next lemma. Once this is proved, $m(D_x) \in \mathcal{L}(L^p(\mathbb{R}^n))$ for all $1 < p \leq 2$ by the Marcinkiewicz interpolation theorem and the case $2 < p < \infty$ will follow by duality.

**Lemma B.1** Let $M \equiv m(D_x)$ be as in Theorem 4.1. Then

$$|\{x \in \mathbb{R}^n : |Mf(x)| > t\}| \leq \frac{C\|f\|_{L^1(\mathbb{R}^n)}}{t} \quad \text{for all } t > 0$$

(B.1)

for some $C > 0$ independent of $t > 0$, i.e., $M \in \mathcal{L}(L^1(\mathbb{R}^n), L^1_{\text{weak}}(\mathbb{R}^n))$.

In order to prove Lemma B.1, an essential ingredient will be the following kernel estimate:

**Proposition B.2** Let $m : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ be as in Theorem 4.1. Then there is a locally integrable function $k : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ such that

$$m(D_x)f(x) = \int_{\mathbb{R}^n} k(x-y)f(y) \, dy \quad \text{for all } x \notin \text{supp } f,$$

which satisfies

$$|k(z)| \leq C|z|^{-n} \quad |\nabla k(z)| \leq C|z|^{-n-1}$$

(B.3)

for all $z \neq 0$.

We postpone the proof of Proposition B.2 to the end of this section.

**Corollary B.3** Let $Q$ be a cube and $a \in L^1(\mathbb{R}^n)$ with $\text{supp } a \subseteq Q$ and $\int_Q a(x) \, dx = 0$. Then there is a constant $C > 0$ independent of $a$ and $Q$ such that

$$\int_{\mathbb{R}^n \setminus \tilde{Q}} |m(D_x)a(x)| \, dx \leq B_K\|a\|_1,$$

where $\tilde{Q} = Q^{2\sqrt{n}}$ denotes the cube with same center as $Q$ and $2\sqrt{n}$ times the side-length of $Q$.

**Proof:** Let $a$ denote the center of $Q$. Then $x \notin \tilde{Q} = Q^{2\sqrt{n}}$ and $y \in Q$ implies $|x - a| > 2|y - a|$. Therefore

$$\int_{\mathbb{R}^n \setminus \tilde{Q}} |m(D_x)a(x)| \, dx = \int_{\mathbb{R}^n \setminus \tilde{Q}} |k \ast a(x)| \, dx$$

$$\leq \int_Q \int_{|x-a|>2|y-a|} |k(x-y) - k(x-a)| \, dx |a(y)| \, dy$$

$$\leq C \int_Q |a(y)| \, dy,$$
since \( \int_{Q} a(x) \, dx = 0 \) provided that

\[
\int_{|x|>2|y|} |k(x-y)-k(x)| \, dx \leq C \quad \text{for all } y \in \mathbb{R}^n. \tag{B.4}
\]

In order to prove the latter estimate, we use

\[
k(x-y)-k(x) = -\int_{0}^{1} y \cdot \nabla k(x-ty) \, dt.
\]

If \( |x| > 2|y| \), then

\[
|k(x-y)-k(x)| \leq \sup_{t \in [0,1]} |\nabla_y k(x-ty)| |y| \leq C|x|^{-n-1}|y|
\]

since \( |x-ty| \geq \frac{1}{2}|x| \) for all \( t \in [0,1] \). Hence

\[
\int_{|x|>2|y|} |k(x-y)-k(x)| \, dx \leq C \int_{|x|>2|y|} |x|^{-n-1} \, dx |y| \leq C'
\]

uniformly in \( y \neq 0 \). \( \Box \)

**Proposition B.4** Let \( f \in L^1(\mathbb{R}^n) \) be continuous and let \( t > 0 \). Then there are cubes \( Q_j, j \in \mathbb{N}_0 \), with disjoint interior and parallel to the axis such that

1. \( t < \frac{1}{|Q_j|} \int_{Q_j} |f(y)| \, dy \leq 2^n t. \) \( \tag{B.5} \)

2. \( |f(x)| \leq t \) for (almost) all \( x \notin \bigcup_{j \in \mathbb{N}_0} Q_j \).

