights, where at most $n(\beta)$ of them are equal to β.

(b) is analogous, by considering the algebraic monodromy group of $\mathcal{F} \otimes G$.

(c) It follows from the assumption that the image of $\pi_1(\overline{X'})$ in $\pi_1(\overline{X})$ is of finite index: since the schemes are normal, it suffices to consider the fiber over the generic point η of $\overline{X}$ ($Gal(\overline{k(\eta)}/k(\eta)) \to \pi_1(\overline{X})$ it surjective), and this has a rational point in a finite extension of $k(\eta)$. It follows that, for the corresponding Zariski closures G_1' and G_1, the image of G_1' has finite index in G_1, therefore contains G_1^0. The image of the center Z' of G' centralizes G_1^0, and with the same conclusion as in the proof of 7.14 one can see that it has a finite index in the center Z of G. This implies the claim.

8 Cohomology of curves and L-series

Let X be a smooth geometric irreducible curve over $\mathbb{F}_q$ and let $\mathcal{F}$ be a smooth $\mathbb{Q}_\ell$-sheaf on X, with stalk $V = \mathcal{F}_{\overline{y}}$ in a geometric point $\overline{y}$ of X. Let $\mathcal{F}'$ (respectively $\mathcal{F}''$) be the biggest subsheaf (respectively quotient sheaf) of $\mathcal{F}$, which is constant on $\overline{X}$. By (7.6.1), $\mathcal{F}'$ and $\mathcal{F}''$ are inverse images of sheaves on $\mathrm{Spec}(\mathbb{F}_q)$, i.e., these come from $\mathbb{Q}_\ell$-representations F' and F'' of $Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q)$.

Lemma 8.1 (a) $H^0(\overline{X}, \mathcal{F}) = V^{\pi_1(\overline{X},\overline{y})} = F'$, where $V^{\pi_1(X,\overline{y})}$ denotes the fixed module under $\pi_1(\overline{X}, \overline{y})$.

(b) $H^0_c(\overline{X}, \mathcal{F}) = \begin{cases} H^0(\overline{X}, \mathcal{F}) & \text{if } X \text{ is proper,} \\ 0, & \text{otherwise} \end{cases}$.

(c) $H^2_c(\overline{X}, \mathcal{F}) = V_{\pi_1(\overline{X},\overline{y})}(-1) = F''(-1)$, where $V_{\pi_1(\overline{X},\overline{y})}$ denotes the cofixed module under $\pi_1(\overline{X}, \overline{y})$.

(d) If $\mathcal{F}$ is an arbitrary constructible $\mathbb{Q}_\ell$-sheaf on X and $U \subseteq X$ is open, then we have $H^2_c(\overline{U}, \mathcal{F}) \xrightarrow{\sim} H^2_c(\overline{X}, \mathcal{F})$.

Proof (a) follows from the equivalence of categories between smooth sheaves and representations of the fundamental group. (b) for non-proper X follows from the fact that a smooth sheaf has no sections with support in finitely many points (it suffices to see this for constant sheaves). One can also use Poincaré duality and weak Lefschetz. (c) follows from (a) by Poincaré duality. (d) follows from the relative cohomology sequence for $U \subset X \supset X - U$, since $H^i(\overline{X - U}, \mathcal{F}) = 0$ for $i > 0$ ($\overline{X - U}$ consists of finitely many copies of $\mathrm{Spec}(\overline{\mathbb{F}_q})$).

Corollary 8.2 Let α be an eigenvalue of F on $H^0(\overline{X}, \mathcal{F})$ or $H^0_c(\overline{X}, \mathcal{F})$ (or respectively on $H^2_c(\overline{X}, \mathcal{F})$).

(a) For every $x \in X^0$, $\alpha^{\deg(x)}$ (or respectively $(q^{-1}\alpha)^{\deg(x)}$) is an eigenvalue of F_x on $\mathcal{F}$ (i.e., on V).

(b) The number ι-$w_q(\alpha)$ (or respectively ι-$w_q(\alpha) - 2$) is a ι-determinant weight of $\mathcal{F}$ (i.e., the associated $\overline{Q}_\ell$-sheaf).

For the following we use the Grothendieck-Lefschetz formula

$$(8.3.1) \qquad \prod_{x \in X_0} \det(1 - F_x T^{\deg(x)} | \mathcal{F})^{-1} = \prod_{i \geq 0} \det(1 - FT | H^i_c(\overline{X}, \mathcal{F}))^{(-1)^{i+1}}$$

for a constructible $\mathbb{Q}_\ell$-sheaf $\mathcal{F}$ on a scheme X of finite type over $\mathbb{F}_q$, which follows from Theorem 1.6 and (1.5.2). The left hand side is controlled by

Proposition 8.3 If we have ι-$w_{N(x)}(\alpha) \leq \beta$ for all eigenvalues α of F_x on $\mathcal{F}$, for all $x \in X_0$, then $\iota \prod_{x \in X_0} \det(1 - F_x T^{\deg(x)} | \mathcal{F})^{-1}$ converges absolutely for $|T| < q^{-\frac{\beta}{2} - \dim(X)}$ (i.e., for $Re(s) > \frac{\beta}{2} + \dim(X)$ if $T = q^{-s}$), and hence has neither pole nor zero in this area.

Proof Let $d = \dim(X)$; then there is a finite covering of X such that every member is quasi-finite over an affine space $A^d_{\mathbb{F}_q}$ (Noether normalization). This implies

$$\sharp\{x \in X_0 \text{ with } \deg(x) = n\} \leq C \cdot q^{dn}$$

with a constant $C > 0$ (which estimates the sum of the generic degrees), since $\mathbb{A}^d_{\mathbb{F}_q}(\mathbb{F}_{q^n}) = \mathbb{F}^d_{q^n}$. The convergence thus follows from the convergence of the geometric series

$$\sum_n q^{nd} \, q^{\frac{n\beta}{2}} |T|^n \, .$$

Corollary 8.4 If X is an affine smooth geometric irreducible curve, then we have

$$\iota\text{-}w_q(\alpha) \leq \beta + 2$$

for the eigenvalues α of F on $H^1_c(\overline{X}, \mathcal{F})$.

Proof The right hand side of the formula (8.3.1) is

$$\frac{\det(1 - FT|H^1_c(\overline{X}, \mathcal{F}))}{\det(1 - FT|H^2_c(\overline{X}, \mathcal{F}))}$$

By 8.2 (a) we have $\iota\text{-}w_q(\alpha) \leq \beta + 2$ for the reciprocal zeros α of the denominator, and by 8.3 this also holds for the reciprocal zeros of the whole fraction.

In fact, let α be an eigenvalue of F on $H^2_c(\overline{X}, \mathcal{F})$. By 8.2 (a), $(q^{-1}\alpha)^{\deg(x)}$ is an eigenvalue of F_x on $\mathcal{F}$, therefore by assumption we have

$$\beta \geq \iota\text{-}w_{N(x)}(q^{-1}\alpha)^{\deg(x)} = -2 + \iota\text{-}w_q(\alpha) \, .$$

On the other hand, if α is an eigenvalue of F on $H^1_c(\overline{X}, \mathcal{F})$, then by (8.3.1) and 8.3 $(1 - \iota\alpha T) \neq 0$ for all $|T| < q^{-\frac{\beta}{2}-1}$, hence $|\iota\alpha| \leq q^{\frac{\beta}{2}+1}$, i.e., $\iota\text{-}w_q(\alpha) \leq \beta + 2$.

9 Purity of real $\mathbb{Q}_\ell$-sheaves

This chapter treats an important method which is used in both papers of Deligne about the Weil conjecture.

Definition 9.1 Let $\mathcal{F}$ be a smooth $\overline{\mathbb{Q}_\ell}$-sheaf on a scheme X which is of finite type over $\mathbb{Z}$.
(a) $\mathcal{F}$ is called totally real, if the coefficients of

$$\det(1 - F_x T|\mathcal{F}) := \det(1 - F_x T \mid \mathcal{F}_{\overline{x}})$$

are totally real algebraic numbers for every $x \in X_0$.
(b) $\mathcal{F}$ is called ι-real for $\iota : \overline{\mathbb{Q}_\ell} \hookrightarrow \mathbb{C}$, if

$$\iota \det(1 - F_x T|\mathcal{F})$$

has real coefficients for all $x \in X_0$.

Remarks 9.2 If $\mathcal{F}$ is pure (respectively, ι-pure), then $\mathcal{F}$ is a direct summand of a totally real (respectively, ι-real) sheaf, to wit: of $\mathcal{F} \oplus \mathcal{F}^\vee(-n)$, if n is the weight (respectively, ι-weight) of $\mathcal{F}$: for $\iota\alpha$ with $|\iota\alpha| = N(x)^{\frac{n}{2}}$, $N(x)^n \cdot \iota\alpha^{-1}$ is its complex conjugate.

Theorem 9.3 Let X be a smooth geometric irreducible curve over $\mathbb{F}_q$. Then the composition factors of a smooth, ι-real $\overline{\mathbb{Q}_\ell}$-sheaf on X are ι-pure.

We use:

Lemma 9.4 Let $\mathcal{F}$ be a smooth ι-real $\overline{\mathbb{Q}_\ell}$-sheaf on X and let ϱ be the biggest ι-determinant weight of $\mathcal{F}$. For every $x \in X_0$ and every eigenvalue α of F_x on $\mathcal{F}$ we have ι-$w_{N(x)}(\alpha) \leq \varrho$.

Proof By possibly omitting a point which one does not consider in the moment, X is affine without restriction. Then the Lefschetz formula gives

$$(9.3.1) \qquad \prod_{x \in X_0} \iota \det(1 - F_x T^{\deg(x)} \mid \mathcal{F}^{\otimes 2k})^{-1} = \frac{\iota \det(1 - FT \mid H^1_c(\overline{X}, \mathcal{F}^{\otimes 2k}))}{\iota \det(1 - FT \mid H^2_c(\overline{X}, \mathcal{F}^{\otimes 2k}))}$$

for every positive integer k. Here

$$\iota \det(1 - F_x T^{\deg(x)} \mid \mathcal{F}^{\otimes 2k})^{-1} = \exp(\sum_{n \geq 1} \iota Tr F_x^n \mid \mathcal{F}^{\otimes 2k}) \frac{T^{n \cdot \deg(x)}}{n}$$

is a formal power series with non-negative real coefficients, since by assumption

$$\iota Tr(F_x^n \mid \mathcal{F}^{\otimes 2k}) = \iota Tr(F_x^n \mid \mathcal{F})^{2k}$$

is non-negative real. By 7.16 (b), the ι-determinant weights of $\mathcal{F}^{\otimes 2k}$ are at most equal to $2k\varrho$, and hence by 8.2 (b), the right hand side of the Lefschetz formula (9.3.1) has no pole for $|T| < q^{-\frac{1}{2}(2k\varrho+2)}$ (i.e., for the reciprocal zeros α' of the denominator we have ι-$w_q(\alpha') \leq 2k\varrho + 2$). By the following Lemma,

$$\iota \det(1 - F_x T^{\deg(x)} \mid \mathcal{F}^{\otimes 2k})^{-1}$$

has no pole for $|T| < q^{\frac{1}{2}(2k\varrho+2)}$ as well. For an eigenvalue α of F_x on $\mathcal{F}$, $\iota\alpha^{-2k/\deg(x)}$ is a pole; this implies

$$|\iota\alpha|^{2k/\deg(x)} \leq q^{(2k\varrho+2)/2},$$

i.e.,

$$|\iota\alpha| \leq N(x)^{(\varrho+\frac{1}{k})/2}.$$

Since this holds for all k, the claim follows.

Lemma 9.5 Let $f_i = \sum_n a_{i,n}T^n$ be a series of formal power series with constant term 1 and non-negative real coefficients. Let the order of $f_i - 1$ tend towards infinity with i, and let $f = \prod_i f_i$. Then the absolute radius of convergence for every f_i is at least as big as for f. If f and the f_i are Taylor expansions of meromorphic functions, we have

$$\inf\{|z| \mid f(z) = \infty\} \leq \inf\{|z| \mid f_i(z) = \infty\}$$

for every i.

Proof If $f = \sum_n a_nT^n$, the first claim follows from the fact that $a_{i,n} \leq a_n$ for all i. For meromorphic functions, the denoted infima are exactly the absolute convergence radii.

Proof of Theorem 9.3 Let $\mathcal{F}$ be a smooth ι-real $\overline{\mathbb{Q}_\ell}$-sheaf on X. For $\beta \in \mathbb{R}$, let $\mathcal{F}(\beta)$ be the sum of composition factors of $\mathcal{F}$ with ι-determinant weights β and let $n(\beta)$ be the rank of $\mathcal{F}(\beta)$. Let $x \in X_0$, and let $\alpha_1^\beta, \ldots, \alpha_{n(\beta)}^\beta$ be the eigenvalues of F_x on $\mathcal{F}(\beta)$. We have to show that $\iota\text{-}w_{N(x)}(\alpha_i^\beta) = \beta$ for all i.

By definition of the determinant weights we have

$$\sum_i \iota\text{-}w_{N(x)}(\alpha_i^\beta) = n(\beta)\beta\,. \tag{9.3.2}$$

Without restriction, let $\mathcal{F}(\beta) \neq 0$, and let N be the sum of those $n(\gamma)$ with $\gamma > \beta$. By 7.16 (a), for the ι-determinant weights ρ of the $(N+1)$-th external power of $\mathcal{F}$ we have $\rho \leq \beta + \sum_{\gamma>\beta} n(\gamma)\gamma$. Since every $\alpha_i^\beta \prod_{\gamma>\beta} \prod_{i=1}^{n(\gamma)} \alpha_i^\gamma$ is an eigenvalue of F_x on $\Lambda^{N+1}\mathcal{F}$, by Lemma 9.4 we have

$$\iota\text{-}w_{N(x)}(\alpha_i) + \sum_{\gamma>\beta}\sum_i \iota\text{-}w_{N(x)}(\alpha_i^\gamma) \leq \beta + \sum_{\gamma>\beta} n(\gamma)\gamma\,.$$

By equation (9.3.2) (for every $\gamma > \beta$) we have

$$\iota\text{-}w_{N(x)}(\alpha_i) \leq \beta\,.$$

By adding over i, one has to obtain equation (9.3.2) for β, therefore the equality holds.

10 The formalism of nearby cycles and vanishing cycles

For induction over dimension, Deligne uses fibrations $f : X \to S$ over a smooth curve S, where f is smooth over an open set $U \subseteq S$, and only over finitely many points $s \in S - U$ has fibers with (mild) singularities. The cohomology $H^i(X, \mathcal{F})$ is studied by the Grothendieck-Leray spectral sequence

$$H^p(S, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}) .$$

To examine the sheaves $R^q f_* \mathcal{F}$ at the bad places $s \in S - U$, one passes to the local ring $\mathcal{O}_{S,s}$ (which is a discrete valuation ring), or rather to its henselization $\mathcal{O}^h_{S,s}$; this is a henselian discrete valuation ring.

For the étale topology, a strict henselian discrete valuation ring A is an analog of the open disk $D = \{z \in \mathbb{C} \mid |z| < 1\}$ in $\mathbb{C}$: We have $\pi_1(D) = 0$ and $\pi_1(D \smallsetminus \{0\}) \cong \mathbb{Z}$; this corresponds to the fact that $\pi_1(\mathrm{Spec}(A)) = 0$ and

$$\pi_1(\mathrm{Spec}(A) - \{s\}) = \prod_{\ell \neq char(k(s))} \mathbb{Z}_\ell ,$$

where s is the closed point of Spec(A). The point s corresponds to the point $0 \in D$, and the generic point η corresponds to a “general point” $t \in D - \{0\}$.

In classical topology, one has the theory of vanishing cycles for a fibration $f : X \to D$, with f smooth on D^* and singular fiber X_0 over 0. In étale topology, one considers the cartesian diagram

$$\begin{array}{ccccc} X_\eta & \hookrightarrow & X & \longleftarrow & X_s \\ \downarrow & & \downarrow & & \downarrow \\ \eta & \longrightarrow & \mathrm{Spec}(A) & \longleftarrow & s \end{array}$$

Preliminary considerations 10.1 Let $T = \mathrm{Spec}\ A$ be for a henselian discrete valuation ring A.

(a) By the decomposition theorem there is a equivalence of categories between the category $Sh(T_{et})$ of the étale sheaves on T and the category of all triples $(\mathcal{F}_0, \mathcal{F}_1, \varphi)$, where

(i) $\mathcal{F}_0$ is a sheaf on the closed point $s \overset{i}{\hookrightarrow} T$,

(ii) $\mathcal{F}_1$ is sheaf on the generic point $\eta \overset{j}{\hookrightarrow} T$, and

(iii) $\varphi : \mathcal{F}_0 \longrightarrow i^* j_* \mathcal{F}_1$ is a morphism of sheaves.

Here, a sheaf $\mathcal{F}$ on T is mapped on the triple

$$(i^*\mathcal{F}, j^*\mathcal{F},\ sp : i^*\mathcal{F} \to i^* j_* j^* \mathcal{F}) ,$$

where one obtains the so-called specialization morphism sp by applying i^* to the adjunction morphism $\mathcal{F} \to j_* j^* \mathcal{F}$.

(b) This has the following reinterpretation via Galois modules: Let $\overline{k(\eta)}$ be a separable closure of $k(\eta)$ and let $\overline{\eta} = \mathrm{Spec}(\overline{k(\eta)}) \longrightarrow T$ be the associated geometric point over η. This defines a geometric point $\overline{s} \to T$ over s as follows. Let $\tilde{A}$ be the integral closure of A in $\overline{k(\eta)}$

, i.e., let $\widetilde{T} = \text{Spec } (\tilde{A})$ be the normalization of T in $\overline{\eta}$. Then $\tilde{A}$ is local and its residue field is a separably closed extension of $k(s)$ and defines a geometric point $\overline{s} \to T$ over s. Further one obtains a surjection

$$G_\eta = Gal(k(\overline{\eta})/k(\eta)) \twoheadrightarrow G_s = Gal(k(\overline{s})/k(s))\,;$$

its kernel I is called the inertia group. The strict henselization $O^h_{T,\overline{s}}$ of T in $\overline{s}$ can be identified with $\tilde{A}^I$.

The triples in (a) thus correspond to triples (M_0, M_1, ϕ), where

(i) M_0 is a discrete G_s-module,

(ii) M_1 is a discrete G_η-module, and

(iii) $\phi : M_0 \to M_1^I$ is a morphism of G_s-modules.

The passage from the triples in (a) to these is obtained by forming the stalks, i.e., via

$$M_0 = \mathcal{F}_{\overline{s}} = (i^*\mathcal{F})_{\overline{s}} \quad \text{and} \quad M_1 = \mathcal{F}_{\overline{\eta}} = (j^*\mathcal{F})_{\overline{\eta}}\,,$$

where one checks that i^*j_* corresponds to forming the fixed modules under I.

(c) It follows easily from the definitions that the composition

$$\mathcal{F}_{\overline{s}} \xrightarrow{sp} \mathcal{F}^I_{\overline{\eta}} \hookrightarrow \mathcal{F}_{\overline{\eta}}$$

is just the specialization map on the stalks, induced by the morphism

$$\mathcal{O}^h_{T,\overline{s}} \hookrightarrow \mathcal{O}^h_{T,\overline{\eta}} = k(\eta)_s$$

(compare 2.4). In particular, $\mathcal{F}$ is locally constant if and only if I acts trivially on $\mathcal{F}_{\overline{\eta}} = M_1$ and sp is an isomorphism.

(d) If now $f : X \to T$ is a morphism and $\mathcal{F}$ is a sheaf on X, then the higher direct image $R^\nu f_*\mathcal{F}$ is described by the triple

$$((R^i f_*\mathcal{F})_{\overline{s}}\ ,\ (R^i f_*\mathcal{F})_{\overline{\eta}},\ sp:\ (R^i f_*\mathcal{F})_{\overline{s}} \longrightarrow (R^i f_*\mathcal{F})^I_{\overline{\eta}}).$$

If f is proper, then by proper base change this can be identified with a triple

$$(H^i(X_{\overline{x}}, \mathcal{F}),\ H^i(X_{\overline{\eta}}, \mathcal{F})\ ,\ sp:\ H^i(X_{\overline{s}}, \mathcal{F}) \to H^i(X_{\overline{\eta}}, \mathcal{F})^I)\,,$$

where $X_{\overline{s}} = X \times_T \overline{s} = X_s \times_{k(s)} k(\overline{s})$ and $X_{\overline{\eta}} = X \times_T \overline{\eta} = X_\eta \times_{k(\eta)} k(\overline{\eta})$ are the geometric fibers of f at $\overline{s}$ and $\overline{\eta}$.

10.2 The tool for calculating the specialization map is the general theory of vanishing cycles. For this we consider a cartesian diagram

(10.2.1)
$$\begin{array}{ccccc} X_\eta & \xrightarrow{j} & X & \xleftarrow{i} & X_s \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ \eta & \longrightarrow & T & \longleftarrow & s\,, \end{array}$$

where f can be arbitrary of finite type. Let $k(\overline{\eta})^I \subset L \subset k(\overline{\eta})$ be any intermediate field and let B be the integral closure of A in L, i.e., $\overline{T} = \text{Spec } B$ the normalization of T in Spec L (in the literature both $L = K(\overline{\eta})^I$, i.e., $B = O^h_{T,\overline{s}}$ and $L = K(\overline{\eta})$, i.e., $B = \tilde{A}$ = integral closure of A in $K(\overline{\eta})$ are considered). If $\overline{X} = X \times_T \overline{T}$, then we obtain a commutative diagram

(10.2.2)
$$\begin{array}{ccccc} X_{\overline{\eta}} & \xrightarrow{\bar{j}} & \overline{X} & \xleftarrow{\bar{i}} & X_{\overline{s}} \\ \downarrow & & \downarrow f & & \downarrow \\ \overline{\eta} & \longrightarrow & \overline{T} & \longleftarrow & \overline{s} \end{array}$$

which arises from (10.2.1) by base change with the bottom row, and in which both squares are cartesian.

Definition/Lemma 10.3 Let Y be a scheme over a field k with separable closure $\overline{k}$, let $\overline{Y} = Y \times_k \overline{k}$ and let $u : G \to Gal(\overline{k}/k)$ be a homomorphism of topological groups.

(a) A G-sheaf on $\overline{Y}$ is a sheaf $\mathcal{F}$ on $\overline{Y}$ with a continuous discrete action of G, which is compatible with the (right) action of $Gal(\overline{k}/k)$ on $\overline{Y}$; i.e., for every $\sigma \in G$ one has a morphism

$$\sigma_* : \mathcal{F} \longrightarrow (\mathrm{Spec}(u(\sigma))_*\mathcal{F}\,,$$

such that $\tau_*\sigma_* = (\tau\sigma)_*$, and G operates discretely on $\mathcal{F}(\overline{U}) = \mathcal{F}((\mathrm{Spec}\,\sigma)^{-1}\overline{U})$ for every quasi compact étale $U \to Y$, $\overline{U} = U \times_k \overline{k}$. Let $Sh(\overline{Y}, G)$ be the category of G-sheaves on $\overline{Y}$.

(b) Let $\pi \cdot \overline{Y} \to Y$ be the projection. Then there is an equivalence of categories (where sheaf always means ètale sheaf)

$$\begin{array}{ccc} Sh(Y) = (\text{sheaves on } Y) & \leftrightarrow & Sh(\overline{Y}, Gal(\overline{k}/k)) = (Gal(\overline{k}/k)\text{-sheaves on } Y) \\ \mathcal{F} & \mapsto & \pi^*\mathcal{F} \\ (\pi_*\mathcal{G})^{Gal(\overline{k}/k)} & \leftarrow\!\shortmid & \mathcal{G}\,, \end{array}$$

For the proof of (b) see SGA7 XIII 1.1. We note that the morphism

$$\sigma_* : \pi^*\mathcal{F} \longrightarrow (\mathrm{Spec}\,\sigma)_*\pi^*\mathcal{F} \overset{(1)}{=} (\mathrm{Spec}\,\sigma)_*(\mathrm{Spec}\,\sigma)^*\pi^*\mathcal{F}$$

is the adjunction morphism (equality (1) follows from the fact that $\pi = \pi\mathrm{Spec}\,\sigma$).

This allows the definition of the following category and functors.

Definition 10.4 Let $Sh(X_{\overline{s}} \times_s T)$ be the abelian category of the triples $(\mathcal{F}_0, \mathcal{F}_1, \varphi)$, where
(i) $\mathcal{F}_0$ is a G_s-sheaf on $X_{\overline{s}}$,
(ii) $\mathcal{F}_1$ is a G_η-sheaf on $X_{\overline{s}}$ (with respect to $G_\eta \to G_s$), and
(iii) $\varphi : \mathcal{F}_0 \to \mathcal{F}_1$ is a G_η-equivariant morphism.

