3.4 Intermediate Spaces and Reiteration

**Definition 3.27** Let \( \theta \in [0,1] \), let \((X_0, X_1)\) be an admissible pair of Banach spaces and let \( Y \) be an intermediate space with respect to \((X_0, X_1)\). Then

1. \( Y \) is said to belong to the class \( J_\theta \) if there is a constant \( C > 0 \) such that
   \[
   \|x\|_Y \leq C\|x\|_X^{1-\theta}\|x\|_{X_1}^\theta \quad \text{for all } x \in X_0 \cap X_1.
   \]
   In this case we write \( Y \in J_\theta(X_0, X_1) \).

2. \( Y \) is said to belong to the class \( K_\theta \) if there is a constant \( k > 0 \) such that
   \[
   K(t, x) \leq kt^{\theta}\|x\|_Y \quad \text{for all } x \in X_0 \cap X_1.
   \]
   In this case we write \( Y \in K_\theta(X_0, X_1) \).

Note that, if \( \theta \in (0,1) \), then \( Y \in K_\theta(X_0, X_1) \) means that \( Y \hookrightarrow (X_0, X_1)_{\theta,\infty} \) continuously.

**Proposition 3.28** Let \( \theta \in (0,1) \) and let \( Y \) be an intermediate space with respect to an admissible couple of Banach spaces \((X_0, X_1)\). Then \( Y \in J_\theta(X_0, X_1) \) if and only if \((X_0, X_1)_{\theta,1} \hookrightarrow Y \).

Hence we have \( Y \in J_\theta(X_0, X_1) \cap K_\theta(X_0, X_1) \) if and only if

\[
(X_0, X_1)_{\theta,1} \hookrightarrow Y \hookrightarrow (X_0, X_1)_{\theta,\infty}.
\]

In particular, \( (X_0, X_1)_{\theta,p} \in J_\theta(X_0, X_1) \cap K_\theta(X_0, X_1) \) for every \( p \in [1, \infty] \).

**Example 3.29** \( C^k(\mathbb{R}^n) \in J_{1/2}(C^{k-1}(\mathbb{R}^n), C^{k+1}(\mathbb{R}^n)) \cap K_{1/2}(C(\mathbb{R}^n), C^2(\mathbb{R}^n)) \) for all \( k \in \mathbb{N} \). More generally, \( C^k(\mathbb{R}^n) \in J_{(m_1/m_2-1)/(m_2-m_1)}(C^{m_1}(\mathbb{R}^n), C^{m_2}(\mathbb{R}^n)) \cap K_{(m_1/m_2-1)/(m_2-m_1)}(C^{m_1}(\mathbb{R}^n), C^{m_2}(\mathbb{R}^n)) \) for all \( m_1, m_2, k \in \mathbb{N} \) with \( m_0 < k < m_1 \).

**Remark:** It can be shown that \( C^1(\mathbb{R}^n) \) is not an interpolation space with respect to \((C(\mathbb{R}^n), C^2(\mathbb{R}^n))\), cf. [2, Example 1.3.3]!

The main result of this section is the following fundamental theorem:

**Theorem 3.30 (Reiteration Theorem)**
Let \((X_0, X_1)\) be an admissible pair of Banach spaces and let \( 0 \leq \theta_0, \theta_1 \leq 1 \) with \( \theta_0 \neq \theta_1 \). Moreover, let \( \theta \in (0,1) \) and set \( \omega = (1-\theta)\theta_0 + \theta\theta_1 \). Then:
1. If \( Y_j \in K_{\theta_j}(X_0, X_1) \) for \( j = 0, 1 \), then

\[
(Y_0, Y_1)_{\theta,p} \hookrightarrow (X_0, X_1)_{\omega,p} \quad \text{for all } 1 \leq p \leq \infty.
\]

2. If \( Y_j \in J_{\theta_j}(X_0, X_1) \) for \( j = 0, 1 \), then

\[
(X_0, X_1)_{\omega,p} \hookrightarrow (Y_0, Y_1)_{\theta,p} \quad \text{for all } 1 \leq p \leq \infty.
\]

In particular, if \( Y_j \in J_{\theta_j}(X_0, X_1) \cap K_{\theta_j}(X_0, X_1) \) for \( j = 0, 1 \), then

\[
(Y_0, Y_1)_{\theta,p} = (X_0, X_1)_{\omega,p} \quad \text{for all } 1 \leq p \leq \infty
\]

with equivalent norms.