**Proof:** In the following \( D_k, k \in \mathbb{Z} \), denotes the set of all “dyadic cubes” with side length \( 2^{-k} \) meaning the collection of all (closed) cubes \( Q \) with corners on neighboring points of the lattice \( 2^{-k} \mathbb{Z}^n \). Moreover, we set \( \mathcal{D} = \bigcup_{k \in \mathbb{Z}} D_k \).

Let \( \mathcal{C}_t' \) for given \( t > 0 \) be the set of all \( Q \in \mathcal{D} \) satisfying the condition

\[
t < \frac{1}{|Q|} \int_{Q} |f(x)| \, dx
\]

and let \( \mathcal{C}_t \) be the subset of all \( Q \in \mathcal{C}_t' \) that are maximal with respect to inclusion in \( \mathcal{C}_t' \). Every \( Q \in \mathcal{C}_t' \) is contained in some \( Q' \in \mathcal{C}_t \) since \( |Q| \leq t^{-1} \|f\|_1 \)
for all $Q \in C'$. Next, if $Q \in C_l \cap D_k$ and $Q \subset Q' \in D_{k-1}$, then by the maximality of $Q$ we have $Q' \not\subset C'$, i.e.,

$$\frac{1}{|Q'|} \int_{Q'} |f(x)| \, dx \leq t.$$ 

Moreover, since $|Q'| = 2^n |Q|$, we get

$$t < \frac{1}{|Q|} \int_Q |f(x)| \, dx \leq 2^n \frac{1}{|Q'|} \int_{Q'} |f(x)| \, dx \leq 2^n t$$

for all $Q \in C_l$.

Hence $C_l = \{Q_j : j \in \mathbb{N}_0\}$, where $Q_j, j \in \mathbb{N}_0$, are non-overlapping and (B.5) is satisfied for all $j \in \mathbb{N}_0$.

Now let $F := \mathbb{R}^n \setminus \bigcup_{j=0}^\infty Q_j$. If $x \in F$, then $\frac{1}{|Q|} \int_Q f(y) \, dy \leq t$ for every $Q \in D$ such that $x \in Q$. Hence, choosing a sequence of cubes $Q_k \in D_k$ with $x \in Q_k$, we obtain

$$|f(x)| = \lim_{k \to \infty} \left| \frac{1}{|Q_k|} \int_{Q_k} f(y) \, dy \right| \leq t \quad \text{for all } x \in F.$$  

\[\blacksquare\]

**Remark B.5** The latter proposition holds for a general $f \in L^1(\mathbb{R}^n)$ if one applies Lebesgue’s differentiation theorem in the last step of the proof.

**Proof of Lemma B.1:** First of all, since $C_0^\infty(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$, it is enough to consider $f \in C_0^\infty(\mathbb{R}^n)$. Moreover, let $t > 0$ be fixed and let $Q_j, j \in \mathbb{N}_0$ be the cubes due to Proposition B.4, $\Omega = \bigcup_{j \in \mathbb{N}_0} Q_j$ and $F = \mathbb{R}^n \setminus \Omega$. We define $g, b \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in F \\ \frac{1}{|Q_j|} \int_{Q_j} |f(y)| \, dy & \text{if } x \in Q_j \end{cases}$$

and $b(x) = f(x) - g(x)$. Note that this implies

1. $|g(x)| \leq 2^n t$ almost every in $\mathbb{R}^n$,

2. $b(x) = 0$ for every $x \in F$ and $\int_{Q_j} b(x) \, dx = 0$ for each $j \in \mathbb{N}_0$. 

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Then
\[ |\{ x : |Tf(x)| > t\} | \leq |\{ x : |Tg(x)| > t/2\} | + |\{ x : | Tb(x)| > t/2\}| \]
and it is sufficient to estimate each term separately.