Definition 10.5 Let $\pi : \overline{X} \to X$, $\pi_0 : X_{\overline{s}} \to X_s$ and $\pi_1 : X_{\overline{\eta}} \to X_\eta$ be the projections. Then define

$$\begin{array}{cccll} \Psi_{\overline{s}} & : & Sh(X_s) & \longrightarrow & Sh(X_{\overline{s}}, G_s) \\ \Psi_{\overline{\eta}} & : & Sh(X_\eta) & \longrightarrow & Sh(X_{\overline{s}}, G_\eta) \qquad \text{(operation with respect to } G_\eta \to G_s) \\ \Psi & : & Sh(X) & \longrightarrow & Sh(X_{\overline{s}} \times_s T) \end{array}$$

by

$$\begin{array}{rcl}\Psi_{\overline{s}}\mathcal{F} & = & \pi_0^*\mathcal{F}\\ \Psi_{\overline{\eta}}\mathcal{F} & = & \overline{i}^*\overline{j}_*\pi_1^*\mathcal{F}\\ \Psi\mathcal{F} & = & (\overline{i}^*\pi^*\mathcal{F},\, \overline{i}^*\overline{j}_*\overline{j}^*\pi^*\mathcal{F},\, \overline{i}^*\pi^*\mathcal{F} \stackrel{\overline{i}^*ad}{\longrightarrow} \overline{i}^*\overline{j}_*\overline{j}^*\pi^*\mathcal{F})\\ & = & (\pi_0^*\, i^*\mathcal{F},\, \overline{i}^*\overline{j}_*\overline{j}^*\pi_1^*j^*\mathcal{F},\, \varphi_F\ = \overline{i}^*ad)\\ & = & (\Psi_{\overline{s}}i^*\mathcal{F},\, \Psi_{\overline{\eta}}j^*\mathcal{F},\, \varphi_F)\,.\end{array}$$

These functors are additive, left exact and have right derivatives $R^i\Psi_{\overline{s}}$, $R^i\Psi_{\overline{\eta}}$ and $R^i\Psi$, resp. $R\Psi_{\overline{s}}$, $R\Psi_{\overline{\eta}}$ and $R\Psi$ in the derived categories: If $\mathcal{F} \hookrightarrow I^\bullet$ is an injective resolution, then $R\Psi\mathcal{F}$ is represented by $\Psi I^\bullet$ (unique up to unique homotopy), and is called the complex of nearby cycles, and $R^i\Psi\mathcal{F} = H^i(\Psi I^\bullet)$ (i-th homology object, unique up to unique isomorphism) is called the i-th sheaf of nearby cycles, similarly one has $R^i\Psi_{\overline{\eta}}$ and $R\Psi_{\overline{\eta}}$, whereas $\Psi_{\overline{s}}$ is exact and has no higher derivatives.

One can interpret a complex in $Sh(X_{\overline{s}} \times_s T)$ as an object $(\mathcal{F}_0^\bullet, \mathcal{F}_1^\bullet, \varphi)$, where $\mathcal{F}_0^\bullet$ is a complex in $Sh(X_{\overline{s}}, G_s)$, $\mathcal{F}_1^\bullet$ is a complex in $Sh(X_{\overline{s}}, G_\eta)$ and $\varphi : \mathcal{F}_0^\bullet \to \mathcal{F}_1^\bullet$ is an equivariant morphism of complexes. If we define the functor

$$sp^* : Sh(X_{\overline{s}}, G_s) \longrightarrow Sh(X_{\overline{s}}, G_\eta)$$

by $sp^*\mathcal{F}_0 = \mathcal{F}_0$, with G_η-operation via $G_\eta \to G_s$, then we can also interpret φ as a morphism

$$\varphi : sp^*\mathcal{F}_0^\bullet \longrightarrow \mathcal{F}_1^\bullet$$

of complexes in $Sh(X_{\overline{s}}, G_\eta)$. To each triple $(\mathcal{F}_0^\bullet, \mathcal{F}_1^\bullet, \varphi)$ one can functorially assign a short exact sequence

$$0 \to \mathcal{F}_1^\bullet \to Cone(\varphi) \to sp^*\mathcal{F}_0^\bullet[1] \to 0$$

where $Cone(\varphi)$ is the cone of φ (see [Mi]S.174,167). If we set

$$\Phi(\mathcal{F}_0^\bullet, \mathcal{F}_1^\bullet, \varphi) = Cone(\varphi)\,,$$

then Φ maps quasi-isomorphisms to quasi-isomorphisms, and for a sheaf $\mathcal{F}$ with injective resolution $\mathcal{F} \hookrightarrow I^\bullet$, the complex

$$R\Phi\mathcal{F} \ := \ \Phi R\Psi\mathcal{F} \quad (= \Phi\Psi I^\bullet)$$

is unique up to unique quasi-isomorphism, hence well-defined in the derived category of $S(X_{\overline{s}}, G_\eta)$.

Definition 10.6 $R\Phi\mathcal{F}$ is called the complex of vanishing cycles. Set

$$R^i\Phi\mathcal{F} = H^i(R\Phi\mathcal{F}) \quad (= H^i(\Phi\Psi I^\bullet))$$

for the i-th sheaf of vanishing cycles of $\mathcal{F}$.

By construction, for every $\mathcal{F}$ in $Sh(X_{et})$ we have a distinguished triangle of complexes in $Sh(X_{\overline{s}}, G_\eta)$

(10.6.1) $$sp^*i^*\mathcal{F} \longrightarrow R\Psi_{\overline{\eta}}\mathcal{F} \longrightarrow R\Phi\mathcal{F} \longrightarrow sp^*i^*\mathcal{F}[1]\ ,$$

which is well-defined in the derived category of $Sh(X_{\overline{s}}, G_\eta)$ and functorial in $\mathcal{F}$. Here, we should have written $sp^*\Psi_{\overline{s}} i^*\mathcal{F}$, but we identify $Sh(X_s)$ and $Sh(X_{\overline{s}}, G_s)$ via $\Psi_{\overline{s}}$ and we also write sp^* for $sp^*\Psi_{\overline{s}}$. For an injective resolution $\mathcal{F} \hookrightarrow I^\bullet$, (10.6.1) is represented by

$$\overline{i}^* I^\bullet \stackrel{\overline{i}^* ad}{\longrightarrow} \overline{i}^* \overline{j}_* \overline{j}^* I^\bullet \longrightarrow Cone(\overline{i}^* ad) \longrightarrow$$

where we suppressed π^*. In the following, we often omit π^*, π_0^* and π_1^*. Note that $\overline{i}^*\mathcal{F}$ is quasi-isomorphic to $\overline{i}^* I^\bullet$.

Somewhat more imprecisely we can also write (10.6.1) as

$$\overline{i}^*\mathcal{F} \longrightarrow \overline{i}^* R\overline{j}_* \overline{j}^* \mathcal{F} \longrightarrow R\Phi\mathcal{F} \longrightarrow \ .$$

But by this it is not so clear that we consider complexes of G_η-sheaves; and apart from that, one can not define $R\Phi\mathcal{F}$ by this: note that the formation of cones is not well-defined in the derived category. The use of triples as above overcomes this problem and gives a more rigidified version.

10.7 In the formalism of vanishing cycles, the operation of the inertia group $I \subseteq G_\eta$ is described by the so-called variation: If $\sigma \in I$, then, by the trivial operation of I on $sp^*i^*\mathcal{F}$, the endomorphism $\sigma - 1$ of $R\Psi_{\overline{\eta}}\mathcal{F}$ factorizes over $R\Phi\mathcal{F}$, and we obtain a canonical commutative diagram

(10.7.1)
$$\begin{array}{ccccccc}
sp^*i^*\mathcal{F} & \longrightarrow & R\Psi_{\overline{\eta}}\mathcal{F} & \longrightarrow & R\Phi\mathcal{F} & \longrightarrow & \\
\downarrow {\scriptstyle 0} & & {\scriptstyle \sigma-1} \downarrow & \swarrow {\scriptstyle Var(\sigma)} & \downarrow {\scriptstyle \sigma-1} & & \\
sp^*i^*\mathcal{F} & \longrightarrow & R\Psi_{\overline{\eta}}\mathcal{F} & \longrightarrow & R\Phi\mathcal{F} & \longrightarrow & ,
\end{array}$$

which is functorial in $\mathcal{F}$. The induced morphism

$$Var(\sigma) : R\Phi\mathcal{F} \to R\Psi_{\overline{\eta}}\mathcal{F}$$

(and the map induced herby in the cohomology) is called the variation of σ. Because of the trivial formula

$$(\sigma\tau - 1) = (\sigma - 1) + (\tau - 1) + (\sigma - 1)(\tau - 1)$$

one has

(10.7.2)
$$\begin{aligned} Var(\sigma\tau) &= Var(\sigma) + Var(\tau) + (\sigma - 1)Var(\tau) \\ &= Var(\sigma) + Var(\tau) + Var(\sigma)(\tau - 1) \ . \end{aligned}$$

The theory of vanishing cycles has the following application: By forming the long exact cohomology sequence on $X_{\overline{s}}$ for (10.6.1), one obtains a long exact sequence of G_η-modules

(10.8.1)
$$\to H^\nu(X_{\overline{s}}, i^*\mathcal{F}) \stackrel{\gamma}{\to} H^\nu(X_{\overline{s}}, R\Psi_{\overline{\eta}}\mathcal{F}) \to H^\nu(X_{\overline{s}}, R\Phi\mathcal{F}) \to H^{\nu+1}(X_{\overline{s}}, \mathcal{F}) \to \dots \ .$$

Furthermore we have the following properties.

Lemma 10.8 (i) There are canonical homomorphisms

(a) $\Psi : H^\nu(X_{\overline{\eta}}, \overline{j}^*\mathcal{F}) \to H^\nu(X_{\overline{s}}, R\Psi_{\overline{\eta}}\mathcal{F})$ for all sheaves $\mathcal{F}$ and all $\nu \in \mathbb{Z}$

(b) $\Psi' : H^\nu_c(X_{\overline{s}}, R\overline{\Psi}_{\overline{\eta}}\mathcal{F}) \to H^\nu_c(X_{\overline{\eta}}, \overline{j}^*\mathcal{F})$ for all torsion sheaves and all $\nu \in \mathbb{Z}$.

(ii) (a): Ψ is contravariantly functorial for open immersions $\mu : U \hookrightarrow X$, i.e., the diagram

$$\begin{array}{ccc} H^\nu(X_{\overline{\eta}}, \overline{j}^*\mathcal{F}) & \xrightarrow{\Psi_X} & H^\nu(X_{\overline{s}}, R\Psi_{\overline{\eta}}\mathcal{F}) \\ \downarrow \mu^* & & \downarrow \mu^* \\ H^\nu(U_{\overline{\eta}}, \overline{j}^*\mathcal{F}|_U) & \xrightarrow{\Psi_U} & H^\nu(U_{\overline{s}}, R\Psi_{\overline{\eta}}\mathcal{F}|_U) \end{array}$$

is commutative, and this is compatible with composition of immersions

(b): Ψ' is covariantly functorial for open immersions $\mu : U \hookrightarrow X$, i.e., the diagram

$$\begin{array}{ccc} H^\nu_c(X_{\overline{s}}, R\Psi_{\overline{\eta}}\mathcal{F}) & \longrightarrow & H^\nu_c(X_{\overline{\eta}}, \overline{j}^*\mathcal{F}) \\ \uparrow \mu_! & & \uparrow \mu_! \\ H^\nu_c(U_{\overline{s}}, R\Psi_{\overline{\eta}}\mathcal{F}|_U) & \longrightarrow & H^\nu_c(U_{\overline{\eta}}, \overline{j}^*\mathcal{F}|_U) \end{array}$$

is commutative, and this is compatible with compositions of open immersions.

(iii) If $f : \overline{X} \to \overline{T}$ is proper and $\mathcal{F}$ is a torsion sheaf on X, then (a) and (b) are isomorphisms which are inverse to each other.

(iv) The composition

$$H^\nu(X_{\overline{s}}, \overline{i}^*\mathcal{F}) \overset{sp}{\to} H^\nu(X_{\overline{\eta}}, \overline{j}^*\mathcal{F}) \overset{\Psi}{\to} H^\nu(X_{\overline{s}}, R\Psi_{\overline{\eta}}\mathcal{F})$$

is equal to the map γ in (10.8.1).

Proof (i) (a): For every sheaf $\mathcal{G}$ on $X_{\overline{\eta}}$ define the composition

(10.8.2) $$\Psi : H^\nu(X_{\overline{\eta}}, \mathcal{G}) \overset{(1)}{\to} H^\nu(\overline{X}, R\overline{j}_*\mathcal{G}) \overset{(2)}{\to} H^\nu(X_{\overline{s}}, \overline{i}^* R\overline{j}_*\mathcal{G})$$

Here, (1) is an isomorphism (composition of derived functors) and (2) is the base change morphism, induced by the base change morphism

(10.8.3) $$\overline{i}^* Rf_* R\overline{j}_*\mathcal{G} \to R(f_{\overline{s}})_* \overline{i}^* R\overline{j}_*\mathcal{G}\,.$$

If f is proper and $\mathcal{G}$ a torsion sheaf, then (10.8.3) is an isomorphism, hence (2) and thus Ψ is an isomorphism. For $\mathcal{G} = \overline{j}^*\mathcal{F}$ we obtain (i)(a), since $R\Psi_{\overline{\eta}}\mathcal{F} = \overline{i}^* R\overline{j}_* \overline{j}^*\mathcal{F}$.

(i) (b): Let $\mu : X \hookrightarrow X'$ be an open immersion into a proper S-scheme. It induces a commutative diagram

$$\begin{array}{ccccc} X'_{\overline{\eta}} & \overset{\overline{j}'}{\hookrightarrow} & X' & \overset{\overline{i}'}{\hookleftarrow} & X'_{\overline{s}} \\ \uparrow \mu_{\overline{\eta}} & & \uparrow \mu & & \uparrow \mu_{\overline{s}} \\ X_{\overline{\eta}} & \overset{\overline{j}}{\hookrightarrow} & X & \overset{\overline{i}}{\hookleftarrow} & X_{\overline{s}} \end{array}$$

with cartesian squares.

For every sheaf $\mathcal{G}$ on $X_{\overline{\eta}}$, by definition we have

$$H^\nu_c(X_{\overline{\eta}}, \mathcal{G}) = H^\nu(X'_{\overline{\eta}}, (\mu_{\overline{\eta}})_!\mathcal{G})\,.$$

Furthermore we have the just defined isomorphism for the proper S-scheme X'

$$\Psi : H^\nu(X'_{\overline{\eta}}, (\mu_{\overline{\eta}})_!\mathcal{G}) \xrightarrow{\sim} H^\nu(X'_{\overline{s}}, (\overline{i}')^* R(\overline{j}')_*(\mu_{\overline{\eta}})_!\mathcal{G})\,.$$

Finally we define a canonical homomorphism

$$\Psi_c : H^\nu_c(X_{\overline{s}}, \overline{i}^* R\overline{j}_*\mathcal{G}) \to H^\nu(X'_{\overline{s}}(\overline{i}')^* R(\overline{j}')_*(\mu_{\overline{\eta}})_!\mathcal{G})\,.$$

This is obtained by a morphism

$$(\mu_{\overline{s}})_!\overline{i}^* R\overline{j}_*\mathcal{G} \to (\overline{i}')^* R(\overline{j}')_*(\mu_{\overline{\eta}})_!\mathcal{G} \tag{10.8.4}$$

which we get by the following adjunctions. Since $(\mu_{\overline{s}})_!$ is left adjoint to $\mu^*_{\overline{s}}$, (10.8.4) corresponds to a morphism

$$\overline{i}^* R\overline{j}_*\mathcal{G} \to \mu^*_{\overline{s}}(\overline{i}')^* R(\overline{j}')_*(\mu_{\overline{\eta}})_!\mathcal{G} = \overline{i}^*\mu^* R(\overline{j}')_*(\mu_{\overline{\eta}})_!\mathcal{G}$$

(note $\mu\overline{i} = \overline{i}'\mu_{\overline{s}}$). By definition, we get this morphism by applying $\overline{i}^*$ to the isomorphism

$$R\overline{j}_*\mathcal{G} = \mu^* R(\overline{j}')_*(\mu_{\overline{\eta}})_!\mathcal{G} = R\overline{j}_*\mu^*_{\overline{\eta}}(\mu_{\overline{\eta}})_!\mathcal{G}$$

(note that $\mu^*_{\overline{\eta}}(\mu_{\overline{\eta}})_! = id$).

For a torsion sheaf $\mathcal{G}$ the morphism

$$\Psi' : H^\nu_c(X_{\overline{s}}, \overline{i}^* R\overline{j}_*\mathcal{G}) \to H^\nu_c(X_{\overline{\eta}}, \mathcal{G})$$

is now defined as the composition $\Psi^{-1}\Psi_c$. For $\mathcal{G} = \overline{j}^*\mathcal{F}$, with a torsion sheaf $\mathcal{F}$, we obtain (i) (b). If X is already proper, then Ψ_c is the identity and $\Psi' = \Psi^{-1}$; this shows (iii).

Claim (iv) follows immediately from the definition of the base change morphism.

Claim (ii) (a) follows, since the base change morphism (10.8.3) is compatible with restriction to open subscheme.

For (ii) (b) let $\rho : U \hookrightarrow X$ be another open immersion. The covariance for ρ regarding the isomorphism Ψ follows from the fact that Ψ is covariant for the morphism of sheaves $(\rho_{\overline{\eta}})_!(\rho_{\overline{\eta}})_*\mathcal{G} \to \mathcal{G}$. For the covariance of Ψ_c we need to construct a suitable commutative diagram

$$\begin{array}{ccc} (\mu_{\overline{s}})_!\overline{i}^* R\overline{j}_*\mathcal{G} & \longrightarrow & (\overline{i}')^* R(\overline{j}')_*(\mu_{\overline{\eta}})_!\mathcal{G} \\ \uparrow & & \uparrow \\ (\mu_{\overline{s}})_!(\rho_{\overline{s}})_!(\overline{i}_U)^* R(\overline{j}_U)_*(\rho_{\overline{\eta}})^*\mathcal{G} & \longrightarrow & (\overline{i}')^* R(\overline{j}')_*(\mu_{\overline{\eta}})_!(\rho_{\overline{\eta}})_!\rho^*_{\overline{\eta}}\mathcal{G}\,, \end{array} \tag{10.8.5}$$

with the notations from the commutative diagram

$$\begin{array}{ccccc} U_{\overline{\eta}} & \xhookrightarrow{\overline{j}_U} & U & \xhookleftarrow{\overline{i}_U} & U_{\overline{s}} \\ \big\downarrow{\scriptstyle \rho_{\overline{\eta}}} & & \big\downarrow{\scriptstyle \rho} & & \big\downarrow{\scriptstyle \rho_{\overline{s}}} \\ X_{\overline{\eta}} & \xhookrightarrow{\overline{j}} & \overline{X} & \xhookleftarrow{\overline{i}} & X_{\overline{s}} \\ \big\downarrow{\scriptstyle \mu_{\overline{\eta}}} & & \big\downarrow{\scriptstyle \mu} & & \big\downarrow{\scriptstyle \mu_{\overline{s}}} \\ X'_{\overline{\eta}} & \xhookrightarrow{\overline{j}'} & \overline{X}' & \xhookleftarrow{\overline{i}'} & X'_{\overline{s}}\,. \end{array}$$

Note here that we have a canonical isomorphism of functors

$$\varepsilon_1 : (\rho_{\overline{s}})_!(\overline{i}_U)^* \underset{\sim}{\longrightarrow} (\overline{i})^*\rho_! \,,$$

as well as a canonical morphism of functors

$$\varepsilon_2 : \rho_! R(\overline{j}_U)_* \longrightarrow R\overline{j}_*(\rho_{\overline{\eta}})_! \,, \tag{10.8.6}$$

which, by adjunction, corresponds to the isomorphism

$$R(\overline{j}_U)_* = R(\overline{j}_U)_*(\rho_{\overline{\eta}})^*(\rho_{\overline{\eta}})_! = \rho^* R\overline{j}_*(\rho_{\overline{\eta}})_! \,.$$

Then we define (10.8.5) by the commutative diagram

$$\begin{array}{ccccc}
 & & (\mu_{\overline{s}})_!\overline{i}^* R\overline{j}_*\mathcal{G} & \longrightarrow & (\overline{i}')^* R(\overline{j}')_*(\mu_{\overline{\eta}})_!\mathcal{G} \\
 & \nearrow^{\eta} & \uparrow{\scriptstyle ad} & & \uparrow{\scriptstyle ad} \\
(\mu_{\overline{s}})_!(\rho_{\overline{s}})_!(\overline{i}_U)^* R(\overline{j}_U)_*(\rho_{\overline{\eta}})^*\mathcal{G} & \xrightarrow{\varepsilon} & (\mu_{\overline{s}})_!\overline{i}^* R\overline{j}_*(\rho_{\overline{\eta}})_!(\rho_{\overline{\eta}})^*\mathcal{G} & \longrightarrow & (\overline{i}')^* R(\overline{j}')_*(\mu_{\overline{\eta}})_!(\rho_{\overline{\eta}})_!(\rho_{\overline{\eta}})^*\mathcal{G} \,.
\end{array}$$

Here the commutative rectangle arises from the functoriality of (10.8.4) for the adjunction morphism $(\rho_{\overline{\eta}})_!(\rho_{\overline{\eta}})^*\mathcal{G} \to \mathcal{G}$, ε is induced by ε_1 and ε_2, and η makes the diagram commutative.

If X is proper over T, then, by Lemma 10.8, (10.8.1) becomes an exact sequence

$$\ldots \to H^\nu(X_{\overline{s}}, i^*\mathcal{F}) \overset{sp}{\to} H^\nu(X_{\overline{\eta}}, j^*\mathcal{F}) \to H^\nu(X_{\overline{s}}, R\Phi\mathcal{F}) \to H^{\nu+1}(X_{\overline{s}}, i^*\mathcal{F}) \to \ldots \,, \tag{10.8.7}$$

by replacing $H^\nu(X_{\overline{s}}, R\Psi_{\overline{\eta}}\mathcal{F})$ by $H^\nu(X_{\overline{\eta}}, \mathcal{F})$ via Ψ. Hence the study of *sp* is reduced to the calculation of $R\Phi\mathcal{F}$.

This is a local problem, more precisely: for a geometric point $\overline{a}$ of $X_{\overline{s}}$, the stalk $(R\Phi\mathcal{F})_{\overline{a}}$ in $\overline{a}$ depends only on the strict henselization $O^h_{X,\overline{a}}$ of X in $\overline{a}$, since this holds for $\mathcal{F}_{\overline{a}}$ and $(R\overline{j}_*\overline{j}^*\mathcal{F})_{\overline{a}}$, and one has a an excellent triangle

$$\mathcal{F}_{\overline{a}} \longrightarrow (R\overline{j}_*\overline{j}^*\mathcal{F})_{\overline{a}} \longrightarrow (R\Phi\mathcal{F})_{\overline{a}} \longrightarrow \,.$$

By the next lemma, $R\Phi\mathcal{F}$ is concentrated only in the singular points of f, if $\mathcal{F}$ is locally constant on the smooth locus of f.

Lemma 10.9 If f is smooth and $\mathcal{F}$ is locally constant, then $R\Phi\mathcal{F} = 0$.

Proof Since one can test the vanishing on étale neighborhoods, $\mathcal{F} = \Lambda$ is constant without restriction. We have to show that

$$\overline{i}^*\Lambda \overset{\overline{i}^* ad}{\longrightarrow} \overline{i}^* R\overline{j}_*\overline{j}^*\Lambda \tag{10.9.1}$$

is a quasi isomorphism. We consider the cartesian diagram

$$\begin{array}{ccccc}
X_{\overline{\eta}} & \xrightarrow{\overline{j}} & \overline{X} & \xleftarrow{\overline{i}} & X_{\overline{s}} \\
\downarrow{\scriptstyle f_{\overline{\eta}}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_{\overline{s}}} \\
\overline{\eta} & \xrightarrow{\overline{j}} & \overline{T} & \xleftarrow{\overline{i}} & \overline{s} \,.
\end{array}$$

First we note that the canonical morphisms

$$\Lambda_{\overline{T}} \xrightarrow{ad} \overline{j}_* \overline{j}^* \Lambda_{\overline{T}} \longrightarrow R\overline{j}_* \overline{j}^* \Lambda_{\overline{T}}$$

become isomorphisms after applying $\overline{i}^*$, since for $\pi_1 : \overline{\eta} \to \eta$ we have

$$(R^\nu \overline{j}_* \Lambda)_{\overline{s}} = (R^\nu j_* \pi_{1*} \Lambda)_{\overline{s}} = H^\nu(I, Ind_I(\Lambda)) \,,$$

where $Ind_I(\Lambda)$ denotes the induced module. Furthermore it is known that an induced module is cohomologically trivial, hence $H^\nu(I, Ind_I(\Lambda)) = 0$ for $\nu > 0$, while $H^0(I, Ind_I(\Lambda)) = \Lambda$.