**Remark 3.31** Since \( (X_0, X_1)_{\theta,q} \in J_{\theta}(X_0, X_1) \cap K_{\theta}(X_0, X_1) \) for all \( \theta \in (0, 1) \), \( 1 \leq q \leq \infty \), Theorem 3.30 yields

\[
((X_0, X_1)_{\theta_0,q_0}, (X_0, X_1)_{\theta_1,q_1})_{\theta,p} = (X_0, X_1)_{(1-\theta)\theta_0+\theta_1,p}
\]

for all \( 0 < \theta_0 \neq \theta_1 < 1 \) and \( 1 \leq p, q_0, q_1 \leq \infty \).

As consequences of the previous identity we obtain

\[
(C^{\theta_0}(\mathbb{R}^n), C^{\theta_1}(\mathbb{R}^n))_{\theta,\infty} = (C^{\theta_0}(\mathbb{R}^n), C^{\theta_1}(\mathbb{R}^n))_{(1-\theta)\theta_0+\theta_1,\infty} = C^{(1-\theta)\theta_0+\theta_1}(\mathbb{R}^n),
\]

\[
(W^{\theta_0}_p(\mathbb{R}^n), W^{\theta_1}_p(\mathbb{R}^n))_{\theta,p} = (L^p(\mathbb{R}^n), L^p(\mathbb{R}^n))_{(1-\theta)\theta_0+\theta_1,p} = W^{(1-\theta)\theta_0+\theta_1}(\mathbb{R}^n)
\]

for all \( 0 \leq \theta_0, \theta_1 \leq 1 \), \( \theta \in (0, 1) \), and \( 1 \leq p < \infty \) due to Example 3.10. Moreover, since

\[
C^1(\mathbb{R}^n) \in J_{\frac{1}{2}}(C^0(\mathbb{R}^n), C^2(\mathbb{R}^n)) \cap K_{\frac{1}{2}}(C^0(\mathbb{R}^n), C^2(\mathbb{R}^n)),
\]

cf. Example 3.29, we have

\[
(C^{\theta}(\mathbb{R}^n), C^{\omega}(\mathbb{R}^n))_{\omega,\infty} = (C^{\theta}(\mathbb{R}^n), C^{\omega}(\mathbb{R}^n))_{2\omega,\infty} = C^{2\omega}(\mathbb{R}^n)
\]

for all \( \omega \in (0, \frac{1}{2}) \).

In order to show \( (X_0, X_1)_{\theta} \in K_{\theta}(X_0, X_1) \) we will need the following integral representation for bounded and holomorphic function on the strip:
Lemma 3.32 (Poisson formula)
Let $X$ be a Banach space and let $f : S \to X$ be continuous, bounded, and holomorphic in $S_0$. Then $f(z) = f_0(z) + f_1(z)$, where

$$f_j(z) = \int_{\mathbb{R}} P_j(z, t)f(j + it)\, dt \quad \text{for all } z \in S_0, j = 0, 1,$$

and

$$P_j(x + iy, t) = \frac{e^{\pi(y-t)}\sin(\pi x)}{\sin^2(\pi x)+(\cos(\pi x) - (-1)^j \exp(\pi(y-t)))^2}.$$

Proof: (Sketch) First let $X = \mathbb{C}$, recall the Poisson formula for the unit circle:

$$f(z) = \frac{1}{2\pi} \int_{|w|=1} f(w) \frac{1-|z|^2}{|w-z|^2} \, dw \quad \text{for all } |z| < 1,$$

which can be derived from the Cauchy formula, cf. e.g. Remmert: “Funktionsentheorie Γ”. The representation on $S$ can be derived from the latter Poisson formula with the aid of the mapping $h : S \to \{|z| \leq 1\}$ as in (2.2). Finally, the case that $X$ is a general Banach space can be easily reduced to the scalar case by considering $z \mapsto (f(z), x')$ for an arbitrary $x' \in X'$.