In order to estimate \( Tg \), we use that \(|g(x)| \leq 2^n t\) for almost every \( x \in \mathbb{R}^n \), \( f(x) = g(x) \) for \( x \in \mathcal{F} \), \( t|\Omega| \leq \|f\|_1 \), and that \( T \in \mathcal{L}(L^2(\mathbb{R}^n)) \). More precisely,
\[
|\{ x : |Tg(x)| > t/2\}| \leq \frac{4}{t^2} \int |Tg(x)|^2 \, dx \leq \frac{4}{t^2} \|m\|_\infty \int |g(x)|^2 \, dx \\
\leq C \|m\|_\infty t^{-2} \left( \int_F |f(x)| \, dx + t^2 |\Omega| \right) \\
\leq C t^{-1} \|f\|_1
\]

In order to estimate \( Tb \), we apply Corollary B.3 to \( b_j(x) := b(x) \chi_{\tilde{Q}_j}(x) \) and conclude
\[
\int_{\mathbb{R}^n \setminus \tilde{Q}_j} |Tb_j(x)| \, dx \leq C \|b_j\|_1 \leq 2C \int_{Q_j} |f(x)| \, dx
\]
where \( \tilde{Q}_j = Q_j^{2\sqrt{n}} \). On the other hand, since \( b \in L^2(\mathbb{R}^n) \), \( \sum_{j=0}^{\infty} b_j \) and therefore \( \sum_{j=0}^{\infty} Tb_j \) converge in \( L^2(\mathbb{R}^n) \) to \( b \) and \( Tb \), respectively. Hence
\[
|Tb(x)| \leq \sum_{j=0}^{\infty} |Tb_j(x)| \quad \text{almost everywhere}
\]
and
\[
\int_{\mathbb{R}^n \setminus \tilde{\Omega}} |Tb(x)| \, dx \leq 2C \sum_{j=0}^{\infty} \int_{Q_j} |f(x)| \, dx \leq 2C \|f\|_1,
\]
where \( \tilde{\Omega} = \bigcup_{j=0}^{\infty} \tilde{Q}_j \). Finally,
\[
|\{ x : |Tb(x)| > t/2\} | \leq |\tilde{\Omega}| + \frac{2}{t} \int_{\mathbb{R}^n \setminus \tilde{\Omega}} |Tb(x)| \, dx \leq \frac{C}{t} \|f\|_1,
\]
where we have used that
\[
|\tilde{\Omega}| \leq \sum_{j=0}^{\infty} |\tilde{Q}_j| \leq (2\sqrt{n})^n \sum_{j=0}^{\infty} |Q_j| \leq \frac{C}{t^n} \sum_{j=0}^{\infty} \int_{Q_j} |f(x)| \, dx \leq \frac{C}{t} \|f\|_1
\]
This finishes the proof of (B.1). □

**Proof of Theorem 4.1:** Since \( \|m(D_x)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq \|m\|_{\infty} \) and because of (B.1), we can apply the Marcinkiewicz interpolation theorem to conclude \( m(D_x) \in \mathcal{L}(L^p(\mathbb{R}^n)) \) for all \( 1 < p \leq 2 \). The statement for \( 2 < p < \infty \) follows by duality since

\[
\int_{\mathbb{R}^n} m(D_x)f(x)g(x)\,dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} m(\xi)f(\xi)\tilde{g}(\xi)\,d\xi
\]

\[
= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi)m(\xi)\tilde{g}(\xi)\,d\xi
\]

\[
= \int_{\mathbb{R}^n} f(x)\overline{m(D_x)}g(x)\,dx
\]

for all \( f, g \in C_0^\infty(\mathbb{R}^n) \) due to (4.1), where \( \overline{m}(\xi) = m(\xi) \) satisfies the same estimates as \( m(\xi) \). Therefore \( \overline{m}(D_x) = m(D_x)' \in \mathcal{L}(L^p(\mathbb{R}^n)) \) if \( 2 < p < \infty \) by the first part. □