This implies the claim of 10.9 by applying $\overline{i}^*$ to the base change morphism

$$f^* R\overline{j}_* \Lambda_{\overline{\eta}} \longrightarrow R\overline{j}_* f_{\overline{\eta}}^* \Lambda_{\overline{\eta}} \;=\; R\overline{j}_* \overline{j}^* \Lambda_{\overline{X}} \,,$$

since the latter is a quasi isomorphism by the smooth base change theorem, which we will now recall:

COH 14 = Theorem 10.10 Smooth base change: Let

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y' & \xrightarrow{f} & Y \end{array}$$

be a cartesian diagram with quasi-compact π and smooth f. If $\mathcal{F}$ is a torsion sheaf on X, whose torsion is prime to $char(X)$ (i.e., for all $x \in X$, $char(k(x)) = 0$ holds or the multiplication with $char(k(x))$ is an isomorphism on $\mathcal{F}$), then the base change morphism

$$f^* R^i \pi_* \mathcal{F} \longrightarrow R^i \pi'_* f'^* \mathcal{F}$$

is an isomorphism for all $i \geq 0$.

For the proof see [Mi] VI §4: In the Proof of Lemma 10.9, we have $f = f$ and $\pi = \overline{j}$.

11 Cohomology of affine and projective spaces, and the purity theorem

In this chapter we use the smooth base change theorem for proving three other important theorems which were used by Deligne.

Theorem 11.1 (Homotopy invariance) Let S be a locally noetherian scheme and let $\mathcal{F}$ be an étale torsion sheaf on S whose torsion is prime to the characteristics on S (This means: If $U \to S$ is étale and $a \in \mathcal{F}(U)$ and $m \in \mathbb{N}$ with $m \cdot a = 0$, then m is invertible on S, i.e., invertible in $\Gamma(S, \mathcal{O}_S)$, i.e., invertible in $k(s)$ for every $s \in S$). Then for the morphism $\pi : \mathbb{A}^1_S \to S$, the induced morphism

$$\pi^* : H^i(S, \mathcal{F}) \xrightarrow{\sim} H^i(\mathbb{A}^1_S, \pi^*\mathcal{F}),$$

is an isomorphism for all $i \geq 0$.

(By iteration one obtains $H^i(S, \mathcal{F}) \xrightarrow{\sim} H^i(\mathbb{A}^n_S, \mathcal{F})$ for all $i \geq 0$).

Proof: By considering the spectral sequence

$$E_2^{p,q} = H^p(S, R^q\pi_*\pi^*\mathcal{F}) \Rightarrow H^{p+q}(\mathbb{A}^1_S, \pi^*\mathcal{F})$$

it suffices to show:

(i) $\mathcal{F} \xrightarrow{\sim} \pi_*\pi^*\mathcal{F}$ is an isomorphism,

(ii) $R^\nu\pi_*\pi^*\mathcal{F} = 0$ for $\nu > 0$

(i.e., $\mathcal{F} \to R\pi_*\pi^*\mathcal{F}$ is a quasi isomorphism).

Proof of (i) and (ii): Since any torsion sheaf is a filtered inductive limit of constructible sheaves, we may consider these, and since we can check the vanishing of sheaves on an étale covering, we may assume that $\mathcal{F}$ is constant. Hence we may consider $\mathbb{Z}/r$ with r invertible on S. Furthermore, the claim holds if it holds for all strict Henselizations in all points of S. We use induction over the dimension of S. Let $S = \mathrm{Spec}(R)$ be for a reduced strict henselian ring with closed point s. For $\dim(S) = 0$, $R = k$ is a separably closed field, and, since r is invertible in k, we obtain :

$$\begin{aligned} H^0(A^1_k, \mathbb{Z}/r) &= \mathbb{Z}/r, \\ H^1(A^1_k, \mathbb{Z}/r) &\cong H^1(A^1_k, \mu_r) \xrightarrow{\sim} Pic(A^1_k)[r] = 0, \\ H^\nu(A^1_k, \mathbb{Z}/r) &= 0 \quad \text{for} \quad \nu > 1 \quad \text{by weak Lefschetz.} \end{aligned}$$

For $\dim(S) > 0$ let $U = S \smallsetminus \{s\}$. Then we have $\dim(U) < \dim(S)$ and we can assume that the claim is already proven for U. Let $U \overset{j}{\hookrightarrow} S$ be the open immersion. In the distinguished triangle

(11.1.1) $$\mathbb{Z}/r \to Rj_*j^*\mathbb{Z}/r \to \mathcal{G} \to,$$

$\mathcal{G}$ is concentrated in s (apply j^*), and thus $\mathcal{G} \to R\pi_*\pi^*\mathcal{G}$ is a quasi isomorphism. It thus suffices to show that

(11.1.2) $$Rj_*j^*\mathbb{Z}/r \to R\pi_*\pi^*Rj_*j^*\mathbb{Z}/r$$

is a quasi isomorphism; then by (11.1.1) the claim follows for $\mathbb{Z}/r$ as well. But (11.1.2) is the composition of

$$Rj_*j^*\mathcal{F} \underset{\sim}{\overset{(1)}{\to}} Rj_*R\pi_*\pi'^*j^*\mathcal{F} = R\pi'_*Rj'_*\pi'^*j^*\mathcal{F} \underset{\sim}{\overset{(2)}{\to}} R\pi_*\pi^*Rj_*j^*\mathcal{F}$$

where $\mathcal{F} = \mathbb{Z}/r$, and the morphisms come from the cartesian diagram

(11.1.3)
$$\begin{array}{ccc} \mathbb{A}^1_U & \overset{j'}{\hookrightarrow} & \mathbb{A}^1_S \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ U & \overset{j}{\hookrightarrow} & S\,. \end{array}$$

Now (1) is a quasi isomorphism by validity of the claim on U, and (2) is a quasi isomorphism by smooth base change for (11.1.3) (smooth morphism = p, structural morphism = j).

Next consider projective spaces.

Theorem 11.2 Let $q : P = \mathbb{P}^m_S \to S$ be the m-dimensional projective space over the scheme S, and let $r \in \mathbb{N}$ be invertible on S. There are canonical isomorphisms

$$R^i q_*\mathbb{Z}/r \cong \begin{cases} \mathbb{Z}/r(-j) & , \quad i = 2j \text{ even, } 0 \le i \le 2m, \\ 0 & , \quad \text{otherwise} \end{cases}$$

More precisely, for $m \geq 1$ we have:
(i) $R^2q_*\mathbb{Z}/r \cong \mathbb{Z}/r(-1)$, and
(ii) the cupproduct induces an isomorphism $(R^2q_*\mathbb{Z}/r)^{\otimes j} \xrightarrow{\sim} R^{2j}q_*\mathbb{Z}/r$ for $j \le m$.

Proof The Kummer sequence $0 \to \mu_r \to \mathbb{G}_m \xrightarrow{r} \mathbb{G}_m \to 0$ gives a canonical element $\eta \in H^2(\mathbb{P}^m_S, \mu_r)$, the image of the class of the canonical $\mathcal{O}_P$-Moduls $\mathcal{O}(1)$ under the connecting morphism δ

$$Pic(P) = H^1(P, \mathbb{G}_m) \xrightarrow{\delta} H^2(P, \mu_r)\,.$$

Denote by η also the image of η under the canonical morphism

$$H^2(P, \mu_r) \to H^0(S, R^2q_*\mu_r)$$

(note that $R^2q_*\mu_r$ is the associated sheaf to the presheaf $U \mapsto H^2(\mathbb{P}^m_U, \mu_r)$). We claim that $R^2q_*\mu_r \cong \mathbb{Z}/r$, with base η, and that $R^iq_*\mathbb{Z}/r = 0$ for i odd or $i > 2m$. By proper base change it suffices to prove this on the fibers of q, i.e., for $S = \operatorname{Spec}, k$ k separably closed. Then we have a decomposition

$$H = \mathbb{P}^{m-1}_k \overset{i}{\hookrightarrow} \mathbb{P}^m_k \overset{j}{\hookleftarrow} \mathbb{A}^m_k = \mathbb{P}^m_k - H\,,$$

where H is a hyperplane in $\mathbb{P}^m_k$. The long exact sequence

$$\ldots \to H^\nu_c(\mathbb{A}^m_k) \to H^\nu(\mathbb{P}^m_k) \xrightarrow{i^*} H^\nu(\mathbb{P}^{m-1}_k) \to H^{\nu+1}_c(\mathbb{A}^m_k) \to \ldots$$

(constant coefficients) and the fact that we have canonical isomorphisms

$$H^\nu_c(\mathbb{A}^m_k) \cong H^{2m-\nu}(\mathbb{A}^m_k, \mathbb{Z}/r(m))^\vee = \begin{cases} 0 & , \quad \nu \neq 2m \\ \mathbb{Z}/r(-m) & , \quad \nu = 2m \end{cases}$$

by Poincaré-duality and by Theorem 11.1, show inductively

$$H^2(\mathbb{P}_k^m, \mu_r) \cong \mathbb{Z}/r, \text{ with base } \eta\,,$$

$$H^i(\mathbb{P}_k^m) = \begin{cases} 0 & , \quad i \text{ odd or } i > 0, \\ \mathbb{Z}/r(-j) & , \quad 0 \le i = 2j \le 2m\,. \end{cases}$$

To show (ii), it suffices to show that $H^{2m}(\mathbb{P}_k^m, \mathbb{Z}/r(m))$ is generated by η^m. This follows from the fact that $tr(\eta^m) = \deg(\eta^m) = 1$.

Now we consider the so-called purity.

COH 15= Theorem 11.3 (Purity): Let S be a scheme and let (Y, X) be a smooth S-pair of codimension c, i.e., one has a diagram

$$\begin{array}{ccc} Y & \overset{i}{\hookrightarrow} & X \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\ & S & \end{array}$$

where f and g are smooth, and i is a closed immersion such that the geometric fibers $Y_{\overline{x}} \hookrightarrow X_{\overline{s}}$ have constant codimension c for all $s \in S$. Let $\mathcal{F}$ be a locally constant constructible $\mathbb{Z}/r$-sheaf, with r invertible on S. Then we have

$$R^\nu i^! \mathcal{F} = \begin{cases} 0 & \nu \ne 2c, \\ i^*\mathcal{F} \otimes R^{2c} i^! \mathbb{Z}/r \quad , & \nu = 2c. \end{cases}$$

Furthermore $R^{2c} i^! \mathbb{Z}/r$ is (étale) locally isomorph to $\mathbb{Z}/r(-c)$, and compatible with base change on S.

Equivalent: Let $j : U \hookrightarrow X$ the open complement of Y, then $\mathcal{F} \xrightarrow{\sim} j_*j^*\mathcal{F}, R^i j_* j^* \mathcal{F} = 0$ for $i \ne 0, 2c-1$, and $i^* R^{2c-1} j_* j^* \mathcal{F}$ is locally isomorph to $i^*\mathcal{F}(-c)$, and compatible with base change on S.

Proof The equivalence of the conditions follows from the distinguished triangle

(11.3.1) $$i_* R i^! \mathcal{F} \to \mathcal{F} \to R j_* j^* \mathcal{F} \to \ .$$

We prove the second version. The claim is local on X for the étale topology, therefore without restriction $\mathcal{F} = \Lambda$ is constant and (Y, X) is the smooth S-pair

$$\begin{array}{ccc} \mathbb{A}_S^{m-c} & \hookrightarrow & \mathbb{A}_S^m \\ & \searrow \quad \swarrow & \\ & S\,. & \end{array}$$

By induction it suffices to consider the case $c = 1$, and one can use $\mathbb{A}_S^{m-1}$ as base, i.e., without restriction we consider $S \hookrightarrow \mathbb{A}_S^1$ (the zero section). In the same way we can consider the zero section $S \hookrightarrow \mathbb{P}_S^1$ (note that for $i : S \xrightarrow{i'} \mathbb{A}_S^1 \xrightarrow{j'} \mathbb{P}_S^1$ we have: $i' = (i')^!(j')^*$), i.e., we have to show the claim for the diagram

$$\begin{array}{ccccc} Y = S & \overset{i}{\hookrightarrow} & \mathbb{P}_S^1 & \overset{j}{\hookleftarrow} & \mathbb{A}_S^1 = U \\ & \searrow & \downarrow{\scriptstyle q} & \swarrow{\scriptstyle p} & \\ & & S & & \end{array}\,.$$

For this consider the Leray spectral sequence

$$E_2^{s,t} = R^s q_* R^t j_* \Lambda \quad \Rightarrow \quad R^{s+t} p_* \Lambda\,.$$

For $t > 0$, $R^t j_*\Lambda$ is concentrated in S, since then $j^* R^t j_* \Lambda = 0$ (one has $j^* R j_* \Lambda \cong \Lambda$). For $t = 0$, we claim that $j_*\Lambda = \Lambda$. This can be checked in the stalks at geometric points $\overline{x}$ of X. For $\overline{x}$ over U, the claim is obvious; hence let $\overline{x}$ be over S and let $\tilde{X} = \operatorname{Spec} \mathcal{O}^h_{X,\overline{x}}$ be the strict henselization of X in x. Then we have

$$(j_*\Lambda)_{\overline{x}} = H^0(U \times_X \tilde{X}, \Lambda)\,,$$

and hence we have to show that $U \times_X \tilde{X}$ is connected. Since we took the Henselization in $\overline{x}$, we can assume without restriction that $S = \operatorname{Spec}(A)$ is affine, and consider the situation

$$S \overset{i}{\hookrightarrow} A^1_S = \operatorname{Spec}(A[T]) \overset{j}{\hookleftarrow} \mathbb{G}_{m,S} = \operatorname{Spec}(A[T, T^{-1}])\,.$$

Then $U \times_X \tilde{X} = \operatorname{Spec}(R[T, T^{-1}])$, where $R = \mathcal{O}^h_{X,\overline{x}}$ and T also denotes the image of T in R. Since T is not a zero divisor in R, $D(T) = \operatorname{Spec}(R[T, T^{-1}])$ is dense in $\operatorname{Spec}(R)$ (If $\emptyset \neq D(f) = \operatorname{Spec} R_f$ is a standard-affine set, then fT is not nilpotent, hence $\emptyset \neq D(fT) = D(f) \cap D(T)$). Since R is connected, this also holds for $R[T, T^{-1}]$.

Hence, in the spectral sequence above, we have $R^s q_* R^t j_* \Lambda = 0$ for $s > 0$ and $t > 0$, since $R^s q_* i_* \mathcal{F} = R^s id_* \mathcal{F} = 0$ for $s > 0$ and every sheaf $\mathcal{F}$ on S. Since furthermore $R^{s+t} p_* \Lambda = 0$ for $s + t > 0$ by Theorem 10.1, we have

$$q_* R^t j_* \Lambda \underset{\sim}{\overset{d^{0,t}_{t+1}}{\to}} R^{t+1} q_*(j_*\Lambda) = R^{t+1} q_* \Lambda$$

for $t \geq 1$. Since $R^t j_* \Lambda \overset{\sim}{\to} i_* i^* R^t j_* \Lambda$ for $t \geq 1$, we get

$$i^* R^t j_* \Lambda \overset{\sim}{\to} R^{t+1} q_* \Lambda \cong \begin{cases} \Lambda(-1) & t = 1 \\ 0 & t > 1 \end{cases}$$

by Theorem 11.2, and thus the claim – the base change property follows from the fact that $R^t j_* \Lambda$ is universally locally constructible (i.e., locally constructible in an arbitrary situation $U' \hookrightarrow X'$ obtained by base change (compare [Mi] VI Proof of 2.3, V 1.7).

Remark 11.4 In the situation of 11.3, if $S = \operatorname{Spec} k$ for a field k, one can show that one has a canonical isomorphism

$$R^{2c} i' \Lambda \cong \Lambda(-c)$$

This provides a canonical isomorphism $\Lambda \overset{\sim}{\to} R^{2c} i' \Lambda(c)$, or in other words, a canonical element in $H^0(Y, R^{2c} i' \Lambda(c))$ which is also called the local cycle class of Y.

12 Local Lefschetz theory

In both papers of Deligne the induction principle is given by the theory of Lefschetz pencils. The idea is to fiber a given variety X over a curve – even over the $\mathbb{P}^1$ – in the best possible way, i.e., to to construct a morphism $f : X \to \mathbb{P}^1$ which has mild singularities. Then one can study the cohomology of a sheaf $\mathcal{F}$ on X by using the Leray spectral sequence for f. The calculation of the higher direct image sheaves $R^j f_*\mathcal{F}$ is related to the cohomology on the fibers, which have smaller dimension than X. If the fibers have only mild singularities, then the $R^j f_*\mathcal{F}$ differ only slightly from smooth sheaves, and the cohomology groups $H^i(\mathbb{P}^1, R^j f_*\mathcal{F})$ are accessible for calculation.

Such a morphism f can not be found in general, but the following result, which we will prove in §14, suffices for the application.

Theorem 12.1 Let X be a smooth projective variety over a algebraic closed field k. Then there are morphisms

$$X \xleftarrow{\pi} \widetilde{X} \xrightarrow{f} \mathbb{P}^1_k$$

with the following properties:

(i) $\widetilde{X}$ is a blowing-up of X in a smooth, closed subvariety $A \subset X$ of codimension 2, in particular, $\widetilde{X}$ is smooth and projective.

(ii) The fibers of f are smooth except for some fibers over a finite set Σ of closed points in $\mathbb{P}^1_k$.

(iii) For $s \in \Sigma$ the fibers X_s of f over s have exactly one singular point, which is an ordinary quadratic singularity (see below).

Remark 12.2 The morphisms are constructed as follows. Let $X \hookrightarrow \mathbb{P}^N_k$ be a closed immersion and let $(\mathbb{P}^N_k)^\vee$ be the duale projective space, which parametrizes the hyperplanes in $\mathbb{P}^N_k$: If $\mathbb{P}^N_k$ has the homogeneous coordinates $X_0, \ldots, X_N$, then the point $(a_0 : \ldots : a_N) \in (\mathbb{P}^N_k)^\vee(k)$ corresponds to the hyperplane with the equation $a_0X_0 + \ldots + a_NX_N = 0$. If $L \cong \mathbb{P}^1_k \hookrightarrow (\mathbb{P}^N_k)^\vee$ is a line, then this gives a family

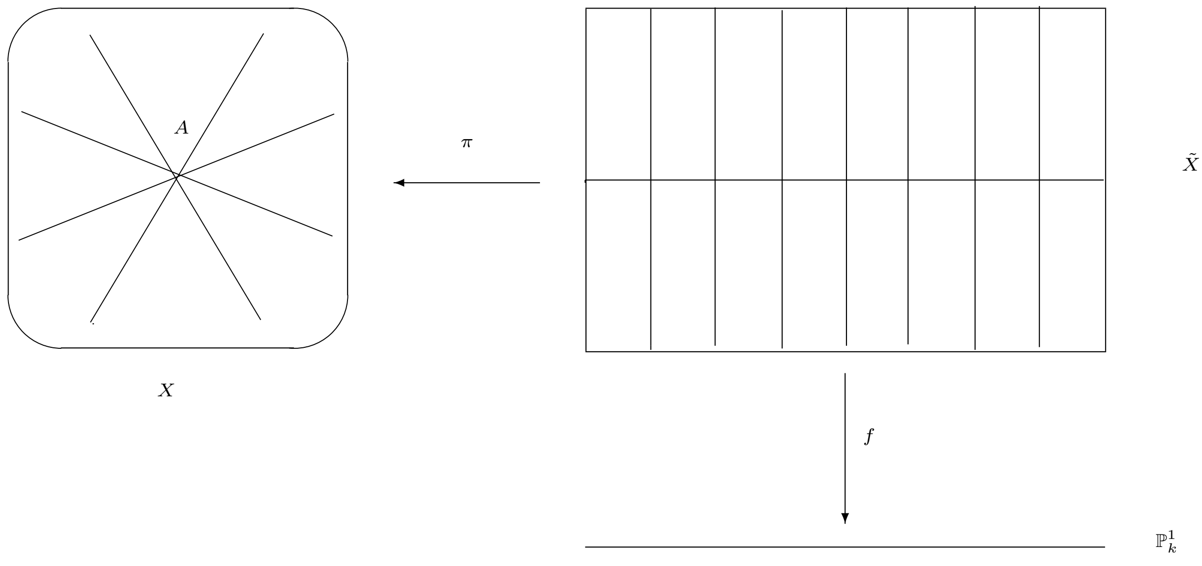
$$(H_t)_{t\in\mathbb{P}^1_k}$$

of hyperplanes, and the hyperplane sections $X_t = X \cdot H_t = X \times_{\mathbb{P}^N_k} H_t$ form a family, for which the following holds if L is suitably chosen: For $t_1 \neq t_2$ set $A = H_{t_1} \cap H_{t_2}$. Then A is independent of t_1, t_2 and of codimension 2 in X, and (X_t) is the family of hyperplane sections which contain A. There is a morphism

$$\begin{array}{c} \tilde{X} \\ \Big\downarrow f \\ \mathbb{P}^1_k \end{array}$$

with fibers $\tilde{X}_t = X_t$, and $\tilde{X}$ is the blowing-up of X in A.

Picture (for surfaces):



Furthermore there are lines $\mathbb{P}^1_k \subset (\mathbb{P}^N_k)^\vee$ which satisfy the properties in 12.1. Then $\pi : \tilde{X} \to \mathbb{P}^1_k$ is called a Lefschetz bundle and A the axis of the bundle.

We recall the following

Definition 12.3 Let k be an algebraically closed field and let Y be a scheme of finite type over k. A closed point $y \in Y$ is called an ordinary quadratic singularity, if the completion $\hat{\mathcal{O}}_{Y,y}$ of the local ring $\mathcal{O}_{Y,y}$ at y is isomorphic to a ring of the form

$$k[[x_0, \ldots, x_n]]/\langle g \rangle ,$$

where $g \in \langle x_0, \ldots, x_n \rangle^2$ and

$$g \equiv Q \mod \langle x_0, \ldots, x_n \rangle^3$$

with a non-trivial quadratic form $Q(x_0, \ldots, x_n)$ which is smooth, i.e., for which the subvariety in $\mathbb{P}^n_k$ defined by $Q = 0$ is smooth.

If k is arbitrary with algebraic closure $\overline{k}$, then y is called ordinary quadratic singularity, if all points of $Y \times_k \overline{k}$ over y are ordinary quadratic singularities.

Remark 12.4 (a) The number n is the local dimension of Y at y.

(b) From the theory of quadratic forms (compare Bourbaki Algèbre Chap. X) and the Jacobi criterion for smoothness it follows easily that the following claims are equivalent:
(i) Q is non-trivial and smooth,
(ii) By linear base change, Q can be brought into the following standard form Q_n:

$$Q_n(x_0, \ldots, x_n) = \begin{cases} x_0x_1 + x_2x_3 + \ldots + x_{n-1}x_n , & \text{if n is odd,} \\ x_0^2 + x_1x_2 + x_3x_4 + \ldots + x_{n-1}x_n , & \text{if n is even.} \end{cases}$$

(c) If $char\, k \neq 2$ or n is odd, then this is also equivalent to:

(iii) the symmetric bilinear form associated to Q

$$B(x,y) = Q(x+y) - Q(x) - Q(y)$$

is non-degenerate, or

(iv) the Hesse matrix $(\frac{\delta^2 Q}{\delta x_i \delta x_j}(0))$ is invertible. If these conditions are satisfied, then Q (and the singularity y) is called non-degenerate.

(d) If $char\, k = 2$ and n is even, then there are no non-degenerate Q and y.

(e) If $n = 1$, then $\hat{O}_{Y,y} \cong k[[x_0, x_1]](x_0 x_1)$, and one calls y an ordinary double point. This notation is sometimes used for arbitrary n.

We now study the morphism $f : \tilde{X} \to \mathbb{P}^1_k = S$ for the Lefschetz bundle at the bad places $s \in \Sigma$. Here we can pass to the henselization $\mathcal{O}^h_{S,s}$; this is a henselian discrete valuation ring.

We consider more generally the following situation. Let $T = \mathrm{Spec}(A)$ for a henselian discrete valuation ring A, let $s \in T$ be the closed point and $\eta \in T$ the generic point. Furthermore let

$$f : X \to T$$

be a proper surjective morphism. By §10, the higher direct image sheaf $R^i f_* \mathcal{F}$ for a sheaf $\mathcal{F}$ on X is described by the triplet

$$((R^i f_* \mathcal{F})_{\bar{s}}, (R^i f_* \mathcal{F})_{\bar{\eta}},\ sp : (R^i f_* \mathcal{F})_{\bar{s}} \to (R^i f_* \mathcal{F})^I_{\bar{\eta}}),$$

where $I \subset G_{k(\eta)}$ is the inertia group. Since f is proper, this can be identified by proper base change with the triplet

$$(H^i(X_{\bar{x}}, \mathcal{F}),\ H^i(X_{\bar{\eta}}, \mathcal{F}), sp : H^i(X_{\bar{s}}, \mathcal{F}) \to H^i(X_{\bar{\eta}}, \mathcal{F})^I),$$

where $X_{\bar{s}} = X \times_T \bar{s} = X_s \times_{k(s)} k(\bar{s})$ and $X_{\bar{\eta}} = X \times_T \bar{\eta} = X_\eta \times_{k(\eta)} k(\bar{\eta})$ are the geometric fibers of f at $\bar{s}$ and $\bar{\eta}$.

The following theorem is the main result of the local Lefschetz theory.