Proposition 3.33 For every $\theta \in (0, 1)$ we have $(X_0, X_1)_{\theta} \in K_0(X_0, X_1)$.

As a consequence we obtain

Theorem 3.34 Let $0 < \theta_0, \theta_1, \theta < 1$ with $\theta_0 \neq \theta_1$ and let $1 \leq q \leq \infty$. Then

$$((X_0, X_1)_{\theta_0}, (X_0, X_1)_{\theta_1})_{\theta, q} = (X_0, X_1)_{(1-\theta)\theta_0 + \theta\theta_1, q}$$

with equivalent norms.

For completeness we note that

$$((X_0, X_1)_{\theta_0}, (X_0, X_1)_{\theta_1})_{\theta} = (X_0, X_1)_{(1-\theta)\theta_0 + \theta\theta_1}$$

if $X_0 \cap X_1$ is dense in $X_0, X_1$, and $(X_0, X_1)_{\theta_0} \cap (X_0, X_1)_{\theta_1}$, cf. [1, Theorem 4.6.1]. Note that, if $X_1 \hookrightarrow X_0$, the latter conditions are valid.

Moreover, for all $0 < \theta_0 \neq \theta_1 < 1$, $0 < \theta < 1$, $1 \leq p_0, p_1 \leq \infty$, and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

we have

$$((X_0, X_1)_{\theta_0, p_0}, (X_0, X_1)_{\theta_1, p_1})_{\theta} = (X_0, X_1)_{(1-\theta)\theta_0 + \theta\theta_1, p},$$

cf. [1, Theorem 4.7.2].

22
3.5 Summary of Further Useful Results

3.5.1 Lorentz Spaces and Real Interpolation

In the following let \((U, \mu)\) be a measure space. Recall that for given measurable \(f: U \to \mathbb{K}\)
\[
m(t; f) = \mu \left( \{ x \in U : |f(x)| > t \} \right), \quad t \geq 0,
\]
is the distribution function of \(f\).

**Definition 3.35** The **decreasing rearrangement** of \(f\) is the function \(f^* : (0, \infty) \to [0, \infty]\) defined by
\[
f^*(t) = \inf \{ \sigma > 0 : m(\sigma; f) \leq t \} . \tag{3.10}
\]

Then \(f^* : (0, \infty) \to [0, \infty)\) is decreasing and continuous from the right. A fundamental property of \(f^*\) is:

**Lemma 3.36**
\[
m(\rho; f^*) = m(\rho, f) \quad \text{for all } \rho \geq 0. \tag{3.11}
\]

**Proof:** First of all, (3.10) implies \(f^*(m(\rho; f)) \leq \rho\). Since \(f^*\) is decreasing, this implies \(m(\rho; f^*) \leq ||[0, m(\rho; f)]|| = m(\rho; f)\). In order to prove the converse inequality, we note that \(f^*(m(\rho; f^*)) \leq \rho\) since \(f^*\) is continuous from the right. Hence \(m(\rho; f) \leq m(\rho, f^*)\) due to (3.10). \(\blacksquare\)

Because of the latter lemma and [3, Theorem 8.16], we obtain
\[
\int_U |f(x)|^p \, d\mu(x) = p \int_0^\infty t^{p-1} m(t; f) \, dt
= p \int_0^\infty t^{p-1} m(t; f^*) \, dt = \int_0^\infty |f^*(t)|^p \, dt
\]
for all \(1 \leq p < \infty\) and \(f \in L^p(U, \mu)\). Moreover,
\[
\|f\|_{L^\infty(U, \mu)} = \inf \{ t > 0 : m(t; f) = 0 \} = \|f^*\|_{L^\infty(0, \infty)}.
\]