It remains to prove Proposition B.2. To this end we will use the so-called **Littlewood-Paley partition of unity**, which is nowadays a standard tool in the theory of function spaces and harmonic analysis. It is frequently used to analyze mapping properties of certain operators. Usually a Littlewood-Paley or dyadic partition of unity on \( \mathbb{R}^n \setminus \{0\} \) is a decomposition of unity \( \phi_j(\xi) \), \( j \in \mathbb{Z} \) of \( \mathbb{R}^n \setminus \{0\} \) such that

\[
\text{supp } \phi_j(\xi) \subseteq \{ \xi \in \mathbb{R}^n : c2^j \leq |\xi| \leq C2^j \} \quad \text{for all } j \in \mathbb{Z}.
\]

Here \( c, C > 0 \) are some suitable fixed numbers often chosen to be \( c = \frac{1}{2}, C = 2 \), which we will assume in the following. Such a partition can be easily constructed by choosing some non-negative \( \psi \in C_0^\infty(\mathbb{R}^n) \) such that \( \psi(\xi) > 0 \) if and only if \( \frac{1}{2} < |\xi| < 2 \). Then defining \( \psi_j(\xi) := \psi(2^{-j}\xi) \), \( j \in \mathbb{Z} \) we have

\[
\Phi(\xi) = \sum_{j \in \mathbb{Z}} \psi_j(\xi) > 0 \quad \text{for all } \xi \neq 0,
\]

where we note that for each \( \xi \neq 0 \) the sum above contains at most two non-vanishing terms. Hence

\[
\phi_j(\xi) = \Phi(\xi)^{-1}\psi_j(\xi)
\]
defines a decomposition of unity with the desired properties. Moreover, in this case \( \varphi_j(\xi) = \varphi_0(2^{-j}\xi) \) since \( \Phi(2^{-j}\xi) = \Phi(\xi) \), which implies that

\[
|\partial^\alpha_\xi \varphi_j(\xi)| \leq C \|\partial^\alpha_\xi \varphi_0\|_{L^\infty(\mathbb{R}^n)} 2^{-|\alpha|j} \tag{B.6}
\]

for all \( \alpha \in \mathbb{N}_0^n \) and \( j \in \mathbb{Z} \).

The idea of the proof of Proposition B.2 is to decompose

\[
m(\xi) = \sum_{j \in \mathbb{Z}} m_j(\xi), \quad \xi \neq 0,
\]

where \( m_j(\xi) = \varphi_j(\xi)m(\xi) \). Then

\[
|\partial^\alpha_\xi m_j(\xi)| \leq \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} |\partial^\alpha_\xi - \beta \varphi_j(\xi)||\partial^\beta_\xi m(\xi)| \leq C 2^{-|\alpha|j} \tag{B.7}
\]

because of (4.2), (B.6), and since \( 2^{-j-1} \leq |\xi| \leq 2^{-j+1} \) on \( \text{supp} \varphi_j \).

For each part \( m_j(\xi) \), we have

\[
m_j(D_x)f = \mathcal{F}^{-1} \left[ m_j(\xi) \hat{f}(\xi) \right] = \int_{\mathbb{R}^n} k_j(x - y)f(y) \, dy,
\]

where

\[
k_j(x) = \mathcal{F}^{-1}[m_j](x) \in C^0(\mathbb{R}^n)
\]

since \( m_j(\xi) \in L^1(\mathbb{R}^n) \). Here we have used that

\[
\mathcal{F}[f * g](\xi) = \hat{f}(\xi)\hat{g}(\xi) \quad \text{for all } \xi \in \mathbb{R}^n, f, g \in L^1(\mathbb{R}^n).
\]

Hence formally

\[
m(D_x)f = \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} k_j(x - y)f(y) \, dy,
\]

where it remains to show that the sum on the right-hand side converges for \( x \notin \text{supp } f \) and that

\[
k(z) = \sum_{j \in \mathbb{Z}} k_j(z), \quad z \neq 0,
\]

converges to a function satisfying (B.3). To this end, we need some suitable uniform estimates of \( k_j(z) \). These are a consequence of the following simple lemma:

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Lemma B.6 Let $N \in \mathbb{N}_0$ and let $g: \mathbb{R}^n \to \mathbb{C}$ be an $N$-times differentiable function, $N \in \mathbb{N}_0$, with compact support. Then

$$|\mathcal{F}^{-1}[g](x)| \leq C_N|\text{supp } g||x|^{-N} \sup_{|\beta|=N} \|\partial^\beta_x g\|_{L^\infty(\mathbb{R}^n)}$$

(B.8)

uniformly in $x \neq 0$ and $g$.