Theorem 12.5 Let $f : X \to T$ be a flat proper morphism of fiber dimension n such that the generic fiber X_η is smooth and the geometric special fiber $X_{\bar{s}}$ has exactly one singular point a; let this be an ordinary quadratic singularity. Finally let $\Lambda = \mathbb{Z}/\ell^r$ for an $r \in \mathbb{N}$ and a prime number ℓ which is invertible on T.

(a) For $\nu \neq n, n+1$, $sp : H^\nu(X_{\bar{x}}, \Lambda) \to H^\nu(X_{\bar{\eta}}, \Lambda)$ is an isomorphism.

(b) If $n = 2m+1$ is odd, then over an étale covering of T we have:

(i) There is an exact sequence of $Gal(k(\bar{\eta})/k(\eta))$-modules

$$0 \to H^n(X_{\bar{s}}, \Lambda) \stackrel{sp}{\to} H^n(X_{\bar{\eta}}, \Lambda) \stackrel{\alpha}{\to} \Lambda(m-n) \to H^{n+1}(X_{\bar{s}}, \Lambda) \stackrel{sp}{\to} H^{n+1}(X_{\bar{\eta}}, \Lambda) \to 0\,.$$

(ii) Let the so-called vanishing cycle $\delta \in H^n(X_{\bar{\eta}}, \Lambda)(m)$ be defined by the fact that $\alpha(x) = < x, \delta >$ for $x \in H^n(X_{\bar{\eta}}, \Lambda)$ where

$$< \, , \, >: H^n(X_{\bar{\eta}}, \Lambda) \times H^n(X_{\bar{\eta}}, \Lambda)(m) \to H^{2n}(X_{\bar{\eta}}, \Lambda)(m) \stackrel{tr}{\to} \Lambda(m-n)$$

is the composition of the cup product and the trace map.

Then there is a $Gal(k(\overline{s})/k(s))$-equivariant character

$$\chi:\ I \to \mathbb{Z}_\ell(1)\,,$$

such that for $x \in H^n(X_{\overline{\eta}}, \Lambda)$ and $\sigma \in I$ we have the following Picard-Lefschetz-formula

$$\sigma x = x + \chi(\sigma)\langle x, \delta\rangle \delta\,. \tag{12.5.1}$$

(iii) We have $\chi = a\ t_\ell$ for some $a \neq 0$ in $\mathbb{Z}$, where

$$t_\ell : I \to \mathbb{Z}_\ell(1)$$

is the fundamental character: its reduction modulo $\ell^r, t_{\ell^r} : I \to \mu_{\ell^r}$, is given by $\chi(\sigma) = \sigma(\sqrt[\ell^r]{\pi})/\sqrt[\ell^r]{\pi}$ for every prime element π in $A = O(T)$.

(c) If $n = 2m$ is even, then, by passing to a finite étale covering of T, there is a non-trivial character

$$\epsilon : G_\eta \to \{\pm 1\}$$

such that the following holds:

(i) There is an exact sequence of $Gal(k(\overline{\eta}/k(\eta))$-modules

$$0 \to H^n(X_{\overline{s}}, \Lambda) \overset{sp}{\to} H^n(X_{\overline{\eta}}, \Lambda) \overset{\alpha}{\to} \Lambda(m-n)(\epsilon) \to H^{n+1}(X_{\overline{s}}, \Lambda) \overset{sp}{\to} H^{n+1}(X_{\overline{\eta}}, \Lambda) \to 0\,,$$

where $\Lambda(m-n)(\epsilon) = \Lambda(m-n) \otimes \Lambda(\epsilon)$. Here, $\Lambda(\epsilon) \cong \Lambda$ as an abelian group, with operation of G_η via ϵ.

(ii) The Picard-Lefschetz-formula here is

$$\sigma x = x \pm \frac{\epsilon(\sigma) - 1}{2}\langle x, \delta\rangle \delta\,;$$

in particular, we have $\sigma x = x$ for $\sigma \in Ker(\epsilon)$.

We only prove (a) and (b) (i) and (ii), for odd $n = 2m+1$, since we need only this below. We use the following local description of ordinary quadratic singularities.

Lemma 12.6 Let $T = \operatorname{Spec} A$ for a strictly henselian ring A and let $f : X \to T$ be a flat morphism of finite type. Let the relative dimension n of f be odd. Then the fiber X_s over the closed point $s \in T$ has an ordinary quadratic singularity in a closed point $y \in X_s$ if and only if there is a $\lambda \neq 0$ in the maximal ideal $\mathfrak{m}$ of A, so that X at y as an A-scheme is locally isomorphic to $X_{n,\lambda} = \operatorname{Spec}\, R_{n,\lambda}$ at y_n for the étale topology, where

$$R_{n,\lambda} = A[x_0, \ldots, x_n]/(Q_n + \lambda)$$

and $y_n = \operatorname{Spec}\, (R_{n,\lambda}/\mathfrak{m} + (x_0, \ldots, x_n))$ (i.e., over A there are isomorphic étale neighborhoods of the geometric points y and y_n, i.e., the strict henselizations $O^h_{X,y}$ and $O_{X_{n,\lambda}, y_n}$ are isomorphic over A). Here we have

$$Q_n(x_0, \ldots, x_n) = x_0x_1 + x_2x_3 + \ldots + x_{n-1}x_n$$

(compare 12.4 (b)).

Proof: See SGA 7, 2, XV 1.3.2.

Now we use the theory of the vanishing cycles from §10.

In particular, by Lemma 10.9, for an isolated ordinary quadratic singularity as in 12.5, the complex $R\Phi\Lambda$ is concentrated in the singular point a. By 12.6, for calculating the stalk at a it suffices to consider the following situation, where, as in 12.6, $T = \operatorname{Spec} A$.

Let $X \subset \mathbb{P}_T^{n+1}$ be the projective quadric of relative dimension n, which is defined via the equation

$$Q_n(X_0, \ldots, X_n) + \lambda X_{n+1}^n = 0 \qquad , 0 \neq \lambda \in \mathfrak{m}\,,$$

(with Q as above, hence with n odd). Let $Y \subset X \subset \mathbb{P}^{n+1}$ be the hyperplane section with the hyperplane $X_{n+1} = 0$, i.e., defined by

$$Q_n(X_0, \ldots, X_n) = 0 \qquad \text{in } < X_{n+1} = 0 > \cong \mathbb{P}_T^n\,,$$

and let $\overset{\circ}{X} = X - Y$ be the open complement, which is the singular quadric

$$Q_n(x_0, \ldots, x_n) + \lambda = 0\,.$$

in an affine space $A_T^{n+1} = \mathbb{P}_T^{n+1} - H$ with coordinates $x_0 = \frac{X_0}{X_{n+1}}, \ldots, x_n = \frac{X_n}{X_{n+1}}$. Then $\overset{\circ}{X}$ has exactly one ordinary quadratic singularity in the point $a = (0, \ldots, 0)$ of the special fiber $\overset{\circ}{X}_s$.

Lemma 12.7 The following canonical maps (for Ψ and Ψ' see 10.8) are isomorphisms for all i, where $\Lambda = \mathbb{Z}/\ell^r$ for $r \in \mathbb{N}$ and a prime ℓ invertible on T.

(a) $H^i(\overset{\circ}{X}_{\overline{\eta}}, \Lambda) \overset{\Psi}{\longrightarrow} H^i(\overset{\circ}{X}_{\overline{s}}, R\Psi_{\overline{\eta}}\Lambda)$,

(b) $H^i(\overset{\circ}{X}_{\overline{s}}, R\Psi_{\overline{\eta}}\Lambda) \overset{\sim}{\to} H^i(\{\overline{a}\}, R\Psi_{\overline{\eta}}\Lambda) = (R^i\Psi_{\overline{\eta}}\Lambda)_{\overline{a}}\,.$

(c) $H_c^i(\overset{\circ}{X}_{\overline{s}}, R\Psi_{\overline{\eta}}\Lambda) \overset{\Psi'}{\to} H_c^i(\overset{\circ}{X}_{\overline{\eta}}, \Lambda)\,,$

(d) $H_{\{a\}}^i(\overset{\circ}{X}_{\overline{s}}, R\Psi_{\overline{\eta}}\Lambda) \to H_c^i(\overset{\circ}{X}_{\overline{s}}, R\Psi_{\overline{\eta}}\Lambda)$.

Proof of Lemma 12.7 (b): By Lemma 10.9, $R\Phi\Lambda$ is concentrated in the point a; hence we have (b) for $R\Phi\Lambda$ instead of $R\Psi_{\overline{\eta}}\Lambda$. By the distinguished triangle

$$sp^*\Lambda_{\overset{\circ}{X}_{\overline{s}}} \longrightarrow R\Psi_{\overline{\eta}}\Lambda \longrightarrow R\Phi\Lambda \longrightarrow$$

and the five lemma it suffices to consider $\Lambda_{\overset{\circ}{X}_{\overline{s}}}$, and the claim (b) follows with $Z = \overset{\circ}{X}_{\overline{s}}$ and $f = Q$ from the first claim of the following lemma.

Lemma 12.8 Let k be a field and let $Z \subseteq \mathbb{A}_k^{n+1}$ be defined by a homogeneous equation $f(x_0, \ldots, x_n) = 0$. Then, for every $r \in \mathbb{N}$ invertible in k and all i, the restriction maps

(1) $H^i(Z, \mathbb{Z}/r) \longrightarrow H^i(\{0\}, \mathbb{Z}/r)$

(2) $H_{\{0\}}^i(Z, \mathbb{Z}/r) \to H_c^i(Z, \mathbb{Z}/r)$

are isomorphisms.

Proof (1): Consider the morphisms

(12.8.1)
$$\begin{array}{ccccc} Z & \overset{f_0}{\underset{f_1}{\rightrightarrows}} & Z \times \mathbb{A}^1 & \xrightarrow{m} & Z \\ & & \downarrow{\scriptstyle p} & & \\ & & Z & & \end{array}$$

where $f_0(x) = (x, 0)$, $f_1(x) = (x, 1)$, $m((x_0, \ldots, x_n), y) = (yx_0, \ldots, yx_n)$ and p is the first projection. By Theorem 11.1,

$$p^* : H^i(Z, \mathbb{Z}/r) \longrightarrow H^i(Z \times \mathbb{A}^1, \mathbb{Z}/r)$$

is an isomorphism. Since $pf_0 = pf_1$, we conclude that $f_0^* = f_1^*$ on $H^i(Z \times \mathbb{A}^1, \mathbb{Z}/r)$, hence we also have $(mf_0)^* = f_0^* m^* = f_1^* m^* = (mf_1)^*$. But mf_0 is the map sending everything to $0 = (0, \ldots, 0)$, and mf_1 is the identity. This implies the claim: for the structural morphism $\pi : Z \longrightarrow Spec\, k$ and the rational point given by the zero section i_0: $Spec\ k \longrightarrow Z$ we have $\pi i_0 = id$, therefore $i_0^* \pi^* = id$; on the other hand we also have $\pi^* i_0^* = (i_0 \pi)^* = (mf_0)^* = (mf_1)^* = id$.

(2): By assumption $Z \subseteq \mathbb{A}_k^{n+1}$ is described by a homogeneous equation $f(x_0, \ldots, x_n) = 0$. Let $W \subseteq \mathbb{P}_k^{n+1}$ be described by $f(X_0, \ldots, X_n) = 0$ and let

$$V = W \cap < X_{n+1} = 0 > \ .$$

Then $Z = W - V \subseteq W$, and the diagram (12.8.1) extends to a diagram

(12.8.2)
$$\begin{array}{ccccc} W & \overset{f_0}{\underset{f_1}{\rightrightarrows}} & W \times \mathbb{A}^1 & \xrightarrow{\mu} & W \\ & & \downarrow{\scriptstyle p} & & \\ & & W & & \end{array}$$

where $f_0 = (id, 0)$ is again the zero section and $f_1 = (id, 1)$ is the unit section, and where $\mu((X_0 : \ldots : X_{n+1}), \lambda) = (X_0 : \ldots : X_n : \lambda X_{n+1})$. Let $W' = W - \{(0 : \ldots : 0 : 1)\}$. For the morphisms $\pi : W' \to V, x \mapsto (X_0 : \ldots : X_n : 0)$, and $V \overset{i}{\hookrightarrow} W'$ we have $\pi i = id$ and thus $i^* \pi^* = id$ in the cohomology, and furthermore $i\pi = \mu f_0$ and thus $\pi^* i^* = f_0^* \mu^* = f_1^* \mu^* = (\mu f_1)^* = id$, where the equalities $f_0^* = f_1^*$ follows as in (1). Hence i^* is an isomorphism. The claim now follows with the commutative exact diagram

$$\begin{array}{ccccccc} \ldots \longrightarrow & H^\nu_{\{(0:\ldots:0:1)\}}(W) & \longrightarrow & H^\nu(W) & \longrightarrow & H^\nu(W') & \longrightarrow \ldots \\ & \downarrow & & \| & & \downarrow{\scriptstyle i^*} & \\ \longrightarrow & H^\nu_c(Z) & \longrightarrow & H^\nu(W) & \longrightarrow & H^\nu(V) & \longrightarrow \ldots \ . \end{array}$$

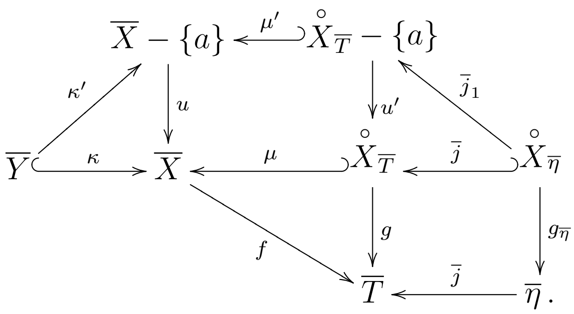
Proof of Lemma 12.7 (a): We want to show the bijectivity of the canonical maps

$$\Psi : H^i(\overset{\circ}{X}_{\overline{\eta}}, \Lambda) \longrightarrow H^i(\overset{\circ}{X}_{\overline{s}}, R\Psi_{\overline{\eta}}\Lambda)\,.$$

By construction of Ψ it suffices to show that the base change morphism for $g : \mathring{X}_{\overline{T}} \longrightarrow \overline{T}$,

$$\overline{i}^* Rg_* R\overline{j}_* \Lambda \longrightarrow R(g_{\overline{s}})_* \overline{i}^* R\overline{j}_* \Lambda$$

is a quasi-isomorphism. We have a commutative diagram



If we set $g_1 = gu'$, then, by smooth base change, the base change morphism

$$g_1^* R\overline{j}_* \Lambda_{\overline{\eta}} \xrightarrow{\sim} R(\overline{j}_1)_* \Lambda$$

is a quasi-isomorphism. For simplicity, we assume that $\overline{T} = \operatorname{Spec} \overline{A}$ where $\overline{A}$ is the integral closure of A in $K(\overline{\eta})$. Then $\overline{j}$ is an open immersion and $R\overline{j}_* \Lambda_{\overline{\eta}} = \overline{j}_* \, \Lambda_{\overline{\eta}} = \Lambda$, hence

$$R\overline{j}_* \Lambda = R(u')_* R(\overline{j}_1)_* \Lambda \cong R(u')_* g_1^* \Lambda = R(u')_* \Lambda \,.$$

Hence we have to show that the base change morphism

$$\overline{i}^* Rg_* Ru'_* \Lambda \longrightarrow R(g_{\overline{s}})_* \overline{i}^* Ru'_* \Lambda$$

is a quasi-isomorphism. There is a distinguished triangle

$$\mathcal{E} \longrightarrow \Lambda \longrightarrow Ru'_* \Lambda \longrightarrow ,$$

in which $\mathcal{E}$ is concentrated in $\{a\}$. Since $\{0\} \longrightarrow \overline{\mathring{X}} \xrightarrow{g} \overline{T}$ is proper, base change holds for $\mathcal{E}$; by the five lemma we can therefore replace $Ru'_* \Lambda$ by Λ. Since f is proper, base change holds for Rf_*, and we have to show base change for $R\mu_* \Lambda$ and $\overline{i}^*$. By the distinguished triangle

$$\kappa_* R\kappa^! \Lambda \longrightarrow \Lambda \longrightarrow R\mu_* \mu^* \Lambda \longrightarrow ,$$

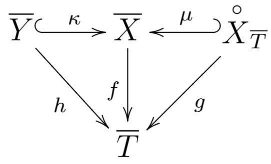
it suffices to show base change for $R\kappa^! \Lambda$ and $\overline{i}^*$. But we have $R\kappa^! \Lambda = \; R(\kappa')^! u^* \Lambda = R(\kappa')^! \Lambda$, and the claim follows by purity for the smooth $\overline{T}$-pair $(\overline{Y}, \overline{X} - \{a\})$, which says that $R(\kappa')^! \Lambda$ is locally isomorphic to $\Lambda[-1]$.

Proof of 12.7 (d): We have a distinguished triangle

$$sp^* \Lambda \to R\Psi_{\overline{\eta}} \Lambda \to R\Phi\Lambda \to ,$$

where $R\Phi\Lambda$ is concentrated in a. Thus it suffices to consider Λ, and the claim follows from 12.8 (2).

Proof of 12.7 (c): The diagram of schemes

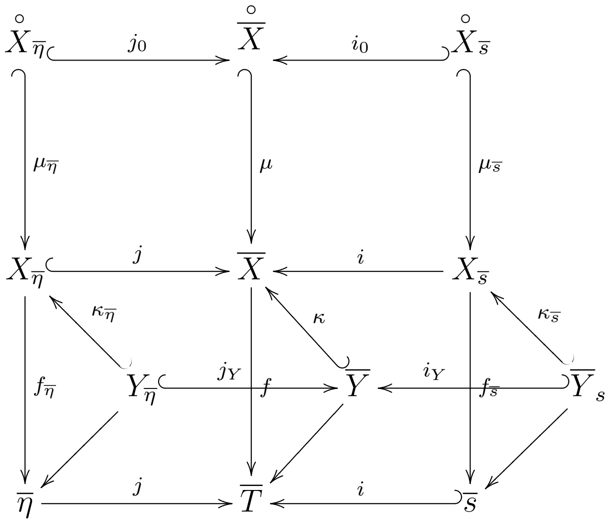


induces the following morphism of long exact cohomology sequences

$$
\begin{array}{rcl}
 & \downarrow & & \downarrow \\
\Psi'_{\overset{\circ}{X}} : H^i_c(\overset{\circ}{X}_{\overline{s}}, R\Psi_{\overline{\eta}}\Lambda) & \longrightarrow & H^i_c(\overset{\circ}{X}_{\overline{\eta}}, \Lambda) \\
\downarrow & (1) & \downarrow \\
\Psi_X^{-1} = \Psi'_X : H^i(X_{\overline{s}}, R\Psi_{\overline{\eta}}\Lambda) & \xrightarrow{\sim} & H^i(X_{\overline{\eta}}, \Lambda) \\
\downarrow & (2) & \downarrow \\
\tilde{\Psi}^{-1} = \tilde{\Psi}' : H^i(Y_{\overline{s}}, \kappa^*_{\overline{s}} R\Psi_{\overline{\eta}}\Lambda) & \xrightarrow{\sim} & H^i(Y_{\overline{\eta}}, \Lambda) \\
\downarrow & & \downarrow
\end{array}
$$

The rectangles (1) exist by 10.8 (ii) (b), and Ψ'_X is an isomorphism since X is proper over T.

With the notations from the following diagram



the rectangles (2) are induced by the following commutative diagrams of complexes (by taking global sections):

$$
\begin{array}{ccc}
R(f_{\overline{s}})_* \overline{i}^* R\overline{j}_* \overline{j}^* \Lambda & \xleftarrow[\sim]{\Psi_X} & \overline{i}^* R f_* R\overline{j}_* \overline{j}^* \Lambda \\
\downarrow & & \downarrow \\
R(f_{\overline{s}})_* \overline{i}^* R\overline{j}_* (\kappa_{\underline{\eta}})_* (\kappa_{\overline{\eta}})^* j_X^* \Lambda & \xleftarrow[\sim]{\tilde{\Psi}} & \overline{i}^* R f_* R\overline{j}_* (\kappa_{\overline{\eta}})_* (\kappa_{\overline{\eta}})^* j_X^* \Lambda
\end{array}
$$

Here, Ψ_X and $\tilde{\Psi}$ are isomorphisms by proper base change. By definition, the bottom right hand corner gives the cohomology $H^i(Y_{\overline{\eta}}, (\kappa_{\overline{\eta}})^* j_X^* \Lambda) = H^i(Y_{\overline{\eta}}, \Lambda)$.

For the bottom left hand corner, we have an isomorphism

$$\begin{aligned} \overline{i}^* R\overline{j}_*(\kappa_{\overline{\eta}})_* \kappa_{\overline{\eta}}^* \overline{j}\Lambda &\overset{(1)}{=} \overline{i}^* \kappa_* R(j_Y)_* \kappa_{\overline{\eta}}^* \overline{j}^* \Lambda \\ &\overset{(2)}{=} (\kappa_{\overline{s}})_* i_Y^* R(j_Y)_* \Lambda \overset{(3)}{=} (\kappa_{\overline{s}})_* \Lambda \,, \end{aligned}$$

in which (1) comes from the composition of derived functors (where $\kappa_* = R\kappa_*$, since κ_* is exact), (2) comes from proper base change, and (3) comes from the canonical morphism $\Lambda \to i_Y^* R(j_Y)_* \Lambda = R\Psi_{\overline{\eta}}^Y \Lambda$ into the nearby cycles of Y, which is an isomorphism, since $Y \to T$ is smooth.

In the same way, the canonical morphism

$$\Lambda \to R\Psi_{\overline{\eta}} \Lambda$$

induces an isomorphism

$$\Lambda = \kappa_{\overline{s}}^* \Lambda \to \kappa_{\overline{s}}^* R\Psi_{\overline{\eta}} \Lambda$$

into the nearby cycles of X, since X is smooth at all points $x \in Y_{\overline{s}}$. Hence the bottom left hand corner gives the cohomology $H^i(Y_{\overline{s}}, \kappa_{\overline{s}}^* R\Psi_{\overline{\eta}} \Lambda)$, which shows the commutativity of the rectangle (2) .

Both columns are exact, and this implies that $\Psi'(\overset{\circ}{X})$ is an isomorphism as well.

Now we calculate the vanishing cycles. By 12.7 (a) and (b) we have isomorphisms

$$H^i(\overset{\circ}{X}_{\overline{\eta}}, \Lambda) \quad \overset{\sim}{\longrightarrow} \quad (R^i \Psi_{\overline{\eta}} \Lambda)_{\overline{a}}, \;;$$

therefore we have to calculate the geometric cohomology of the smooth quadric $\overset{\circ}{X}_\eta$ over η.

Hence let k be a field, let $X \subseteq \mathbb{P}_k^{n+1}$ be a smooth quadric of dimension n, given by

$$q(x_0, \ldots, x_{n+1}) = 0 \,,$$

let $Y = X \cap H$ a smooth hyperplane section and let $\overset{\circ}{X} = X - Y$ be the open complement. Moreover let

$$\eta \in H^2(\mathbb{P}_{\overline{k}}^{n+1}, \Lambda(1))$$

be the class of H (i.e., of the associated canonical bundle $O(1)$), and denote by η also the image of η in $H^2(X_{\overline{k}}, \Lambda(1)$ and $H^2(Y_{\overline{k}}, \Lambda(1))$.

Theorem 12.9 We have the following $Gal(\overline{k}/k)$-isomorphisms:

(i) For X

(a)

$$H^\nu(X_{\overline{k}}, \Lambda) \cong \begin{cases} 0 & , \nu \text{ odd}, \\ \Lambda(-\mu) & , \nu = 2\mu \neq n, 0 \leq \mu \leq n. \end{cases}$$

(b) There is a basis ξ_μ of $H^{2\mu}(X_{\overline{k}}, \Lambda(\mu)) \quad (2\mu \neq n)$ with

$$\eta^\mu = \begin{cases} \xi_\mu & , 0 \leq 2\mu < n, \\ 2\xi_\mu & , n < 2\mu \leq 2n \ . \end{cases}$$

(c) If $n = 2m$ is even, then, by passing to a finite separable extension of k, one has Galois isomorphisms

$$H^{2m}(X_{\overline{k}}, \Lambda) \cong \Lambda(-m) \oplus \Lambda(-m)\,,$$
$$H^{2m}(X_{\overline{k}}, \Lambda(m))/\Lambda\eta^m \cong \Lambda\,.$$

(ii) For $\overset{\circ}{X}$: If $n = 2m+1$ is odd, then we have

$$H^\nu(\overset{\circ}{X}_{\overline{k}}, \Lambda) = \begin{cases} 0 & \nu \neq 0, n, \\ \Lambda & \nu = 0, \\ (H^{2m}(Y_{\overline{k}}, \Lambda(m))/\Lambda\eta^m)^\vee(-m-1) & \nu = n = 2m+1. \end{cases}$$

Proof (Sketch) (i) One has a long exact cohomology sequence

$$\ldots \to H^\nu_c(\mathbb{P}^{n+1}_{\overline{k}} - X_{\overline{k}}) \to H^\nu(\mathbb{P}^{n+1}_{\overline{k}}) \to H^\nu(X_{\overline{k}}) \to H^{\nu+1}_c(\mathbb{P}^{n+1}_{\overline{k}} - X_{\overline{k}}) \to \ldots,$$

(with coefficients Λ), and by weak Lefschetz ($\mathbb{P}^{n+1} - X$ is affine, since X is defined by one equation) we have $H^\mu_c(\mathbb{P}^{n+1}_{\overline{k}} - X_{\overline{k}}) = 0$ for $\mu < n+1$. Hence we get isomorphisms

$$H^\nu(\mathbb{P}^{n+1}_{\overline{k}}) \xrightarrow{\sim} H^\nu(X_{\overline{k}}) \quad \text{for} \quad \nu < n$$

and therefore the claim (a) for $\nu < n$. For $\nu > n$, (a) follows by Poincaré duality:

$$H^\nu(X_{\overline{k}}) \cong \ H^{2n-\nu}(X_{\overline{k}})^\vee(-n) \quad .$$

(b) follows from the fact that $tr_X\eta^n =< H^n \cdot X >= deg\ X = 2$: In fact, if, for $2\mu > n$, one chooses a generator ξ_μ with $< \eta^{n-\mu} \cdot \xi_\mu >= tr_X\ \eta^{n-\mu}\xi_\mu = 1$, then $\eta^\mu = 2\xi_\mu$. (Obviously (a) and (b) hold more generally for a hyperplane X of degree d in $\mathbb{P}^{n+1}_k$, if one replaces the number 2 in (b) by d; compare the calculation for complete intersections in SGA 4,XI 1.6).