**Definition 3.37** Let \(1 \leq p, q \leq \infty\). Then the **Lorentz space** \(L^{p,q}(U, \mu)\) is defined as
\[
L^{p,q}(U, \mu) = \{ f : U \to \mathbb{K} \text{ measurable : } \|f\|_{L^{p,q}} < \infty \}
\]
\[
\|f\|_{L^{p,q}} = \left( \int_0^\infty \left( \int_{[0, \infty]} t^{\frac{p-1}{q}} f^*(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}.
\]
As a direct consequence of the definition we obtain
\[ L^{p,p}(U, \mu) = L^p(U, \mu), \quad L^{p,\infty}(U, \mu) = L^{p}_{\text{weak}}(U, \mu) \]
for all \( 1 \leq p \leq \infty \). Moreover, since \( f^* \) is monotone, one obtains
\[ L^{p,1}(U, \mu) \hookrightarrow L^{p,q_1}(U, \mu) \hookrightarrow L^{p,q_2}(U, \mu) \hookrightarrow L^{p,\infty}(U, \mu) \]
(3.12)
for every \( 1 \leq q_1 \leq q_2 \leq \infty \) similarly as in the proof of Lemma 3.6.

The Lorentz spaces occur naturally as real interpolation spaces of \( L^p \)-spaces:

**Theorem 3.38** Let \( 1 \leq p_0, p_1, q_0, q_1, r \leq \infty \). Then for every \( \theta \in (0, 1) \) we have
\[ (L^{p_0,q_0}(U, \mu), L^{p_1,q_1}(U, \mu))_{\theta,r} = L^{p,r}(U, \mu) \]
with equivalent norms if \( p_0 \neq p_1 \), where \( \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \). If \( p_0 = p_1 \), the same conclusion holds if additionally \( \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \).

In the case \( 1 < p_0 \neq p_1 < \infty \), the latter theorem follows the Reiteration Theorem 3.30 and the following special case:

**Theorem 3.39** For every \( \theta \in (0, 1) \)
\[ (L^1(U, \mu), L^\infty(U, \mu))_{\theta,r} = L^{1-r\theta}(U, \mu). \]
The proof is based on the identity
\[ K(t, f; L^1, L^\infty) = \int_0^t f^*(\tau) \, d\tau \quad \text{for all } t > 0 \]
the monotonicity of \( f^*(t) \) and the Hardy inequality. We refer to [2] for the details.

For the general case, a proof of Theorem 3.38 can be found in [1, Theorem 5.3.1].

**3.5.2 Dual Spaces**

**Theorem 3.40** Let \( (X_0, X_1) \) be an admissible couple of Banach spaces such that \( X_0 \cap X_1 \) is dense in \( X_0 \) and \( X_1 \) and let \( 1 \leq p < \infty, \theta \in (0, 1) \). Then
\[ ((X_0, X_1)_{\theta,p})' = (X_0', X_1')_{\theta,p'}, \]
\[ ((X_0, X_1)_{\theta})' = (X_0', X_1')_{\theta,1}. \]

24
\textbf{Proof:} See [2, Theorem 1.18].

We note that, if $X_0 \cap X_1$ is dense in $X_0$ and $X_1$, then

$$(X_0 \cap X_1)' = X_0' + X_1', \quad (X_0 + X_1)' = X_0 \cap X_1$$

with same norms, cf. [1, Theorem 2.7.1]. Therefore $((X_0, X_1)_\theta,p)'$ and $(X_0', X_1')_{\theta,p'}$ are intermediate spaces with respect to $(X_0', X_1')$ and the equalities above make sense.

\textbf{THEOREM 3.41} Let $(X_0, X_1)$ be an admissible couple of Banach spaces such that $X_0 \cap X_1$ is dense in $X_0$ and $X_1$ and let $\theta \in (0,1)$. Moreover, let $X_0$ or $X_1$ be reflexive. Then

$$((X_0, X_1)_\theta)[_]' = (X_0', X_1')_{\theta}[_]$$

\textbf{Proof:} See [1, Corollary 4.5.2].