Proof: Let $\beta \in \mathbb{N}_0^n$ with $|\beta| = N$. Then

$$(-ix)^\beta \mathcal{F}^{-1}[g] = \mathcal{F}^{-1}[\partial^\beta_x g]$$

and therefore

$$|x^\beta||\mathcal{F}^{-1}[g](x)| \leq \|\partial^\beta_x g\|_{L^1(\mathbb{R}^n)} \leq |\text{supp } g| \sup_{|\beta|=N} \|\partial^\beta_x g\|_{L^\infty(\mathbb{R}^n)}.$$

Since $\beta \in \mathbb{N}_0^n$ with $|\beta| = N$ was arbitrary,

$$|x|^N|\mathcal{F}^{-1}[g](x)| \leq C_N|\text{supp } g| \sup_{|\beta|=N} \|\partial^\beta_x g\|_{L^\infty(\mathbb{R}^n)}$$

for all $x \in \mathbb{R}^n$, which completes the proof. □

Corollary B.7 Let $m$ be as in the assumptions of Theorem 4.1 and let $m_j(\xi) = m(\xi)\varphi_j(\xi)$, $j \in \mathbb{Z}$, where $\varphi_j$, $j \in \mathbb{Z}$, is the dyadic decomposition of unity of $\mathbb{R}^n \setminus \{0\}$ as above. Then $k_j(x) := \mathcal{F}^{-1}_{\xi \to x}[m_j]$ satisfies

$$|\partial^\alpha_x k_j(z)| \leq C2^{j(n+|\alpha|-M)}|z|^{-M}$$

(B.9)

for all $M = 0, \ldots, n+2$, $\alpha \in \mathbb{N}_0^n$.

Proof: First of all,

$$\partial^\alpha_x k_j(z) = \mathcal{F}^{-1}[(i\xi)^\alpha m_j(\xi)].$$

Using (B.7) and $2^{j-1} \leq |\xi| \leq 2^{j+1}$ on supp $\varphi_j$, one easily verifies

$$|\partial^\beta_x ((i\xi)^\alpha m_j(\xi))| \leq C_{\alpha,\beta}2^{j(|\alpha|-|\beta|)}$$

for all $|\beta| \leq n+2$. Hence, applying Lemma B.6 to $(i\xi)^\alpha m_j$ with $N = M = 0, \ldots, n+2$, we obtain

$$|\partial^\alpha_x k_j(z)| \leq C_{n,m}|\text{supp } \varphi_j|2^{j(|\alpha|-M)}|z|^{-M} \leq C_{n,m}'2^{j(n+|\alpha|-M)}|z|^{-M},$$
which finishes the proof.

Proof of Proposition B.2: Firstly, we will show that \( \sum_{j \in \mathbb{Z}} \partial_{z}^{\alpha} k_j(z), |\alpha| \leq 1 \) converges absolutely and uniformly on every compact subset of \( \mathbb{R}^n \setminus \{0\} \) to a function \( \partial_{z}^{\alpha} k(z) \) satisfying (B.3). The main idea of the proof is to split for given \( z \neq 0 \) the sum \( \sum_{j \in \mathbb{Z}} k_j(z) \) into the two parts

\[
\sum_{2^j \leq |z|^{-1}} \partial_{z}^{\alpha} k_j(z) \quad \text{and} \quad \sum_{2^j > |z|^{-1}} \partial_{z}^{\alpha} k_j(z)
\]

and to show convergence and the estimate (B.3) separately.