For (c), we note that, after passing to a finite (separable) extension of k, we can assume via linear change of coordinates that

$$q(x_0, \ldots, x_{2m+1}) = \sum_{i=0}^m x_i x_{m+1+i}\,.$$

The linear subspace $\mathbb{P}^m_k \cong D \overset{\alpha}{\hookrightarrow} \mathbb{P}^{2m+1}_k$ defined by

$$x_0 = x_1 = \ldots = x_m = 0$$

is contained in X (In the terminology of SGA 4, XII 2.7, D is called a "génératrice"). For the complement we have a well-defined morphism

$$\begin{array}{rcl} p : X - D & \to & \mathbb{P}^m_k \\ (x_0 : \ldots : x_{2m+1}) & \mapsto & (x_0 : \ldots : x_m)\,. \end{array}$$

For every standard affine variety $U_i =< x_i \neq 0 >\subseteq \mathbb{P}^m_k$ $(0 \leq i \leq m)$, we have

$$p^{-1}(U_i) \xrightarrow{\sim} A^m_k \times U_i$$

$$x = (x_0 : \ldots : x_{2m+1}) \mapsto ((\frac{x_{m+1}}{x_i}, \frac{x_{m+2}}{x_i}, \ldots, (\frac{\widehat{x_{m+1+i}}}{x_i}), \ldots, \frac{x_{2m+1}}{x_i}, p(x)),$$

where $\hat{a}$ denotes the omission of a. This shows that $X - D$ is an affine fiber bundle over $\mathbb{P}^m_k$ and p has the same properties as p in Theorem 11.1, i.e.,

$$p^* : H^i(\mathbb{P}^m_{\overline{k}}, \Lambda) \xrightarrow{\sim} H^i(X_{\overline{k}} - D_{\overline{k}}, \Lambda)$$

is an isomorphism. The relative cohomology sequence

$$\ldots \to H^{\nu-1}(D_{\overline{k}}) \to H^\nu_c(X_{\overline{k}} - D_{\overline{k}}) \to H^\nu(X_{\overline{k}}) \to H^\nu(D_{\overline{k}}) \to H^{\nu+1}_c(X_{\overline{k}} - D_{\overline{k}}) \to \ldots$$

(coefficients Λ) and Poincaré duality

$$H^\nu_c(X_{\overline{k}} - D_{\overline{k}}) \cong H^{4m-\nu}(X_{\overline{k}} - D_{\overline{k}})^\vee(-2m) \cong H^{4m-\nu}(\mathbb{P}^m_{\overline{k}})^\vee(-2m)$$

provide a commutative diagram with exact row

$$\begin{array}{ccccccccc}
 & & \Lambda(-m) & & & & \Lambda(-m) & & \\
 & & \| & & & & \| & & \\
0 & \longrightarrow & H^{2m}(\mathbb{P}^m_{\overline{k}})^\vee(-2m) & \longrightarrow & H^{2m}(X_{\overline{k}}) & \longrightarrow & H^{2m}(D_{\overline{k}}) & \longrightarrow & 0 \\
 & & & & \uparrow & \nearrow_{\alpha^*} & & & \\
 & & & & H^{2m}(\mathbb{P}^{n+1}_{\overline{k}}) & & & &
\end{array}$$

Here the restriction map α^* is an isomorphism, since D is a linear subspace of $\mathbb{P}^{n+1}_{\overline{k}}$ (this follows immediately from Theorem 11.2). Furthermore, the image of $H^{2m}(\mathbb{P}^{n+1}_{\overline{k}}, \Lambda(m)) \to H^{2m}(X_{\overline{k}}, \Lambda(m))$ is equal to $\Lambda\eta^m$, and we obtain (c).
(ii) now follows from the relative cohomology sequence

$$\ldots \to H^{\nu-1}(X_{\overline{k}}) \to H^{\nu-1}(Y_{\overline{k}}) \to H^\nu_c(\overset{\circ}{X}_{\overline{k}}) \to H^\nu(X_{\overline{k}}) \xrightarrow{\beta} H^\nu(Y_{\overline{k}}) \to \ldots \quad .$$

In fact, by the commutative diagram

$$\begin{array}{ccc}
H^{2\mu}(X_{\overline{k}}) & \longrightarrow & H^{2\mu}(Y_{\overline{k}}) \\
\uparrow & & \uparrow \\
H^{2\mu}(\mathbb{P}^{n+1}_{\overline{k}}) & \xrightarrow{\sim} & H^{2\mu}(H_{\overline{k}})
\end{array} \qquad (\mu < n) \quad ,$$

and by (i), β is injective for $\nu = 2m$ and bijective for odd ν and for even $\nu \neq 2m$, $\nu < 2n = 4m + 2$ (for $\ell = 2$, consider first $\Lambda = \mathbb{Z}/2^r$ for $r \geq 2$ and then $\mathbb{Z}/2$). From this it follows immediately that $H^\nu_c(\overset{\circ}{X}_{\overline{k}}) = 0$ for $\nu \neq 2m + 1, 2n$, $H^{2n}_c(\overset{\circ}{X}_{\overline{k}}) \cong \Lambda(-n)$ and

$$H^{2m+1}_c(\overset{\circ}{X}_{\overline{k}}) \cong Coker(H^{2m}(X_{\overline{k}}) \to H^{2m}(Y_{\overline{k}})) = (H^{2m}(Y_{\overline{k}}, \Lambda(m))/\Lambda\eta^m)(-m)\,.$$

The claim now follows with Poincaré duality for $\overset{\circ}{X}_k$.

Lemma 12.10 In the general situation of vanishing cycles, for every closed point $a \in X_{\overline{\eta}}$ one has a canonical pairing

$$(\ , \ ) \ : \ (R^\nu \Psi_{\overline{\eta}} \Lambda)_{\overline{a}} \times H^{2n-\nu}_{\{a\}}(X_{\overline{s}}, R\Psi_{\overline{\eta}}\Lambda) \to \Lambda(-n)\,.$$

For every étale neighborhood U of $\overline{a}$ in X, this pairing is compatible with the Poincaré pairing on $U_{\overline{\eta}}$ i.e., the diagram

$$\text{(12.10.1)} \qquad \begin{array}{ccccc}
(R^\nu \Psi_{\overline{\eta}}\Lambda)_{\overline{a}} & \times & H^{2n-\nu}_{\{a\}}(X_{\overline{s}}, R^\nu\Psi_{\overline{\eta}}\Lambda) & \longrightarrow & \Lambda(-n) \\
\uparrow & & \downarrow & & \| \\
H^\nu(U_{\overline{s}}, R^\nu\Psi_{\overline{\eta}}\Lambda) & & H^{2n-\nu}_c(U_{\overline{s}}, R^\nu\Psi_{\overline{\eta}}\Lambda) & & \| \\
\uparrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \Psi'} & & \| \\
H^\nu(U_{\overline{\eta}}, \Lambda) & \times & H^{2n-\nu}_c(U_{\overline{\eta}}, \Lambda) & \longrightarrow & \Lambda(-n)
\end{array}$$

is commutative.

Proof: Since

$$(R^\nu \Psi_{\overline{\eta}}\Lambda)_{\overline{a}} = (R^\nu \overline{j}_*\Lambda)_{\overline{a}} = \varinjlim_{U} H^\nu(U_{\overline{\eta}}, \Lambda)\,,$$

where the limit runs over the étale neighborhoods of a, the pairing can be defined by passing to the limit in the diagrams 12.10.1 – note that for $U' \to U$ the diagram of Poincaré pairings

$$\begin{array}{ccccc}
H^\nu(U'_{\overline{\eta}}, \Lambda) & \times & H^{2n-\nu}_c(U'_{\overline{\eta}}, \Lambda) & \longrightarrow & \Lambda(-n) \\
\uparrow & & \downarrow & & \| \\
H^\nu(U_{\overline{\eta}}, \Lambda) & \times & H^{2n-\nu}_c(U_{\overline{\eta}}, \Lambda) & \longrightarrow & \Lambda(-n)
\end{array}$$

commutes.

Lemma 12.11 In the situation of Lemma 12.7, the canonical pairing

$$(R^\nu \Psi_{\overline{\eta}}\Lambda)_{\overline{a}} \times H^{2n-\nu}_{\{a\}}(\overset{\circ}{X}_{\overline{\eta}}, R^\nu\Psi_{\overline{\eta}}\Lambda) \to \Lambda(-n)$$

is non-degenerate.

Proof: By 12.7 (a) - (d), the vertical morphisms in (12.10.1) are isomorphisms for $U = \overset{\circ}{X}$, furthermore the Poincaré pairing is non-degenerate for $U_{\overline{\eta}}$, since this is smooth.

12.12 Now we collected all tools for the **Proof of Theorem 12.5** (local Lefschetz theory) (a) and (b) (i) +(ii), for $n = 2m+1$ odd:

Let $f : X \to T$ be flat with precisely one ordinary quadratic singularity $a \in X_{\overline{s}}$. By the theory of vanishing cycles, we have a long exact sequence of G_η-modules (see (10.8.2))

$$\text{(12.12.1)} \quad \to H^\nu(X_{\overline{s}}, \Lambda) \xrightarrow{sp} H^\nu(X_{\overline{\eta}}, \Lambda) \to (R^\nu\Phi\Lambda)_{\overline{a}} \to H^{\nu+1}(X_{\overline{s}}, \Lambda) \xrightarrow{sp} H^{\nu+1}(X_{\overline{\eta}}, \Lambda) \to \ .$$

The calculation of $(R^\nu\Phi\Lambda)_{\overline{a}}$ is a local problem, and by Lemma 12.6, we can assume, by passing to an étale covering of T, that f has standard form, i.e., that X is defined in $\mathbb{P}^{n+1}_T$ by the equation

$$Q_n(x_0,\dots,x_n)+\lambda x_{n+1}^2 \;=\; 0\,,$$

for a $0\neq\lambda\in\mathfrak{m}$. By Lemma 12.7 (a) and (b), we have isomorphisms

$$H^i(\overset{\circ}{X}_{\overline{\eta}},\Lambda)\xrightarrow{\sim}(R^i\Psi_{\overline{\eta}}\Lambda)_{\overline{a}}$$

for all i. Here $\overset{\circ}{X}_\eta$ is the complement of the smooth hyperplane section Y_η in X_η, which is defined in $< x_{n+1}=0 > \cong\ \mathbb{P}^n_\eta$ by

$$Q_n(x_0,\dots,x_n)\;=\;0$$

By 12.9 (d), we have

$$H^\nu(\overset{\circ}{X}_{\overline{\eta}},\Lambda)\;\cong\;\begin{cases}0 & \nu\neq 0,n\\ \Lambda & \nu=0\,,\\ \Lambda(-m-1) & \nu=n=2m+1\,.\end{cases}$$

In fact, for $n=2m+1$, $Q_n(x_0,\dots,x_n)$ has the form $\sum\limits_{i=0}^{m}\ x_i\ x_{m+1+i}$ assumed in proof of 12.9 (c), and therefore $H^{2m}(Y_{\overline{\eta}},\Lambda(m))/\Lambda\eta^m\ \cong\ \Lambda$. Together with the exact sequence

$$0\to\Lambda\to R^0\psi_{\overline{\eta}}\Lambda\to R^0\Phi\Lambda\to 0\quad,$$

and the isomorphisms

$$R^\nu\psi_{\overline{\eta}}\Lambda\xrightarrow{\sim}R^\nu\ \Phi\Lambda\qquad(\nu>0)$$

(implied by the distinguished triangle $\quad\Lambda\to R\psi_{\overline{\eta}}\Lambda\to R\Phi\Lambda\to$) we obtain

(12.12.2) $$(R^\nu\Phi\Lambda)_{\overline{a}}=\begin{cases}0 & \nu\neq n\\ \Lambda(-m-1) & \nu=n=2m+1\end{cases}\quad.$$

This gives isomorphisms

(12.12.3) $$H^\nu(X_{\overline{s}},\Lambda)\underset{\sim}{\xrightarrow{sp}}\ H^\nu(X_{\overline{\eta}},\Lambda)\qquad \nu\neq n,n+1$$

and an exact sequence of G_η-modules

(12.12.4) $$0\to H^n(X_{\overline{s}},\Lambda)\xrightarrow{sp}H^n(X_{\overline{\eta}},\Lambda)\xrightarrow{\alpha}\Lambda(m-n)\to H^{n+1}(X_{\overline{s}},\Lambda)\to H^{n+1}(X_{\overline{\eta}},\Lambda)\to 0\ ,$$

i.e., 12.5 (a) and (b) (i).

For the operation of the inertia group I on $H^n(X_{\overline{\eta}},\Lambda)$, defined by 10.7, we consider the variation

$$Var(\sigma)\ :\ R\Phi\Lambda\ \longrightarrow\ R\Psi_{\overline{\eta}}\Lambda\qquad(\sigma\in I)\ .$$

Since $R\Phi\Lambda$ is concentrated in a, the map induced in the cohomology factors as follows:

$$\begin{array}{ccc} H^n(X_{\overline{s}},R\Phi\Lambda) & \xrightarrow{Var(\sigma)} & H^n(X_{\overline{s}},R\Psi_{\overline{\eta}}\Lambda)\\ \downarrow & & \uparrow\\ (R^n\Phi\Lambda)_{\overline{a}} & \xrightarrow{Var_{\overline{a}}(\sigma)} & H^n_{\{a\}}(X_{\overline{s}},R\Psi_{\overline{\eta}}\Lambda)\,,\end{array}$$

By (12.10.1) we have a commutative diagram of pairings

$$\begin{array}{ccccc} H^n(X_{\overline{\eta}},\Lambda) & \times & H^n(X_{\overline{\eta}},\Lambda) & \xrightarrow{<\,,\,>} & \Lambda(-n) \\ {\scriptstyle \alpha=\Psi_a}\downarrow & & \uparrow{\scriptstyle \Psi'_a} & & \| \\ (R^n\Phi_{\overline{\eta}}\Lambda)_{\overline{a}} \overset{\sim}{\to} (R^n\Psi_{\overline{\eta}}\Lambda)_{\overline{a}} & \times & H^n_{\{a\}}(X_{\overline{s}}, R\Psi_{\overline{\eta}}\Lambda & \xrightarrow{(\,,\,)} & \Lambda(-n)\,, \end{array}$$

where the upper pairing is the non-degenerate Poincaré pairing, and the bottom is non-degenerate by 12.11. Let ρ be a generator of $(R^n\psi_{\overline{\eta}}\Lambda)_{\overline{a}}(m+1)$ ($\cong \Lambda$), let ξ be the generator of $H^n_{\{a\}}(X_{\overline{s}}, R\psi_{\overline{\eta}}\Lambda)(m)$ with

(12.12.5) $$(\rho,\xi) \;=\; 1\,,$$

and let

$$\delta \;=\; \Psi'_a(\xi) \;\in\; H^n(X_{\overline{\eta}},\Lambda(m)) \quad .$$

Then we have $(\Psi_a x,\xi) \;=<x,\delta>\in \Lambda(m-n)$ for $x\in H^n(X_{\overline{\eta}},\Lambda)$, and hence

$$\alpha(x) \;=\; \Psi_a(x) \;=<x,\delta>\cdot\eta \quad .$$

In the exact sequence (12.12.4) we used the identification

$$H^n(X_{\overline{\eta}},\Lambda) \;\underset{\sim}{\xrightarrow{\Psi}}\; H^n(X_{\overline{s}}, R\Psi_{\overline{\eta}}\Lambda)\,,$$

and we note that this is the inverse of

$$H^n(X_{\overline{s}}, R\Psi_{\overline{\eta}}\Lambda) \;\underset{\sim}{\xrightarrow{\Psi'}}\; H^n(X_{\overline{\eta}},\Lambda)\,.$$

This implies the formula

$$\begin{array}{rcl} (\sigma-1)\;x & = & \Psi^{-1}\;(\sigma-1)\;\Psi\;x \\ & = & \Psi^{-1}\;Var(\sigma)\;Res\Psi\;x \\ & = & \Psi'_a\;Var_a(\sigma)\;\Psi_a\;x \end{array}$$

for $x\in H^n(X_{\overline{\eta}},\Lambda)$, where $Res:\; H^n(X_{\overline{s}}, R\Psi_{\overline{\eta}}\Lambda)\to H^n(X_{\overline{s}}, R\Phi\Lambda)$ is the restriction.

We now recall formula 10.8.2

$$Var(\sigma\tau) \;=\; Var(\sigma) \;+\; Var(\tau) \;+\; Var(\sigma)(\tau-1)\;.$$

Since I operates trivial on $(R^n\Phi\Lambda)_{\overline{a}} \;\cong\; \Lambda(m-n)$, the last term is zero, and thus

$$\begin{array}{rcl} Var_{\overline{a}}:\; I & \mapsto & Hom_\Lambda\;((R\Phi\Lambda)_{\overline{a}},\; H^n_{\{a\}}(X_{\overline{s}},\; R\;\Psi_{\overline{\eta}}\;\Lambda)) \\ \sigma & \mapsto & Var_{\overline{a}}(\sigma) \end{array}$$

is a homomorphism. If we identify the target with

$$Hom_\Lambda\;((R\Phi\Lambda)_{\overline{a}}(m+1), H^n_{\{a\}}(X_{\overline{s}}, R\Psi_{\overline{\eta}}\;\Lambda(m))(1) \;=\; \Lambda(1)\cdot\;\varphi \quad ,$$

$\varphi(\rho) \;=\; \xi$ (see 12.12.5), and $Var_{\overline{a}}$ with a character

(12.12.6) $$\chi:\; I\longrightarrow\;\Lambda(1)\;,\quad Var_{\overline{a}}(\sigma)\;=\;\chi(\sigma)\cdot\varphi\,,$$

then we obtain the formula

$$(\sigma - 1)\; x \;=\; \Psi'_a\; \chi(\sigma)\;\; < x, \delta >\; \xi \;=\; \chi(\sigma)\;\; < x, \delta >\; \delta$$

i.e., 12.5 (b)(ii).

Remark 12.13 One can show that the isomorphism

$$(R^n \Psi_{\overline{\eta}} \Lambda)_{\overline{a}}(m+1) \cong \Lambda$$

is unique up to sign by the construction in 12.9. By this, the vanishing cycle δ is determined up to sign as well. For given $\chi \neq 0$, δ as well is determined up to sign by the construction in the proof and the formula

$$(\sigma - 1)x = \chi(\sigma) < x, \delta > \delta\,.$$

Compare 12.5 (iii) -for the χ in (12.5.5), one can actually show that we have:

$$\chi(\sigma) \;=\; (-1)^{m+1}\; v(\lambda)\; t_\ell \quad ,$$

where v is the normed valuation of A and λ is the element of 12.6.

13 Proof of Deligne's theorem

After the reductions in §6 (see the reductions 6 and 7) it suffices to show:

Theorem 13.1 Let X be a smooth projective, geometric irreducible variety of even dimension d over $\mathbb{F}_q$. For every $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ the eigenvalues of the Frobenius on $H^d(\overline{X}, \mathbb{Q}_\ell)$ are of ι-weight $\leq d+1$.

Proof We use induction over d (even). The case $d = 0$ is trivial; so let $d = n+1 \geq 2$, $n = 2m+1$. By possible base extension of $\mathbb{F}_q$ we may assume that X has a Lefschetz pencil $f : \check{X} \to P \cong \mathbb{P}^1$, defined over $\mathbb{F}_q$, where, with the notations of Theorem 12.1,

(i) all assumptions of 12.1 hold,

(ii) $U = P - \Sigma$ has an $\mathbb{F}_q$-rational point u,

(iii) $X_u = H_u \cdot X$ has a smooth hyperplane section Y_u, defined over $\mathbb{F}_q$.

By a theorem about the cohomology of blowing-ups (see SGA 5VII §8), $H^d(\overline{X}, \mathbb{Q}_\ell)$ is a direct factor of $H^d(\overline{\tilde{X}}, \mathbb{Q}_\ell)$; more explicitly we have

$$H^d(\overline{\tilde{X}}, \mathbb{Q}_\ell) \cong H^d(\overline{X}, \mathbb{Q}_\ell) \oplus H^{d-2}(\overline{X \cap A}, \mathbb{Q}_\ell)(-1) ,$$

where A is the axis of the Lefschetz pencil. Since $X \cap A$ is of dimension $d-2$, by induction, $H^{d-2}(\overline{X \cap A}, \mathbb{Q}_\ell)$ has eigenvalues of ι-weight $\leq d-2+1 = d-1$, and hence $H^{d-2}(\overline{X \cap A}, \mathbb{Q}_\ell)(-1)$ has eigenvalues of ι-weight $\leq d+1$. Therefore it suffices to consider the Frobenius eigenvalues on $H^d(\overline{\tilde{X}}, \mathbb{Q}_\ell)$. We have the Leray spectral sequence

$$E_2^{p,q} = H^p(\overline{P}, R^q f_* \mathbb{Q}_\ell) \Rightarrow H^{p+q}(\overline{\tilde{X}}, \mathbb{Q}_\ell) , \tag{13.1.1}$$

which is obtained from the spectral sequences

$$E_2^{p,q} = H^p(\overline{P}, R^q f_* \mathbb{Z}/\ell^\nu\mathbb{Z}) \Rightarrow H^{p+q}(\overline{\tilde{X}}, \mathbb{Z}/\ell^\nu\mathbb{Z}) \tag{13.1.2}$$

by passing to a projective limit over $\mathbb{Z}/\ell^\nu\mathbb{Z}$ and by tensoring with $\mathbb{Q}_\ell$ (over the ring $\mathbb{Z}_\ell = \lim\limits_{\leftarrow,\nu} \mathbb{Z}/\ell^\nu\mathbb{Z}$). By the proper base change theorem, all groups in (13.1.2) are finite, and the projective limit is exact on projective systems of finite groups; therefore one obtains again a spectral sequence (13.1.1) from the spectral sequences (13.1.2).

By (13.1.1) it suffices to show that the Frobenius-eigenvalues on $E_2^{p,q}$ are of ι-weight $\leq d+1 = n+2$ for all (p,q) with $p+q = d = n+1$. Since $H^p(\overline{P}, -) = 0$ for $p \neq 0,1,2$, these are the groups $E_2^{0,n+1}$, $E_2^{1,n}$ and $E_2^{2,n-1}$.

(A): Consider $E_2^{2,n-1} = H^2(\overline{P}, R^{n-1} f_* \mathbb{Q}_\ell)$: Let $\Lambda = \mathbb{Z}/\ell^\nu\mathbb{Z}$. For every closed point s of P and the generic point η of P we noted that the specialization morphism

$$(R^\nu f_* \Lambda)_{\overline{s}} \to (R^\nu f_* \Lambda)_{\overline{\eta}} \tag{13.1.3}$$

can be identified with the specialization morphism

$$H^\nu(X_{\overline{s}}, \Lambda) \stackrel{sp}{\to} H^\nu(X_{\overline{\eta}}, \Lambda) \tag{13.1.4}$$

from Theorem 12.5 (proper base change; see the remarks before 12.5). By 12.5 (a), (13.1.4) is an isomorphism for $\nu \neq n, n+1$, therefore in particular for $\nu = n-1$. Hence (13.1.3) is an isomorphism for $\nu = n-1$, and since this holds for all s, it follows that $R^{n-1}f_*\Lambda$ is locally constant (Lemma 2.6). Therefore $R^{n-1}f_*\mathbb{Q}_\ell$ is smooth, and corresponds to a $\mathbb{Q}_\ell$-representation of $\pi_1(P,\overline{\eta})$. But one knows that

$$\pi_1(\overline{P},\overline{\eta}) = \{1\}\,. \tag{13.1.5}$$

(This follows from the Riemann-Hurwitz formula). Hence $R^{n-1}f_*\mathbb{Q}_\ell$ is constant on $\overline{P}$, and by Lemma 8.1 (a) we have

$$H^0(\overline{P}, R^{n-1}f_*\mathbb{Q}_\ell) = (R^{n-1}f_*\mathbb{Q}_\ell)_{\overline{y}}$$

for every geometric point $\overline{y}$ of P. By using $\overline{y} = \overline{u}$ one obtains

$$(R^{n-1}f_*\mathbb{Q}_\ell)_{\overline{u}} = H^{n-1}(X_{\overline{u}},\mathbb{Q}_\ell) = H^{n-1}(\overline{X}_u,\mathbb{Q}_\ell)\,,$$

by proper base change. Furthermore we obtain an injection from weak Lefschetz

$$H^{n-1}(\overline{X}_u,\mathbb{Q}_\ell) \hookrightarrow H^{n-1}(\overline{Y}_u,\mathbb{Q}_\ell)\,,$$

and by induction over dimension ($\dim Y = \dim X - 2 = n-1$), $H^{n-1}(\overline{Y}_u,\mathbb{Q}_\ell)$ is of ι-weight $\leq n < n+2$.

(B): Consider $E_2^{0,n+1} = H^0(\overline{P}, R^{n+1}f_*\mathbb{Q}_\ell)$: From the local Lefschetz theorem we get an exact sequence for $j : U \hookrightarrow P$:

$$\bigoplus_{s\in\sum} \mathbb{Q}(m-n)_s \to R^{n+1}f_*\mathbb{Q}_\ell \to j_*j^*R^{n+1}f_*\mathbb{Q}_\ell \to 0\,, \tag{13.1.6}$$

where $j_*j^*R^{n+1}f_*\mathbb{Q}_\ell$ is constant on $\overline{P}$ and $\mathbb{Q}_\ell(m-n)_s$ denotes the sheaf $\mathbb{Q}_\ell(m-n)$, concentrated in s. This follows from the exact sequence

$$\mathbb{Q}_\ell(m-n) \to (R^{n+1}f_*\mathbb{Q}_\ell)_{\overline{s}} \overset{sp}{\to} (R^{n+1}f_*\mathbb{Q}_\ell)_{\overline{\eta}} \to 0 \tag{13.1.7}$$

(see Theorem 12.5 (b) (i)), and from the fact that $f|_U : \tilde{X}\times_p U \to U$ is smooth and proper, and hence $R^\nu f_*\mathbb{Q}_\ell|_U$ is smooth for all ν by the same argument as in (A) (since then the vanishing cycles are zero, sp is an isomorphism for $s \in U$). Moreover, by the surjectivity of sp in (13.1.7) for every $s \in P$, the inertia group $I_s \subset G_{\overline{\eta}}$ operates trivially, so that the operation of $\pi_1(\overline{U},\overline{\eta})$ factorizes over $\pi_1(\overline{P},\overline{\eta}) = \{1\}$, which corresponds to a constant sheaf on $\overline{P}$, with value $H^{n+1}(X_{\overline{u}},\mathbb{Q}_\ell) = H^{n+1}(\overline{X}_u,\mathbb{Q}_\ell)$. Since the functor i_* is exact for $i : \Sigma \hookrightarrow P$, we have $H^p(\overline{P}, i_*\mathcal{G}) \cong H^p(\overline{\Sigma},\mathcal{G}) = 0$ for $p > 0$ and every sheaf $\mathcal{G}$ on Σ, therefore for every sheaf $\mathcal{F}$ on P, which is concentrated on Σ ($\Leftrightarrow j^*\mathcal{F} = 0 \Leftrightarrow \mathcal{F} = i_*i^*\mathcal{F}$), hence also for every quotient of $\oplus_s\mathbb{Q}_\ell(m-n)_s$. This gives an exact sequence

$$\bigoplus_{s\in\Sigma} \mathbb{Q}_\ell(m-n) \to E_2^{0,n+1} \to H^{n+1}(\overline{X}_u,\mathbb{Q}_\ell) \to 0 \tag{13.1.8}$$

By weak Lefschetz one has a surjection

$$H^{n-1}(\overline{Y}_u,\mathbb{Q}_\ell)(-1) \twoheadrightarrow H^{n+1}(\overline{H}_u,\mathbb{Q}_\ell) \quad ,$$

and hence $E_2^{0,n+1}$ is enclosed between sheaves, which have ι-weights $-2m+2n = -2m+4m+2 = 2m+2 = d+1$ and $\leq n-1+1+2 = d+1$, respectively

(C): Consider $E_2^{1,n} = H^1(\overline{P}, R^n f_* \mathbb{Q}_\ell)$. For this case we need some preparations.

Definition 13.2 Let Z be an irreducible normal scheme and let $D \subseteq Z$ be a divisor. Let $\overline{\eta}$ be a generic geometric point of $V = Z - D$.

(a) For a geometric point $\overline{a}$ over a generic point a of D let $\mathcal{O}^h_{Z,a}$ be the henselization of Z in a (since Z is normal and $\dim \mathcal{O}_{Z,a} = 1$, this is a henselian discrete valuation ring), and set $Z(a) = \mathrm{Spec}(\mathcal{O}^h_{Z,a})$ and $\overset{\circ}{Z}(a) = Z(a) - \{a\} = \mathrm{Spec}(K_a)$, where $K_a = Quot(\mathcal{O}^h_{Z,a})$ (which is a henselian discrete valuation field). Let $\mathcal{O}^{sh}_{Z,\overline{a}}$ be the strict henselization of $\mathcal{O}_{Z,a}$ and let $\overline{k(\eta)}$ be a separable closure of $k(\eta)$. For every specialization map

$$\begin{array}{ccc} \mathrm{Spec}(\overline{k(\eta)}) & \longrightarrow & \mathrm{Spec}(\mathcal{O}^{sh}_{Z,\overline{a}}) \\ & \searrow \quad \swarrow & \\ & Z & \end{array}$$

from $\overline{\eta}$ to $\overline{a}$ (see Definition 2.4; also called a path from $\overline{\eta}$ to $\overline{a}$), the image of the homomorphism

$$\pi_1(\overset{\circ}{Z}(a), \overline{\eta}) \to \pi_1(V, \overline{\eta}),$$

induced by the factorization $\mathcal{O}_{Z,a} \to K_a \to \overline{k(\eta)}$, is called a decomposition group at a (which is then well determined up to conjugation in $\pi_1(V, \overline{\eta})$). Correspondingly, the image of the inertia group of $\pi_1(\overset{\circ}{Z}(a), \overline{\eta})$ is called an inertia group at a. (One can also form this by choosing different geometric "base" points $\overline{s}$ and $\overline{t}$ instead of $\overline{\eta}$ for $\pi_1(V, -)$ and $\pi_1(\overset{\circ}{Z}(a), -)$ – in virtue of the isomorphisms $\pi_1(V, \overline{\eta}) \xrightarrow{\sim} \pi_1(X, \overline{s})$ and $\pi_1(\overset{\circ}{Z}(a), \overline{\eta}) \xrightarrow{\sim} \pi_1(\overset{\circ}{Z}(a), \overline{t})$, which one obtains via specializations of $\overline{\eta}$ to $\overline{s}$ resp. $\overline{t}$ and which are unique up to conjugation)

(b) An étale covering V' of V is called tamely ramified along D, if for all geometric points a of D the operation of the inertia groups at a on the $\pi_1(V, \overline{\eta})$-set $V'_{\overline{\eta}} = Hom_V(\overline{\eta}, V')$ factorizes over the tame quotient of the inertia group.

Remark 13.3 Let V' be a Galois covering of V, with Galois group G, and let Z' be the normalization of Z in V' (resp. in the function field of V'). Then the decomposition groups over a are the groups $\{\sigma \in G \mid \sigma a' = a'\}$ $(= \{\sigma \in G \mid \sigma \overline{a}' = \overline{a}'\})$ for a point a' of V' over a ($\overline{a}' \in Hom_Z(\overline{a}, Z')$, respectively). V' is tamely ramified at a, if the order of all inertia groups over a in G is prime to $char\, K(a)$. Sometimes, one calls Z' a tamely ramified covering of Z along D.

It follows that there is a quotient $\pi_1^t(Z, D, \overline{\eta})$ of $\pi_1(V, \overline{\eta})$, which classifies all tamely ramified coverings on V along D: this is the quotient by the normal subgroup, which is generated by all decomposition groups over all generic points a of D. An étale covering V' of V is tamely ramified along D if and only if the operation of $\pi_1(V, \overline{\eta})$ on $V'_{\overline{\eta}}$ factorizes over $\pi_1^t(Z, D, \overline{\eta})$. One has surjections

$$\pi_1(V, \overline{\eta}) \twoheadrightarrow \pi_1^t(Z, D, \overline{\eta}) \twoheadrightarrow \pi_1(Z, \overline{\eta}).$$

Lemma 13.4 (Lemma of Abhyankhar) Let $Z = Spec\ A$ be for a regular local ring A, let $f_1, \ldots, f_r$ be a part of a regular parameter system and let $D \subset Z$ be defined by the ideal $(f_1 \cdot f_2 \ldots f_r)$ (this implies that D is a divisor with normal crossings). Let V' be an étale covering of $V = Z - D$, which is tamely ramified along D. Then there are $n_1, \ldots, n_r \in \mathbb{N}$, which are prime to the residue characteristic p of$A$, such that for

$$Z_1 \;=\; Spec(A[T_1, \ldots, T_r]/(T_1^{n_1} - f_1, \ldots, T_r^{n_r} - f_r))$$

the normalization Z_1' of Z_1 in $V \times_z Z_1$ is an étale covering of Z_1:

$$\begin{array}{ccc} V_1' = V' \times Z_1 & \hookrightarrow & Z_1' \\ \downarrow & & \downarrow \text{étale} \\ V_1 = V \times_Z Z_1 & \hookrightarrow & Z_1 \\ \downarrow & & \downarrow \\ V = Z \smallsetminus D & \hookrightarrow & Z \end{array}$$

($Z_1' =$ normalization of Z_1 in V_1'). Z_1' is regular.

The proof is easy, see SGA 1 XIII 5.2.

Corollary 13.5 If, in 13.4, A is strictly henselian, then every connected tamely ramified covering $Z' \to Z$ is a quotient of a Kummer covering Z_1, as described in 13.4. In particular there is a canonical isomorphism

$$t_{Z,D} \;:\; \prod_{\ell \neq p} \mathbb{Z}_\ell(1) \;\xrightarrow{\sim}\; \Pi_1^t(Z, D, \overline{\eta}) \;,$$

where p is the residue characteristic of A.

Proof Since, in the situation of 13.4, the morphism $Z_1 \to Z$ is finite, $\Gamma(Z_1, \mathcal{O}_{Z_1})$ is again strictly henselian ([Mi] I 4.3). Therefore Z_1' is the disjoint sum of copies of Z_1, and the first claim follows. The second claim follows from the fact that the inverse image V_1 of V in Z_1 is Galois over V with a Galois group

$$\begin{array}{ccl} G & \xrightarrow{\sim} & \prod_{i=1}^{r} \mu_{n_i} \\ \sigma & \mapsto & (\sigma(T_i)/T_i\ ) = (\sigma \sqrt[n_i]{f_i} / \sqrt[n_i]{f_i}) \end{array} \quad .$$

It follows that the factor μ_{n_i} can be identified with the (!) decomposition group at the generic point of $Spec\, A/(f_i) \subseteq D$.

Now we return to our Lefschetz bundle for even $n + 1 = dim\, X$. We can assume that the dimension of the dual variety $X^\vee \subseteq (\mathbb{P}^N)^\vee$ is $N - 1$: If $dim X^\vee < N - 1$, then, by 14.16, there is a line $P \subseteq (\mathbb{P}^N)^\vee$ with $P \cap X^\vee = \emptyset$, then $U = P$ and $f : \widetilde{X} \to P$ is smooth. Then all $R^\nu f_* \mathbb{Q}_\ell$ are smooth, therefore constant sheaves on $\overline{P}$ by (13.1.5), hence $E_2^{1,n} = 0$, since $H^1(\overline{P}, \mathbb{Q}_\ell) = Hom(\pi_1(\overline{P}), \mathbb{Q}_\ell) = 0$. For a generic geometric point $\overline{\eta}$ of P, we consider the $\pi_1(U, \overline{\eta})$-$\mathbb{Q}_\ell$-representation ($\ell \neq char(k)$)

$$V \;=\; (R^n f_* \mathbb{Q}_\ell)_{\overline{\eta}} \quad .$$

Proposition 13.6 By the assumptions, the operation of $\pi_1(U, \overline{\eta})$ on V factorizes over $\pi_1^t(P, \Sigma, \overline{\eta})$, i.e., V is tamely ramified along Σ.

Proof For every $s \in \Sigma$, the operation of an inertia group at s is given by the local Lefschetz theorem 12.5 (b) (ii) for the morphism $\widetilde{X} \times_P Spec\ \mathcal{O}^h_{P,s} \to Spec\ \mathcal{O}^h_{P,s}$. Since with those notations the operation depends only on a character

$$\chi \ : \quad I \quad \longrightarrow \quad \mathbb{Z}_\ell(1)$$

which is necessarily tame (since ℓ is different from the characteristic p of $k(s)$), the claim follows.

For a choice of a path from $\overline{\eta}$ to $\overline{s}$ (compare Definition 13.2 (a)), let

$$\gamma_s \ : \prod_{\ell \neq p} \mathbb{Z}_\ell(1) \ \underset{\sim}{\overset{t_{P(s),s}}{\longrightarrow}} \pi_1^t(P(s), \{s\}, \overline{\eta}) \longrightarrow \ \pi_1^t(P, \Sigma, \overline{\eta})$$

($p = char(k)$) be the composition of the induced homomorphism with the inverse of the isomorphism of 13.5. Then γ_s is well-defined up to conjugation in $\pi_1^t(P, \Sigma, \overline{\eta})$.

Proposition 13.7 We assume that P and $X^\vee$ only intersect in the smooth locus of $X^\vee$, and that this intersection is transversal (one can assume this by a version of the Bertini Theorem 14.16 c)). Let k be separably closed and let

$$r: \ \pi_1^t(P, \Sigma, \overline{\eta}) \ \longrightarrow \ Aut(V)$$

be the homomorphism which describes the operation on V. Then the maps

$$r \circ \gamma_s \quad \cdot \quad \prod_{\ell \neq p} \mathbb{Z}_\ell(1) \ \to \ Aut\ (V)$$

are conjugate in $Im(r)$ for $s \in \Sigma$.

Proof the cartesian diagram

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & H_X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ P & \hookrightarrow & (\mathbb{P}^N)^\vee \end{array}$$

and proper base change gives an isomorphism

$$V = (R^n f_* \mathbb{Q}_\ell)_{\overline{\eta}} \ \overset{\sim}{\longrightarrow} \ (R^n g_* \mathbb{Q}_\ell)_{\overline{\eta}} \quad ,$$

which is compatible with the operations of $\pi_1(U, \overline{\eta})$ and $\pi_1((\mathbb{P}^N)^\vee - X^\vee, \overline{\eta})$, via the homomorphism

$$\pi_1(P - \Sigma, \overline{\eta}) \ \longrightarrow \ \pi_1((\mathbb{P}^N)^\vee - X^\vee, \overline{\eta})$$

(if follows as in (B) that $R^n f_* \mathbb{Q}_\ell$ is smooth over $(\mathbb{P}^N)^\vee - X^\vee$). By applying the local Lefschetz theorem to the morphism $H_X \times_{(\mathbb{P}^N)^\vee}\ Spec\ (\mathcal{O}^h_{(\mathbb{P}^N)^\vee, a_0}) \to Spec\,(\mathcal{O}^h_{(\mathbb{P}^N)^\vee}, a_0)$, for the generic point a_0 of $X^\vee$ ($X^\vee$ is an irreducible divisor) it follows as in 13.6 that $(R^n g_* \mathbb{Q}_\ell)_{\overline{\eta}}$ is tamely ramified along $X^\vee$. Then the representation of $\pi_1^t(P, \Sigma, \overline{\eta})$ on V factorizes over

$$q : \pi_1^t(P, \Sigma, \overline{\eta}) \ \longrightarrow \ \pi_1^t((\mathbb{P}^N)^\vee, X^\vee, \overline{\eta}) \quad .$$

Thus it suffices to show:

Lemma 13.8 (a) For $s \in \Sigma$, the morphisms $q\gamma_s$ are conjugate in $\pi_1^t((\mathbb{P}^N)^\vee, X^\vee, \overline{\eta})$.
(b) $q : \pi_1^t(P, \Sigma, \overline{\eta}) \to \pi_1^t((\mathbb{P}^N)^\vee, X^\vee, \overline{\eta})$ is surjective.

Proof Since $X^\vee \subseteq (\mathbb{P}^N)^\vee \cong \mathbb{P}^N$ is a hypersurface, this obviously follows from the following result claimed in SGA 7.2, XVIII, 6

Proposition 13.8' Let $Z \hookrightarrow \mathbb{P}_k^r$ be a hypersurface in the projective space over an algebraically closed field k, where $r \geq 2$. Let $P \subseteq \mathbb{P}_k^r$ be a projective line which intersects Z transversally in its smooth locus. Let $\underline{a}$ be a geometric point of $P \smallsetminus (P \cap Z)$, and let $\pi_1^t(P, P \cap Z, \overline{a})$ be the quotient of $\pi_1(P \smallsetminus (P \cap Z), \overline{a})$ which corresponds to the étale coverings of $P \smallsetminus (P \cap Z)$ which are tamely ramified at all points in $P \cap Z$. Then the following holds, if Z is sufficiently general:

(a) The canonical morphism

$$q : \pi_1^t(P, P \cap Z, \overline{a}) \to \pi_1^t(\mathbb{P}_k^r, Z, \overline{a})$$

is surjective.

(b) For $s \in P \cap Z$, the morphisms

$$q\gamma_s : \prod_{\ell \neq p} \mathbb{Z}_\ell \xrightarrow[\gamma_s]{\substack{\text{SGA} \\ 6.1.2.1}} \pi_1^t(P, P \cap Z, \overline{a}) \xrightarrow[q]{} \pi_1^t(\mathbb{P}_k^r, Z, \overline{a})$$

are conjugate to each other, if Z is irreducible.

Proof of Proposition 13.8':

(a): By the commutative diagram with surjective vertical arrows

$$\begin{array}{ccc} \pi_1^t(P, P \cap Z, \overline{a}) & \xrightarrow{q} & \pi_1^t(\mathbb{P}_k^r, Z, \overline{a}) \\ \uparrow & & \uparrow \\ \pi_1(P \smallsetminus P \cap Z, \overline{a}) & \xrightarrow{q'} & \pi_1(\mathbb{P}_k^r \smallsetminus Z, \overline{a}) \end{array},$$

it suffices to show the surjectivity of the lower map q'. In SGA 7.2 XVIII, 6 the readers are referred to a suitable Bertini theorem to be contained in a volume EGA V which never appeared. Fortunately, it appeared in the book "Théorèmes de Bertini et Applications" by Jean-Pierre Jouanolou (Progress in Mathematics, 1983) as the Theorem 6.3. Jouanolou considers a subscheme in an affine space $\mathbb{A}_k^m$ over an algebraically closed field k, but $\mathbb{P}_k^t \smallsetminus Z$ is affine and hence can be embedded in a suitable affine space.

For the surjectivity of q' we have to show that, for every connected étale covering E of $\mathbb{P} \smallsetminus Z$, the pull-back of E to $P \smallsetminus P \cap Z$ is (geometrically) irreducible. Since $P \smallsetminus P \cap Z \hookrightarrow \mathbb{P}_k^r \smallsetminus Z$ is a closed immersion and hence unramified, the claim follows from Jouanolou's theorem, if Z is sufficiently general.

(b): Let T be the subscheme of the Grassmann variety of lines in $\mathbb{P}^r$ which

(i) pass through $\overline{a}$, and

(ii) intersect Z transversally in its smooth locus.

Let η be the generic point of T, let $i_\eta : P_\eta \to \mathbb{P}^r_k \otimes k(\eta)$ be the corresponding line, choose a geometric point $\overline{\eta}$ over η, and let $D_{\overline{\eta}}$ be the corresponding line, and let 0 be the point of T corresponding to the given line.

By SGA 1, XIII 2.8, there is a specialization morphism, depending on the choice of a path between $\overline{\eta}$ and 0 in T, and therefore defined up to conjugation,

$$sp : \pi_1^t(P_{\overline{\eta}}, P_{\overline{\eta}} \cap Z, \overline{a}) \to \pi_1^t(P_0, P_0 \cap Z, \overline{a})$$

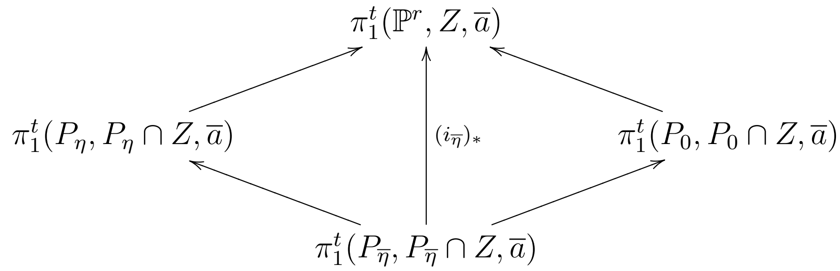
such that the following holds:

(i) The morphism γ_s is obtained as the composition of the local monodromy morphism

$$\underset{\ell \neq p}{\pi} \mathbb{Z}_\ell(1) \to \pi_1^t(P_{\overline{\eta}}, P_{\overline{\eta}} \cap Z, \overline{a})$$

and sp.

(ii) The following diagram is commutative:



By (i) and the commutativity of (ii), it suffices to show that the local monodromy morphisms

$$\prod_{\ell \neq p} \mathbb{Z}_\ell(1) \to \pi_1^t(P_{\overline{\eta}}, P_{\overline{\eta}} \cap Z, \overline{a})$$

become conjugate to each other after composition with

$$\pi_1^z(P_{\overline{\eta}}, P_{\overline{\eta}} \cap Z, \overline{a}) \to \pi_1^t(P_\eta, P_\eta\, Z, \overline{a})\,.$$

For this we note that for every finite Galois extension $K/k(\eta), k(\eta) \subset K \subset k(\overline{\eta})$, the scheme

$$P_K := P_\eta \times_{k(\eta)} K$$

is an étale covering of P_η, and that then $P_K \smallsetminus P_K \cap Z$ is a tame covering of $P_\eta \smallsetminus P_\eta \cap Z$, which gives an action of $\pi_1^t(P_{\overline{\eta}}, P_{\overline{\eta}} \cap Z, \overline{a})$ on $P_K \cap Z$. If K is sufficiently big, then all points in $P_K \cap Z$ are K-rational (note that $P_\eta \cap Z$ is étale over $k(\eta)$).

It remains to show that $\pi_1^t(P_\eta, P_\eta \cap Z, \overline{a})$ operates transitively on the points in $P_K \cap Z$.

But the acion of $\pi_1^t(P_\eta, P_\eta \cap Z, \overline{a})$ on $P_k \cap Z$ factorizes through the canonical quotient

$$\pi_1^t(P_\eta, P_\eta \cap Z, \overline{a}) \to \pi_1(\eta, \overline{\eta}) = Gal(k(\overline{\eta})/k(\eta))\,,$$

and the action of this Galois group on $P_K \cap Z$ is the one which is compatible with the inclusion

$$P_K \cap Z \hookrightarrow \mathbb{P}^r_k \times_k K\,,$$

where the Galois group acts on the second factor. This shows that our claim is equivalent to the fact that the scheme $D_\eta \cap Z$ only has one point (i.e., is the spectrum of a field).

If Z is described by one affine equation $F = 0$, and one supposes that $\overline{a}$ is the origin, then our claim amounts to the following claim, whose proof is left to the readers:

Claim: Let k be an algebraically closed field, $r \geq 2$, and $F(X_1, \dots, X_r) \in k[X_1, \dots, X_r]$ an irreducible polynomial. Then the polynomial

$$G(T) = F(X_1T, \dots, X_rT) \in k(X_1, \dots, X_r)[T]$$

is irreducible as well.

Lemma 13.9 $\pi_1(\overline{U}, \overline{\eta})$ (and thus $\pi_1^t(\overline{P}, \overline{\Sigma}, \overline{\eta})$) is topologically generated by the conjugates of all inertia groups over $s \in \Sigma$ (where $\overline{U} = U \times_k k_s$ etc. ...)

Proof Otherwise there would exist a non-trivial étale covering of $\mathbb{P}^1_{k_s}$ – contradiction to (13.1.5).