For the first sum we use (B.9) with \( |\alpha| \leq 1 \) and \( M = 0 \). Then

\[
\sum_{-N \leq j \leq \text{ld}|z|^{-1}} |\partial_{z}^{\alpha} k_j(z)| \leq C \sum_{-N \leq j \leq \text{ld}|z|^{-1}} 2^{j+n+|\alpha|} \leq C'|z|^{-n-|\alpha|}
\]

for all \( N \in \mathbb{N} \), where \( \text{ld} \) denotes the logarithm with respect to basis 2. For the second sum we apply (B.9) with \( |\alpha| \leq 1 \) and \( M = n + |\alpha| + 1 \) and obtain

\[
\sum_{\text{ld}|z|^{-1} < j \leq N} |\partial_{z}^{\alpha} k_j(z)| \leq 2 \sum_{\text{ld}|z|^{-1} < j \leq N} 2^{-j} |z|^{-n-|\alpha|-1} \leq C'|z|^{-n-|\alpha|}
\]

for all \( N \in \mathbb{N} \). Hence \( \sum_{j \in \mathbb{Z}} \partial_{z}^{\alpha} k_j(z) \) converges absolutely and uniformly on every closed subset of \( \mathbb{R}^n \setminus \{0\} \) to a function \( k(z) \) that satisfies (B.3) for all \( |\alpha| \leq 1 \).

Finally, it remains to show that \( k(z) \) satisfies (B.2). First of all,

\[
m(\xi) \hat{f}(\xi) = \sum_{j \in \mathbb{Z}} m_j(\xi) \hat{f}(\xi), \quad f \in C_{0}^{\infty}(\mathbb{R}^n)
\]

since \( \sum_{j \in \mathbb{Z}} \varphi_j(\xi) \) is locally finite. Moreover, since \( |m_j(\xi)| \) is uniformly bounded w.r.t. \( \xi \) and \( \hat{f}(\xi) \in L^2(\mathbb{R}^n) \), the sum on the right-hand side converges in \( L^2(\mathbb{R}^n) \) to the left-hand side by Lebesgue’s theorem on dominated convergence. Hence

\[
m(D_x)f = \lim_{k \to \infty} \sum_{|j| \leq N_k} m_j(D_x)f = \lim_{k \to \infty} \sum_{|j| \leq N_k} \int_{\mathbb{R}^n} k_j(x-y)f(y) \, dy,
\]

in \( L^2(\mathbb{R}^n) \) and almost everywhere for every \( f \in C_{0}^{\infty}(\mathbb{R}^n) \) and some \( N_k \to \infty \) as \( k \to \infty \) since Fourier transformation is bounded operator from \( L^2(\mathbb{R}^n) \) to
$L^2(\mathbb{R}^n)$. Therefore it only remains to interchange the summation and integration in last term above provided that $x \notin \text{supp } f$. But this can be done since $\sum_{j \in \mathbb{Z}} k_j(z)$ converges absolutely and uniformly on every closed subset of $\mathbb{R}^n \setminus \{0\}$ as shown above. Hence (B.2) follows.

**Comments on the proof of the vector-valued case:** In the case that $m: \mathbb{R}^n \setminus \{0\} \to \mathcal{L}(H_0, H_1)$ satisfies the assumptions of Theorem 4.1 for general Hilbert spaces $H_0, H_1$, we still conclude immediately that

$$m(D_x): L^2(\mathbb{R}^n; H_0) \to L^2(\mathbb{R}^n; H_1)$$

is a bounded operator due Plancharel theorem.\(^5\) The rest of the proof can be modified in a straight-forward manner to the operator-valued case since all arguments are based on direct estimates involving only the size of $f(x)$, $m(\xi)$, and $k(z)$. One just has to replace the pointwise absolute value $|.|$ by the corresponding norms, i.e., $|.|_{H_0}$, $|.|_{H_1}$, and $|.|_{\mathcal{L}(H_0, H_1)}$. Finally, we note that the Marcinkiewicz interpolation theorem holds for general $X$-valued $L^p$-spaces for general Banach spaces $X$ since its proof is only based on arguments involving the size of the function.

\(^5\) Actually, this is the only step in the proof, where it is needed that $H_0, H_1$ are Hilbert space and not general Banach spaces.