Now we consider $E_2^{1,n} = H^1(\overline{P}, R^n f_* \mathbb{Q}_\ell)$. There are two cases:
(1) If for one $s \in S$ an inertia group operates trivially on $V = (R^n f_* \mathbb{Q}_\ell)_{\overline{\eta}}$, then, by 13.7, this also holds for all others, for arbitrary $s \in \Sigma$. This implies that $R^n f_* \mathbb{Q}_\ell$ is smooth on P and thus constant on $\overline{P}$, and we have $E_2^{1,n} = H^1(\overline{P}, R^n f_* \mathbb{Q}_\ell) = 0$ $(H^1(\overline{P}, \mathbb{Q}_\ell) = 0)$.
(2) In the other case, all inertia groups I_s operate non-trivially over $s \in \Sigma$. By the Picard-Lefschetz formula

$$\sigma x - x = \chi_s(\sigma)\ (x, \delta_s)\ \delta_s\,,$$

for $\sigma \in I_s$ and $x \in V$, where $\chi_s : I_s \to \mathbb{Z}_\ell(1)$ is a character and

$$(\ ,\ ):\ V \times V \to \mathbb{Q}_\ell(-n)$$

is the Poincaré-pairing on $V = (R^n f_* \mathbb{Q}_\ell)_{\overline{\eta}} = H^n(\tilde{X}_{\overline{\eta}}, \mathbb{Q}_\ell)$, all vanishing cycles δ_s are non-zero and conjugate to each other under $\pi_1^t(\overline{P}, \overline{\Sigma}, \overline{\eta})$. By 13.9, the $\mathbb{Q}_\ell$-vector space generated by the $\delta_s(-m)$,

$$E \subseteq V\,,$$

is a $\pi_1^t(\overline{P}, \overline{\Sigma}, \overline{\eta})$ submodule. E corresponds to a smooth sheaf

$$\mathcal{E} \subseteq j^* R^n f_* \mathbb{Q}_\ell\,,$$

the **sheaf of vanishing cycles**.

From the Picard-Lefschetz formula and the exact sequences

$$\begin{array}{rcl} 0 \to (R^n f_* \mathbb{Q}_\ell)_{\overline{s}} \to (R^n f_* \mathbb{Q}_\ell)_{\overline{\eta}} & \to & \mathbb{Q}_\ell(m-n) \to 0 \\ x & \mapsto & \langle x, \delta_s \rangle \end{array}$$

for all $s \in \Sigma$ it follows that

$$(R^n f_* \mathbb{Q}_\ell)_{\overline{s}} \xrightarrow{\sim} V^{I_s} = \delta_s^\perp$$

for these s, and thus

$$R^n f_* \mathbb{Q}_\ell \xrightarrow{\sim} j_* j^* R^n f_* \mathbb{Q}_\ell \qquad (13.1.9)$$

Let $\mathcal{E}^\perp$ be the smooth sheaf in $j^*R^nf_*\mathbb{Q}_\ell$, which corresponds to the orthogonal complement $E^\perp$ of E with respect to (,). Again, we have two cases.

(i) One (and thus all) $\delta_s \in E^\perp$ (one can show later that this case does not occur). Then $\mathcal{E} \subset \mathcal{E}^\perp$ and thus $\mathcal{G} = j_*(j^*R^nf_*\mathbb{Q}_\ell/\mathcal{E}^\perp)$ is constant on $\overline{P}$: the inertia group I_s always operates trivially on V/E and hence also on $V/E^\perp$. One has an exact sequence

$$0 \to j_*\mathcal{E}^\perp \to j_*j^*R^nf_*\mathbb{Q}_\ell \to \mathcal{G} \to \bigoplus_{s\in\Sigma} \mathbb{Q}_\ell(m-n)_s \to 0\ , \tag{13.1.10}$$

where $j_*\mathcal{E}^\perp$ is constant on $\overline{P}$, since, for $s \in \Sigma$, one has an exact sequence of stalks

$$0 \to E^\perp \to V^{I_s} \to V/E^\perp \to \mathbb{Q}_\ell(m-n) \to 0\,,$$

which shows that I_s operates trivially on $E^\perp$, so that $E^\perp$ is unramified and hence constant by (13.1.5). If one splits (13.1.10) in two short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & j_*\mathcal{E}^\perp & \to & R^nf_*\mathbb{Q}_\ell & \to & \mathcal{H} & \to & 0 \\ 0 & \to & \mathcal{H} & \to & \mathcal{G} & \to & \bigoplus_{s\in\Sigma} \mathbb{Q}_\ell(m-n) & \to & 0\,, \end{array}$$

then in the cohomology this gives exact sequences

$$0 \to E_2^{1,n} = H^1(\overline{P}, R^nf_*\mathbb{Q}_\ell) \to H^1(\overline{P}, \mathcal{H})$$

$$\bigoplus_{s\in\Sigma} \mathbb{Q}_\ell(m-n) \to H^1(\overline{P}, \mathcal{H}) \to 0\,,$$

and the claim of (c) follows, since $\mathbb{Q}_\ell(m-n)$ is of ι-weight $-2m+2n = d+1$.

(ii) This is the most important and most difficult case: There is no δ_s in $E^\perp$. Then $E \not\subseteq \delta_s^\perp = (R^nf_*\mathbb{Q}_\ell)_{\overline{\Sigma}}$, and thus

$$E + (R^nf_*\mathbb{Q}_\ell)_{\overline{s}} = V\ ,$$

since $\delta_s^\perp$ has codimension 1 in V. Then the surjectivity of

$$j_*j^*R^nf_*\mathbb{Q}_\ell \to j_*(j^*R^nf_*\mathbb{Q}_\ell/\mathcal{E})\,,$$

follows, since the stalks in $s \in \Sigma$ give surjections

$$(R^nf_*\mathbb{Q}_\ell)_{\overline{s}} \to V/E\,.$$

Furthermore the morphism

$$j_*\mathcal{E} \to j_*(\mathcal{E}/\mathcal{E} \cap \mathcal{E}^\perp)$$

is surjective: one has to show that

$$E \cap \delta_s^\perp = E^{I_s} \to (E/E \cap E^\perp)^{I_s}$$

is surjective for all $s \in \Sigma$. But if we have

$$\sigma x - x \in E \cap E^\perp\ ,$$

for $x \in E$ and $\sigma \in I_s$, then the Picard-Lefschetz formula implies

$$< x, \delta_s > \delta_s \in E \cap E^\perp$$

(since $\chi \neq 0$). If $< x, \delta_s > \neq 0$, then $\delta_s \in E \cap E^\perp$, in contradiction to the assumption. Hence we have $x \in E^{I_s}$.

From the consideration above we obtain exact sequences

$$0 \to j_*\mathcal{E} \to R^n f_* \mathbb{Q}_\ell \to \mathcal{F} \to 0\,,$$

$$0 \to j_*(\mathcal{E} \cap \mathcal{E}^\perp) \to j_*\mathcal{E} \to j_*(\mathcal{E}/\mathcal{E} \cap \mathcal{E}^\perp) \to 0\,,$$

with constant sheaves $\mathcal{F} = j_*(j^* R^n f_* \mathbb{Q}_\ell / \mathcal{E})$ and $j_*(\mathcal{E} \cap \mathcal{E}^\perp)$, and in the cohomology we obtain exact sequences

$$H^1(\overline{P}, j_*\mathcal{E}) \to H^1(\overline{P}, R^n f_* \mathbb{Q}_\ell) = E_2^{1,n} \to 0\,,$$

$$0 \to H^1(\overline{P}, j_*\mathcal{E}) \to H^1(\overline{P}, j_*(\mathcal{E}/\mathcal{E} \cap \mathcal{E}^\perp))\,.$$

It suffices to consider the cohomology of $\mathcal{E}/(\mathcal{E} \cap \mathcal{E}^\perp)$. For this the next theorem is essential

Theorem 13.10 For every $\iota : \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$, $\mathcal{E}/(\mathcal{E} \cap \mathcal{E}^\perp)$ is purely of ι-weight n.

If we show this, then the estimate follows as wanted: It follows from 8.5 that

$$H^1(\overline{P}, j_*(\mathcal{E}/(\mathcal{E} \cap \mathcal{E}^\perp))$$

has ι-weights $w \leq\ n + 2 = d + 1$, as quotient of

$$H^1_c(\overline{U}, \mathcal{E}/(\mathcal{E} \cap \mathcal{E}^\perp))\,.$$

Proof of Theorem 13.10 First we show

Lemma 13.11 $E/(E \cap E^\perp) \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$ is an irreducible smooth $\overline{\mathbb{Q}}_\ell$-sheaf on $\overline{U}$.

Proof We calculate with $\overline{\mathbb{Q}}_\ell$-coefficients and we write again $E, E^\perp$ etc. Let $W \subset E$ be a $\pi_1^t(\overline{P}, \overline{\Sigma}, \overline{\eta})$ submodule, which is not contained in $E \cap E^\perp$. Then there is a $w \in W$ and a δ_s with $(x, \delta_s) \neq 0$. From the Picard-Lefschetz formula

$$\sigma x - x \ = \ \chi_s(\sigma)(x, \delta_s)\delta_s$$

for a $\sigma \in I_s$ with $\chi_s(\sigma)\ \neq\ 0$ we get $\delta_s \in W$ and thus $E \subset W$, since all vanishing cycles are conjugate to each other.

13.12 By Theorem 9.3, every ι-real irreducible smooth $\overline{\mathbb{Q}}_\ell$-sheaf is pure. By 13.11, is suffices to show that $\mathcal{E}/(\mathcal{E} \cap \mathcal{E}^\perp)$ is ι-real over a finite extension of $\mathbb{F}_q$, because then, in view of the non-degenerate Poincaré-pairing

$$(\ ,\ ):\ E/E \cap E^\perp \times E/(E \cap E^\perp)\ \to\ \mathbb{Q}_\ell(-n)\,,$$

the ι-weight β is necessarily equal to n.

By the Lefschetz formula and proper base change, for a geometric point $\overline{t}$ over a closed point t of U we have

$$\begin{aligned} Z(\tilde{X}_t, T) &= \prod_{i=0}^{2n} \det\,(1 - FT \mid H^i(\tilde{X}_{\overline{t}}, \mathbb{Q}_\ell))^{(-1)^{i+1}} \\ &= \prod_{i=0}^{2n} \det\,(1 - FT \mid (R^i f_*\,\mathbb{Q}_\ell)_{\overline{t}})^{(-1)^{i+1}}\,. \end{aligned}$$

This is a product of

$$Z^f = \prod_{i \neq n} \det(1 - F_t T | j^* R^i h_* \mathbb{Q}_\ell)^{(-1)^{i+1}} \det(1 - F_t T \mid (j^* R^n h_* \mathbb{Q}_\ell)/\mathcal{E}) \det(1 - F_t T | \mathcal{E} \cap \mathcal{E}^\perp)$$

and

$$Z^m = \det(1 - F_t\, T \mid \mathcal{E}/(\mathcal{E} \cap \mathcal{E}^\perp)) \ ,$$

where we set $\det(1 - F_t T \mid \mathcal{H}) := \det(1 - F_t T \mid \mathcal{H}_{\bar{t}})$. We saw that all $\mathbb{Q}_\ell$-sheaves which can be found in Z^f are the restriction of smooth $\mathbb{Q}_\ell$-sheaves on P, namely the following:

$$\begin{gathered} R^i\, f_*\, \mathbb{Q}_\ell \quad (i \neq n, n+1), \\ j_*\, j^*\, R^{n+1} f_*\, \mathbb{Q}_\ell, \\ j_*(j^*\, R^n f_*\, \mathbb{Q}_\ell/\mathcal{E}), \\ j_*(\mathcal{E} \cap \mathcal{E}^\perp)\,. \end{gathered}$$

These are constant on $\overline{P}$, hence come by pull-back from representations of $Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q)$. For such a sheaf $\mathcal{K}$ there are ℓ-adic units $\gamma_1, \ldots, \gamma_r \in \overline{\mathbb{Q}}_\ell^\times \quad (r = \dim \mathcal{K})$ with

$$\det(1 - F_t\; T \mid \mathcal{K}) \;=\; \prod_{j=1}^{r} (1 - \gamma_j^{\deg(t)} T)$$

for every $t \in U$ (and even every $t \in P$), since every homomorphism $Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q) \to \mathbb{Q}_\ell^\times = U_\ell \times \mathbb{Z}$ has image in the group U_ℓ of ℓ-adic units. From this we see that there are ℓ-adic units $\alpha_1, \ldots, \alpha_M$ and $\beta_1, \ldots, \beta_N$ in $\overline{\mathbb{Q}}_\ell^\times$, such that for all $t \in U$ we have

$$Z(\tilde{X}_t, T) \;=\; \frac{\prod_i (1 - \alpha_i^{\deg(t)} T)}{\prod_j (1 - \beta_j^{\deg(t)} T)} \det\, (1 - F_t\; T \mid \mathcal{E}/(\mathcal{E} \cap \mathcal{E}^\perp))\,. \tag{13.12.1}$$

Here we can assume that $\alpha_i \neq \beta_j$ for all i, j.

By passing to a finite extension of $\mathbb{F}_q$, we can assume that $\alpha_i^m \neq \beta_j^m$ for all i, j and all $m \in \mathbb{N}$ (the $m \in \mathbb{Z}$ with $\alpha_i^m = \beta_j^m$ for a j form an ideal (n_j), which, by assumption, is not equal to $\mathbb{Z}$). The left hand side of the equation 13.12.1 is ι-real, i.e., lies in $\mathbb{R}(T)$ after embedding of the coefficient (even in $\mathbb{Q}[T]$, by the proven Weil conjecture for curves). Thus it suffices to show that the polynomials $R_t(T) = \prod_i (1 - \alpha_i^{\deg(t)} T)$ and $S_t(T) = \prod_j (1 - \beta_j^{\deg(t)} T)$ are ι-real for all $t \in U_0$.

Lemma 13.13 If $\gamma_1 \ldots, \gamma_r$ are ℓ-adic units in $\overline{\mathbb{Q}}_\ell^\times$, then there is a $t \in |\, U\, |$ such that no linear factor $(1 - \gamma_i^{\deg(t)} T)$ $(i = 1, \ldots, r)$ divides the polynomial $\det(1 - F_t\; T \mid \mathcal{E}/(\mathcal{E} \cap \mathcal{E}^\perp))$.

Proof Otherwise we consider

$$\pi_1' \;=\; \{\sigma \in \pi_1(U, \overline{\eta}) \mid \sigma \text{ has an eigenvalue } \gamma_i^{\deg(\sigma)} \text{ on } E\}\,.$$

Here, $\deg(\sigma) \in \mathbb{Z}$ is the image of σ under the surjection

$$\deg: \; \pi_1(U, \overline{\eta}) \;\to\; Gal\; (\overline{\mathbb{F}}_q/\mathbb{F}_q) \; \underset{\sim}{\overset{can}{\to}} \; \widehat{\mathbb{Z}}\,;$$

note that for an ℓ-adic unit $\gamma \in \overline{\mathbb{Q}}_\ell^\times$ the power γ^a is defined for every $a \in \widehat{\mathbb{Z}}$. Then π_1' is closed in $\pi_1(U, \overline{\eta})$, as union of the inverse images under the maps

$$\begin{array}{rcl} \varphi_i : \ \pi_1(U, \overline{\eta}) & \to & Aut\ (E \otimes \overline{\mathbb{Q}}_\ell) \\ \sigma & \mapsto & (e \ \mapsto \ \gamma_i^{-\deg(\sigma)}\ \sigma e) \end{array}$$

of the closed set $\{\alpha \in Aut(E \otimes \overline{\mathbb{Q}}_\ell) \mid \det(\alpha - id) \ = \ 0\}$.

If 13.13 does not hold, then π_1' contains all geometric Frobenius elements over all $t \in \mid U \mid$. Since these generate $\pi_1(U, \overline{\eta})$ by the Čebotarev density theorem, $\pi_1' \ = \ \pi_1(U, \overline{\eta})$. Now we consider inertia groups $I_1, \ldots, I_e$ over $\sum$ such that the associated vanishing cycles $\delta_1, \ldots, \delta_e$ form a basis of E. If $\epsilon_1, \ldots, \epsilon_e$ is the dual basis, then, for $\epsilon = \epsilon_1 + \ldots + \epsilon_e$ and $\sigma_i \in I_i$, we have $(i = 1, \ldots, e)$

$$\prod_{i=1}^{e} \sigma_i\ \epsilon \ = \ \prod_{i=1}^{e} (1 + \chi_i(\sigma_i))\ \epsilon\,.$$

For an appropriate choice of the σ_i (since $\chi_i \neq 0$ for all i), $\overset{e}{\underset{i=1}{\pi}} \sigma_i$ is not in π_1', contradiction!

Applied to $\beta_1, \ldots, \beta_N$, it follows from 13.13 that there is a $t \in U_0$ such that $S_t(T)$ is prime to $\det(1 - F_t\ T \mid \mathcal{E}/(\mathcal{E} \cap \mathcal{E}^\perp))$. Since $S_t(T)$ is also prime to $R_t(T)$, for t, the right hand side of 13.12.1 stands in shortened representation, and thus

$$S_t(T) \ = \ \prod_j (1 - \beta_j^m T) \ \in \ \mathbb{Q}[T]\,,$$

$m = \deg(t)$. By base extension to $\mathbb{F}_{q^m}$, we have $S_t(T) \in \mathbb{Q}[T]$ for *all* $t \in U_0$. Then we also have $R_t(T) \det(1 - F_t\ T \mid \mathcal{E}/(\mathcal{E} \cap \mathcal{E}^\perp)) \ \in \ \mathbb{Q}[T]$ for all $t \in U_0$. In particular, $\alpha_1, \ldots, \alpha_M$ are algebraic numbers, and the application of 13.13 to the finitely many units $\sigma\alpha_i \quad (i = 1, \ldots, M, \sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q}))$ gives a $t \in U_0$ with $R_t(T) \ \in \ \mathbb{Q}[T]$. Then, as before, by base extension we have to $K(t)$: for all $t \in U_0$, $R_t(T) \in \mathbb{Q}[T]$ and thus also $\det(1 - F_t T \mid \mathcal{E}/(\mathcal{E} \cap \mathcal{E}1^\perp)) \in \ \mathbb{Q}[T]$. In particular, by both base extensions, the sheaf $\mathcal{E}/(\mathcal{E} \cap \mathcal{E}^\perp)$ is ι-real for every embedding $\iota : \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$, which we had to show.

14 Existence and global properties of Lefschetz pencils

The existence of Lefschetz pencils is shown with typical methods of projective algebraic geometry (existence of "sufficiently good" hyperplane sections, their "generic" properties, etc.). Here we consider an irreducible smooth projective variety

$$X \overset{i}{\hookrightarrow} \mathbb{P}^N = \mathbb{P}^N_k \qquad (k \text{ a field}).$$

The hyperplanes H in $\mathbb{P}^N$ are parametrized by the points of the dual projective space $(\mathbb{P}^N)^\vee$: to a point $(a_0 : \ldots : a_N)$ in $(\mathbb{P}^N)^\vee$ one associates the hypersurface

$$H: \quad a_0\, x_0 \; + \; a_1\, x_1 \; + \; \ldots \; + \; a_N\, x_N \; = \; 0$$

in the projective space $\mathbb{P}^N$ with coordinates x_i. More generally, the linear subspaces $L \subset \mathbb{P}^N$ of codimension m $(1 \le m \le N)$ correspond to the linear subspaces $L' \subset \mathbb{P}^N$ of codimension $N+1-m$: If one writes coordinate-free $\mathbb{P}^N = \mathbb{P}(V)$ for an $(N+1)$-dimensional vector space V, then $L' \subseteq (\mathbb{P}^N)^\vee = \mathbb{P}(V^\vee)$ consists of all linear forms in the dual space $V^\vee$, which annihilate V.

In particular, the lines $P \cong \mathbb{P}^1 \subseteq (\mathbb{P}^N)^\vee$ correspond to the linear subspaces A of codimension 2 in $\mathbb{P}^N$. The "pencil" $\{H_t\}_{t \in P}$ of the hyperplanes parametrized by P consists of the hyperplanes, which contain the "axis" A, and A is the intersection of any two different hyperplanes H_t, H_s.

Definition 14.1 The family $\{H_t\}_{t \in P}$ is called a Lefschetz pencil for X, if the following conditions hold:

(a) A intersects X transversally,

(b) there exists an open, dense $U \subset P$ such that the hyperplanes H_t and X intersect transversally for all $t \in U$,

(c) for $t \in S = P - U$, H_t and X intersect transversally except for one point, which is an ordinary quadratic singularity of $X \cdot H_t$.

Remark 14.2 The scheme-theoretic intersection $X \cdot H_t$ is the projective variety, which is defined by the equations of X and the linear equation of H_t.

First, we rephrase the conditions (b) and (c) in terms of the dual variety $X^\vee \subset (\mathbb{P}^N)^\vee$. It consists of all hyperplanes H in $\mathbb{P}^N$, which touch X in a point x: this means that H contains the projective tangent space of x. $X^\vee$ can be obtained as follows: Let J be the defining ideal of X and let $\mathcal{N} = (J/J^2)^\vee$ be its normal bundle. Since X is smooth, $\mathcal{N}$ is locally free of rank $N-n$ on X, $n = \dim X$. Let $\mathbb{P}(\mathcal{N})$ be the projective fiber bundle associated to $\mathcal{N}$ over X. Then there is a closed immersion

$$\nu : \mathbb{P}(\mathcal{N}) \;\hookrightarrow\; \mathbb{P}(O_X^{N+1}) \;\cong\; X \times (\mathbb{P}^N)^\vee$$

which is described on the fiber over $x \in X$ as

$$F \mapsto (x, H_F : \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(x) \cdot X_i \;=\; 0) \quad ,$$

where F is a local section of J/J^2. Globally one can describe ν as follows: One has an exact sequence

$$0 \to J/J^2 \xrightarrow{d} \Omega^1_{\mathbb{P}^N|_X} \to \Omega^1_X \to 0$$

of coherent, locally free sheaves on X. On the other hand one has the well-known exact sequence

$$0 \to \Omega^1_{\mathbb{P}^N} \to O_{\mathbb{P}^N}(-1)^{N+1} \to O_{\mathbb{P}^N} \to 0 \quad ,$$

by writing "differential forms in homogeneous coordinates". By restriction to X and dualising, we obtain a surjection

$$O_X(1)^{N+1} \quad \twoheadrightarrow \quad (J/J^2)^\vee \; = \; \mathcal{N} \quad ,$$

which gives the closed immersion

$$\nu : \; \mathbb{P}(\mathcal{N}) \hookrightarrow \mathbb{P}(O_X(1)^{N+1}) \; \cong \; \mathbb{P}(O_X^{N+1}) \; = \; \mathbb{P}^N_X$$

with the local description as stated above.

For an $x \in X$, the annihilator of $T_X(x)$ under the canonical duality between the tangent space $T_{\mathbb{P}^N}(x)$ and $\Omega^1_{\mathbb{P}^N}(x)$ is exactly $(J/J^2)(x)$. Transferred into homogeneous coordinates, it means that a hyperplane H contains the projective tangent space of X at x if and only if the linear form which defines H lies in $(J/J^2)(x)$, via the embedding

$$(J/J^2)(x) \; \hookrightarrow \; (O_X(-1)^{N+1})(x) \; \cong \; V^\vee \quad .$$

This shows that the image of $\mathbb{P}(\mathcal{N})$ under the projection $X \times (\mathbb{P}^N)^\vee \to (\mathbb{P}^N)^\vee$ coincides with the dual variety $X^\vee$. In particular, $X^\vee$ is projective and irreducible, and we have

$$\dim X^\vee \leq \dim \mathbb{P}(\mathcal{N}) = n + (N - n - 1) = N - 1 \; .$$

Lemma 14.3 The morphism $\varphi : \mathbb{P}(\mathcal{N}) \to (\mathbb{P}^N)^\vee$ is unramified at the closed point (x, H) if and only if x is a non-degenerate quadratic singularity of $X \cdot H$. In particular, the subset U' of these points is open in $\mathbb{P}(\mathcal{N})$.

Proof: later.

It may happen that the mentioned subset is empty. But we consider the Segre-embedding of degree d

$$\begin{array}{ccc} \mathbb{P}^N & \hookrightarrow & \mathbb{P}^{N(d)} \\ (x_0 : \ldots : x_N) & \mapsto & (\ldots : x_0^{\alpha_0} \ldots x_N^{\alpha_N} : \ldots) \end{array}$$

where $\alpha_i \in \mathbb{N}_0$, $\sum \alpha_i \; = \; d$, also $N(d) \; + \; 1 = \begin{pmatrix} N+d \\ N \end{pmatrix}$ is the number of all monomial of degree d in x_i. Obviously the hyperplanes in $\mathbb{P}^{N(d)}$ correspond with all hyperplanes of degree d in $\mathbb{P}^N$, and we have

Lemma 14.4 For every closed point $x \in X$ and every $d \geq 2$ there is a hyperplane H of degree d, which touches X in x and for which x is a ordinary quadratic singularity of $X \cdot H$.

Proof By appropriate change of coordinates we can assume that $x = (0 : \ldots : 1)$ and that $x_0/x_N, \ldots, x_{n-1}/x_N$ are local coordinates on X at x. Then the hyperplane with the equation

$$x_n^{d-2}\, Q_{n-1}(x_0, \ldots, x_{n-1}) \;=\; 0$$

has the required property.

Let $F' \subseteq \mathbb{P}(\mathcal{N})$ be the closed complement of U', and let F_1 be the image of F' in $(\mathbb{P}^N)^\vee$. By possibly passing to an embedding of higher degree d we can assume that $F_1(d) \neq X^\vee$ for the corresponding $F_1(d)$.

Lemma 14.5 The set

$$F''(d) \;=\; \{(x, y, H) \;\in\; X \times X \times (\mathbb{P}^{N(d)})^\vee \mid x \neq y\;,\; H \text{ touches } X \text{ in } x \text{ and } y\}$$

is Zariski-closed in $((X \times X)$-diagonal$)\times\mathbb{P}^{N(d)}$, and we have

$$\dim\; F''(d) \;\leq\; N(d) \;-\; d \qquad \text{for} \quad d \geq 3.$$

Proof By the remarks about $X^\vee$, the closedness is obvious. For the dimension it suffices to show that the fiber over $(a, b) \in (X \times X) \smallsetminus \{\text{diagonal}\}$ is at most of dimension $N(d) - 2 - 2n$. But the fiber consists exactly of all hyperplanes H, which touch X in a and b. Obviously it suffices to show:

Claim Let $L, M \subseteq \mathbb{P}^N$ be linear subspaces and $a \in L, b \in M, a \neq b$. Let W be the vector space of the homogeneous equations of degree d $\;(\Rightarrow \dim W = N(d) + 1)$. Then the subset $\widetilde{W}$ of the equations f, whose zero set $H_f \subseteq \mathbb{P}^N$ touches L in a and M in b, is a subvector space of W, and we have

$$\mathit{codim}\; \widetilde{W} \;\geq\; \dim\; L \;+\; \dim\; M \;+\; 2 \quad .$$

Proof Let $a_1, \ldots, a_{\dim\, L}$ or $b_1, \ldots, b_{\dim\, M}$ be independent points of L or M, respectively (i.e., they span no smaller subspaces). The condition on H_f is

(a) $a, b \in H_f$,

(b) $a_i \in T_a\, H_f$, $b_i \in T_b\, H_f$.

These are linear conditions, in fact $\dim\; L + \dim\; M + 2$ many. We have to show that these are linear independent. We only have to show this for $L = M = \mathbb{P}^N$. Without restriction, by change of coordinates we have $a = (1 : 0 : \ldots : 0)$, $b = (0 : 1 : \ldots : 0)$. Then the conditions for $f = \sum\; a_\nu\; X^\nu, \nu = (\nu_0, \ldots, \nu_N), X^\nu = X_0^{\nu_0} \ldots X_N^{\nu_N}, \sum\; \nu_i = d$ are:

(a) the coefficient of X_0^d and X_1^d is zero,

(b) $\partial f/\partial X_j = 0 \quad (j = 0, \ldots, N)$ at a and b, therefore the coefficients of $X_0^{d-1}X_j$ and $X_1^{d-1}X_j$ are zero.

These are $2N + 2$ linearly independent conditions for $d \geq 3$.

Now we consider the proof of Lemma 14.3. First we recall some facts about fitting ideals and Jacobi ideals.

Definition 14.6 Let A be a commutative ring and let M be a finitely presented A-module. Choose a presentation

$$G \xrightarrow{\alpha} F \longrightarrow M \longrightarrow 0$$

with locally free modules F, G of finite rank and define the p-th Fitting ideal $I^p(M) \ = \ I^p_A(M)$ by

$$I^p(M) \ = \ Im(\Lambda^{n-p} G \otimes \Lambda^{n-p} F^\vee \to A) \qquad (p \geq 0) \ ,$$

if the rank of F is equal to n ($F^\vee = Hom_A(F, A)$).

The following properties follow easily from the definition.

Proposition 14.7 (a) The ideals $I^p(M)$ are independent of the chosen presentation.

(b) If F and G are free, then $I^p(M)$ is generated by the determinants of all $(n-p)\times(n-p)$-minors of (a matrix-representation of) α.

(c) If $A \to B$ is a ringhomomorphism, then

$$B \otimes_A A/I^p_A(M) \ \longrightarrow \ B/I^p_B(B \otimes_A M)$$

is an isomorphism. In particular, $I^p_A(M)$ is compatible with localizations on A: for a multiplicative subset S of A we have

$$I^p_{S^{-1}A}(S^{-1}M) \ = \ S^{-1} \ I^p_A(M) \ .$$

(d) We have

$$I^0(M) \subset I^1(M) \subset I^2(M) \subset \dots \qquad ,$$

and for $x \in \operatorname{Spec}(A)$ the following conditions are equivalent:

(i) $I^p(M)_x \ = \ A_x \qquad$ (i.e., $x \notin Supp\ (A/I^p(M))$),

(ii) if $F_x \cong A^n_x$, then G_x contains a submodule A^{n-p}_x, which is mapped on a direct factor of F_x (particularly $Supp\, A/I^0(M) = Supp\ (M)$).

The conditions holds if $p \geq \dim_{\kappa(x)} M(x)$ (where $M(x) = \kappa(x) \otimes_A M$).

By 14.7 (c), the definition of Fitting ideals globalizes: for a quasi-coherent, finitely presented O_X-module $\mathcal{O}$ on a scheme X one obtains quasi coherent ideal sheaves $I^p_X(\mathcal{O})$ by

$$\Gamma(U, I^p_X(\mathcal{O})) \ = \ I^p_{\Gamma(U,O_X)}(\Gamma(U, \mathcal{O}))$$

for $U \subset X$ affine and open. In particular, one defines

Definition 14.8 Let $f : X \to Y$ be a scheme-morphism of finite presentation. For $p \geq 0$ the closed subscheme $J^p(X/Y)$ defined by $I^p_X(\Omega^1_{X/Y})$ is called the p-th Jacobi scheme of X over Y.

Proposition 14.9 (a) For every base change $Y' \to Y$

$$J^p(X'/Y') \longrightarrow J^p(X/Y) \times_Y Y'$$

is an isomorphism (where $X' = X \times_Y Y'$).

(b) $J^0(X/Y) \supset J^1(X/Y) \supset \ldots$, and $x \notin J^p(X/Y)$ for $p \geq \dim_{\kappa(x)}\ \Omega^1_{X/Y}(x)$.

(c) $x \notin J^p(X/Y)$ if and only if there is an open neighborhood U of x and a closed Y-immersion $U \to U'$ for a smooth Y-scheme U' with $p = \dim_x\ U'_{f(x)}$ (the dimension of the fiber over $f(x)$ in U' at x).

Proof It only remains to show (c). If U' exists as stated, then

$$\dim_{\kappa(x)}\ \Omega^1_{U/Y}(x) \leq \dim_{\kappa(x)}\ \Omega^1_{U'/Y}(x) = p$$

and thus $x \notin J^p(X/Y)$ by (a) and (b). Conversely, let $x \notin J^p(X/Y)$. Since the question is local, $X = Spec(B)$ and $Y = Spec(A)$ are affine without restriction. Choose a presentation of B as an A-algebra

$$0 \to J \to P \to B \to 0$$

with $P = A[x_1, \ldots, x_r]$. Then

$$J/J^2 \xrightarrow{d} B \otimes_P \Omega^1_{P/A} \longrightarrow \Omega^1_{B/A} \longrightarrow 0$$

is exact and $B \otimes_P \Omega^1_{P/A}$ is free of rank r, thus

$$\begin{aligned} I^P_B(B/A) &= Im(\Lambda^{r-p}(J/J^2) \otimes_B \Lambda^{r-p}(B \otimes_p \Omega^1_{P/A})^\vee \to B) \\ &= (det(\varphi_j(df_i)) \mid f_1, \ldots, f_{r-p} \in J/J^2, \varphi_1 \ldots, \varphi_{r-p} \in (B \otimes_p \Omega^1_{P/A})^\vee)\ . \end{aligned}$$

Therefore we have $x \in J^p(X/Y)$ if and only if there exist $f_1, \ldots, f_{r-p}$ in J with $det(\frac{\partial f_i}{\partial x_j}(x) \neq 0$, where x_j runs through $r-p$ variables among $x_1, \ldots, x_r$. It follows that

$$U' = Spec\ (P/(f_1, \ldots, f_{r-p})) \longrightarrow Spec\ A$$

is smooth at $x' =$ image of x under the closed immersion

$$U\ =\ Spec\ B \longrightarrow Spec\ (P/(f_1, \ldots, f_{r-p}))\quad .$$

Furthermore the fiber dimension of $U' \to \mathrm{Spec}\, A$ at x is equal to p.

Now we return to hyperplane pencils. Let

$$H_{univ}\ \subseteq\ \mathbb{P}^N\ \times\ (\mathbb{P}^N)^\vee$$

be the incidence relation, i.e., the closed points of H_{univ} are the pairs (x, H) with $x \in H$. H_{univ} is defined scheme theoretically by the equation

$$F = F(x_i, a_i)\ =\ \sum_{i=0}^{N} a_i\ x_i\ =\ 0$$

The diagram

$$\begin{array}{ccc} H_{univ} & \hookrightarrow & \mathbb{P}^N \times (\mathbb{P}^N)^\vee = (\mathbb{P}^N)_{(\mathbb{P}^N)^\vee} \\ & {\searrow}^{f} & \downarrow pr_2 \\ & & (\mathbb{P}^N)^\vee \end{array}$$

identifies H_{univ} with the universal family of hyperplanes - for $t \in (\mathbb{P}^N)^\vee$, the fiber of f at t is the hyperplane H_t, embedded in $\mathbb{P}^N$ via pr_1. Let H_X be the restriction to X, i.e., defined by the cartesian diagram of closed immersions

$$\begin{array}{ccc} H_X & \hookrightarrow & X \times (\mathbb{P}^N)^\vee \\ \downarrow & & \downarrow \\ H_{univ} & \hookrightarrow & \mathbb{P}^N \times (\mathbb{P}^N)^\vee \end{array}$$

Then the fiber over t of

$$g : H_X \longrightarrow (\mathbb{P}^N)^\vee$$

is equal to $H_t \cdot X$. Now we calculate the $(n-1)$-th Jacobian variety of H_X over $(\mathbb{P}^N)^\vee$ (compare SGA 7 XVII Remarque 3.1.5), where $n = \dim X$ as before.

Lemma 14.10 $J^{n-1}(H_X/(\mathbb{P}^N)^\vee) \; = \; \mathbb{P}(\mathcal{N}) \; \hookrightarrow \; X \times (\mathbb{P}^N)^\vee$.

Proof We have an exact sequence

$$i^*(pr_1^* J + (F)) \to i^* \; \Omega^1_{\mathbb{P}^N \times (\mathbb{P}^N)^\vee/(\mathbb{P}^N)^\vee} \to \Omega^1_{H_X/(\mathbb{P}^N)^\vee} \to 0 \; ,$$

where $i : H_X \hookrightarrow \mathbb{P}^N \times (\mathbb{P}^N)^\vee$ is the closed immersion, J is the defining ideal of X in $\mathbb{P}^N$ as above and the middle sheaf is locally free of rank N. This shows that the defining ideal of $J^{n-1}(H_X/(\mathbb{P}^N)^\vee)$ is locally generated by the $(N-n+1) \times (N-n+1)$-minors of the matrix

$$\begin{pmatrix} & \dfrac{\partial f_i}{\partial x_j}, & \begin{array}{l} i = 1, \ldots, N-n \\ j = 0, \ldots, N \end{array} & \\ & & & \\ a_0 & a_1 & \ldots & a_N \end{pmatrix}$$

where $f_1, \ldots, f_{N-n}$ are local generators of J. These minors generate exactly the ideal of $\mathbb{P}(\mathcal{N})$ in $X \times (\mathbb{P}^N)^\vee \; = \; \mathbb{P}\; (\mathcal{O}_X(1)^{N+1})$, as follows from the definition of the injection

$$J/J^2 \hookrightarrow \mathcal{O}_X(-1)^{N+1}$$

and the following elementary lemma.

Lemma 14.11 Let A be a ring and let $M \hookrightarrow A^{N+1}$ be a free submodule of rank s such that A^{N+1}/M is locally free. Let $b_0, \ldots, b_N$ be a basis of A^{N+1} and let $a_0, \ldots, a_N$ be the dual basis. Then the kernel of the ring-epimorphism

$$A[a_0, \ldots, a_N] \; = \; Sym \; (A^{N+1})^\vee \longrightarrow Sym \; M^\vee$$

is generated by the $(m+1) \times (m+1)$-minors of the $(s+1) \times (N+1)$-matrices

$$\begin{pmatrix} & a_j(m_i), & \begin{array}{l} i = 1, \ldots, s \\ j = 0, \ldots, N \end{array} & \\ & & & \\ a_0 & a_1 & \ldots & a_N \, , \end{pmatrix}$$

where $m_1, \ldots, m_s$ is a basis of M.

Proof Obviously these minors lie in the kernel. To show that the kernel is generated by this, by localizing on A we can assume that A^{N+1}/M is free as well, and by base change we have $m_i = b_{i-1}$, $i = 1, \ldots, s$ without restriction. Then the considered matrix is

$$\begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & 0 & \\ & & & 1 & & & \\ a_0 & a_1 & \cdots & a_{s-1} & a_s & \cdots & a_N \end{pmatrix},$$

and the minors $\pm a_s, \ldots, a_N$ obviously generate the kernel.

Corollary 14.12 The morphism $g: H_X \to (\mathbb{P}^N)^\vee$ is smooth at $y \in H_X$ if and only if $y \notin \mathbb{P}(\mathcal{N})$. In particular, g is smooth over $(\mathbb{P}^N)^\vee - X^\vee$.

Proof Since H_X and $\mathbb{P}^N \times (\mathbb{P}^N)^\vee$ are both smooth over k, $i : H_X \hookrightarrow \mathbb{P}^N \times (\mathbb{P}^N)^\vee$ is a regular immersion (SGA 6 VIII 1.2) and thus $g : H_X \to (\mathbb{P}^N)^\vee$ a locally complete intersection (loc. cit. 1.1), of the (relative) virtual dimension

$$\dim_{\kappa(y)} \Omega^1_{\mathbb{P}^N \times (\mathbb{P}^N)^\vee/(\mathbb{P}^N)^\vee}(y) - \dim_{\kappa(y)}(J'/J'^2)(y) \;=\; N - (2N - (N+n-1)) = n-1$$

(loc. cit. 1.9), where J' is the ideal sheaf of the closed immersion i. But we have:

Proposition 14.13 Let $f: Y \to X$ be a local complete intersection of the virtual dimension m. Then f is smooth at $y \in Y$ if and only if $y \notin J^m(Y/X)$.

Proof (see SGA7 VI 5.4) By 14.9 (c), we have $y \in J^m(Y/X)$ if and only if there is a closed X-immersion $i : U \hookrightarrow U'$ into a smooth X-scheme U' for an open neighborhood U of y, with $\dim_y U'_{f(y)} = m$. If f is smooth at y, we can take $U' = U$. Conversely, let $U = \mathit{Spec}\ B$, $U' = \mathit{Spec}\ B'$ and $X = \mathit{Spec}\ A$ be affine without restriction. The exact sequence

$$0 \to I \to B' \to B \to 0$$

induces an exact sequence

$$I/I^2 \xrightarrow{d} B \otimes_{B'} \Omega^1_{B'/A} \to \Omega^1_{B/A} \to 0 \quad .$$

Since $Y \to X$ is a locally complete intersection and $U' \to X$ is smooth, $U \hookrightarrow U'$ is a regular immersion (SGA 6 VIII 1.2). By the independence of the virtual dimension of the chosen presentation we have

$$\begin{aligned} m &= \dim_{\kappa(y)} B \otimes_{B'} \Omega^1_{B'/A}(y) - \dim_{\kappa(y)} I/I^2(y) \\ &= m - \dim_{\kappa(y)} \; I/I^2(y) \quad , \end{aligned}$$

and we get $(I/I^2)_y = 0$, thus $I_y = 0$; i.e., i at y is locally an isomorphism.

Now we come to the

Proof of Lemma 14.3: The morphism

$$\varphi \;:\; \mathbb{P}(\mathcal{N}) \;=\; J^{n-1}(H_X/(\mathbb{P}^N)^\vee) \;\to\; (\mathbb{P}^N)^\vee$$

is unramified at the closed point $y = (x, H_t)$ if and only if the fiber $J^{n-1}(H_t \cdot X/Spec\ k) \to Spec\ k$ is unramified at y. Since $H_t \cdot X$ is defined in X by one equation, $H_t \cdot X \to Spec\ k$ is a locally complete intersection of virtual dimension $n-1$. Furthermore, since $\dim_{\kappa(y)} \Omega^1_{H_t \cdot X/k}(y) \;\leq\; \dim_{\kappa(y)} \Omega^1_{X/k}(y) = n$ for every y, the n-th Jacobian $J^n(H_t \cdot X/k) = \emptyset$. Thus the claim follows from the general

Proposition 14.14 Let Y be a locally complete intersection of virtual dimension m over k. Then the following is equivalent for $y \in J^m(Y/k) \smallsetminus J^{m+1}(Y/k)$:

(a) $\Omega^1_{J^m(Y/k)/k}(y) \;=\; 0$,

(b) for a neighborhood U of y, $J^m(Y/k) \cap U$ consists only of y and is reduced (and thus isomorph to $Spec\ k$),

(c) y is a closed point and a non-degenerate quadratic singularity of Y.

Proof The equivalence of (a) and (b) is obvious ([Mi]I 3.2). For $y \in J^m(Y/k) \smallsetminus J^{m+1}(Y/k)$ there is an open affine neighborhood $U = Spec\ B$ and a closed immersion of U into a smooth affine variety $U' = Spec\ B'$ of the dimension $m+1$, therefore an exact sequence

$$0 \;\to\; I \;\to\; B' \;\to\; B \;\to\; 0$$

and an étale morphism $A[x_0, \ldots, x_m] \;\to\; B'$, which maps the point $x_0 = x_1 = \ldots = x_m = 0$ to y. Since Y is a locally complete intersection of virtual dimension m, we can assume that I is generated by one element f; and necessarily we have $f(y) \;=\; 0$.

From (b) it follows that for sufficiently small U the $\frac{\partial f}{\partial x_j}$ $(j = 0, \ldots, m)$ generate the maximal ideal m_y of y in B. Thus the $\frac{\partial f}{\partial x_j}$ also generate the maximal ideal m'_y of y in B' (and we have an isomorphism $m'_y/(m'_y)^2 \xrightarrow{\sim} m_y/m_y^2$). This implies the claim by completion in m'_y.

Conversely let y be closed and a non-degenerate quadratic singularity. By definition, $\widehat{\mathcal{O}_{Y,y}} \cong k[[x_0, \ldots, x_m]]/(g)$, where $g \equiv Q\ mod\ (x_0, \ldots, x_m)^3$, with a non-degenerate quadratic form $Q(x_0, \ldots, x_m)$. Then the $\frac{\partial g}{\partial x_j}$ $(j = 0, \ldots, m)$ generate the maximal ideal $\widehat{m}_y \;=\; m_y\ \widehat{\mathcal{O}_{Y,y}}$ of $\widehat{\mathcal{O}_{Y,y}}$. Then we have

$$I^m_{\widehat{\mathcal{O}_{Y,y}}}(\Omega^1_{\mathcal{O}_{Y,y}/k}) \;=\; \widehat{m}_y$$

(where $\hat{}$ means m_y-adic completion). Here we used that for a noetherian ring R and an ideal $m \subset R$ one has

$$\widehat{\Omega^1{}_{R/k}} \;=\; \lim_{\overleftarrow{\nu}} \ \Omega^1_{R_\nu/k} \qquad (EGA\ IV1\quad) ,$$

where $R_\nu \;= R/m^\nu$, and that the m-adic completion is exact on R-modules of finite type. For the same reasons

$$I^p_{\widehat{R}}\ (\widehat{M}) \quad=\quad I^p_R\ (M)\ \widehat{R}$$

for such a module M or

$$\widehat{R}/I^p_{\widehat{R}}(\widehat{M}) \quad\xleftarrow{\sim}\quad \widehat{R/I^p_R(M)}$$

respectively. It follows that $I^m_{O_{Y,y}}(\Omega^1_{O_{Y,y}/k}) \;=\; m_y$, therefore (b).

Now consider the embeddings $X \hookrightarrow \mathbb{P}^N, N = N_0(d)$.

Theorem 14.15 For $d \geq 3$ there exists a Lefschetz pencil $\{H_t\}_{t\in P}$ for $X \hookrightarrow \mathbb{P}^N$, and the set of the lines $P \subseteq (\mathbb{P}^N)^\vee$ for which $\{H_t\}_{t\in P}$ is a Lefschetz pencil, is open and dense in the Grassmann variety $Gr(1,(\mathbb{P}^N)^\vee)$ of the lines in $(\mathbb{P}^N)^\vee$.

Proof *for* $n = \dim\ X$ *even*: Let $F_1 = F_1(d)$ be the closed set in $X^\vee$, over which

$$\varphi\ :\ \mathbb{P}(N)\ =\ J^{n-1}(H_x/(\mathbb{P}^N)^\vee) \twoheadrightarrow X^\vee\ \hookrightarrow\ (\mathbb{P}^N)^\vee$$

is ramified and let $F_2 = F_2(d)$ be the image of the closure of the set (defined in 14.5) $F''(d) \subset X \times X \times (\mathbb{P}^N)^\vee$; F_2 is closed as well. By definition we have

$t \notin F_2 \quad \Rightarrow \quad H_t$ touches X in at most one point,

and by 14.3 we have

$t \in X^\vee - F_1 \quad \Leftrightarrow \quad H_t \cdot X$ has only non-degenerate quadratic singularities.

Finally by construction we have

$t \notin X^\vee \quad \Leftrightarrow \quad H_t$ intersects X transversally.

Hence for the properties (a) - (c) of a Lefschetz pencil we have: A hyperplane pencil $\{H_t\}_{t\in P}$ satisfies

(a) $\Leftrightarrow$ the axis A intersects X transversally,

(b) $\Leftrightarrow P \not\subseteq X^\vee$,

(c) $\Leftrightarrow P \cap (F_1 \cup F_2)\ \neq\ \emptyset$.

Now we showed:

$\dim\ X^\vee \leq N-1$,

$\dim\ F_2 \leq N-d$, if $d \geq 3$ (Lemma 14.5),

$\dim\ F_1 \leq N-2$, if $d \geq 2$ and 2 even (Lemma 14.4)

(By Lemma 14.4, $F_1 \neq X^\vee$ for $d \geq 2$, and $X^\vee$ is irreducible). Thus the claim follows from the well-known

Proposition 14.16 Let $Z \subseteq \mathbb{P}^N$ be a projective variety of dimension m, and let $Gr(\ell,\mathbb{P}^N)$ be the Grassmann variety of the linear subspaces of dimension ℓ in $\mathbb{P}^N$.

(a) The subset of $L \in Gr(N-m-1,\mathbb{P}^N)(k)$ with $L \cap Z = \emptyset$ is open, and non-empty for $Z \neq \mathbb{P}^N$.

(b) The subset of $L \in Gr(N-m,\mathbb{P}^N)(k)$ with $\dim\ Z \cap L = 0$ is open and non-empty.

(c) (Bertini) If Z is smooth, then for $N-m \leq t \leq N-1$, the set of $L \in Gr(t,\mathbb{P}^N)$, which intersect Z transversally, is open and non-empty.

Now let $\{H_t\}_{t\in P}$ be a Lefschetz pencil and let

$$h : H_{X,P}\ \longrightarrow\ P$$

be the restriction of the universal family $H_X \longrightarrow (\mathbb{P}^N)^\vee$ on P (the fiber over t is still $H_t \cdot X$). By 14.12, h is smooth over the open dense set $U = P \smallsetminus (P \cap X^\vee)$, while by (c), the fibers $H_t \cdot X$ over the finitely many $t \in \Sigma = P - U$ have exactly one singularity, and this is ordinary quadratic.

We also have the first projection

$$\pi : \; H_{X,P} \longrightarrow X \quad .$$

Lemma 14.17 Via π, $H_{X,P}$ can be identified with the blowing up of X in the smooth, 2-codimensional subvariety $A \cap X$ (A the axis of the Lefschetz pencil).

Proof First we note that the universal hyperplane $H_{univ} \subseteq \mathbb{P}^N \times (\mathbb{P}^N)^\vee$ is defined independently of the choice of coordinates: it corresponds the kernel of the canonical surjection

$$V^\vee \otimes V \longrightarrow k \quad ,$$

or the cokernel of the dual map

$$k \longrightarrow V \otimes V^\vee ,$$

respectively. This shows that for dual bases $\{x_i\}, \{a_i\}$ of V and $V^\vee$ the hyperplane H_{univ} is *always* defined by the equation $\sum a_i \, x_i = 0$. Now we can choose the coordinates in a way such that P is described by the equations $a_2 = \ldots = a_N = 0$ and thus A is described by the equation $x_0 = x_1 = 0$. Then $H_{X,P}$ in $X \times P$ is described by the equation

$$a_0 \, x_0 + a_1 \, x_1 = 0 ;$$

this is the known description of the blowing up of X in the subvariety $A \cap X$ described by the equations $x_0 = x_1 = 0$ (($a_0 : a_1$) are coordinates of P).

Remark 14.18 If we set $\tilde{X} = H_{X,P}$, we obtain morphisms

$$X \overset{\pi}{\leftarrow} \tilde{X} \overset{f}{\rightarrow} P = \mathbb{P}^1_k$$

with the properties as described in Theorem 12.1.