4 Bessel Potential and Besov Spaces

4.1 Mikhlin Multiplier Theorem

Recall that the Fourier transformation $\mathcal{F}$ and the inverse Fourier transformation $\mathcal{F}^{-1}$ are defined by

$$\mathcal{F}[f](\xi) := \hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-ix\xi} f(x) \, dx,$$

$$\mathcal{F}^{-1}[f](x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\xi} f(\xi) \, d\xi,$$

where $f \in L^1(\mathbb{R}^n)$. We note that the definitions are the same for $f \in L^1(\mathbb{R}^n; X)$, where $X$ is an arbitrary Banach space. Moreover, recall that by Planchard’s theorem $\mathcal{F}: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is an isomorphism with inverse $\mathcal{F}^{-1}$ and that

$$\int_{\mathbb{R}^n} f(x)g(x) \, dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\xi) \, d\xi$$

(4.1)

for all $f, g \in L^2(\mathbb{R}^n)$. Here (4.1) holds true for $f \in L^2(\mathbb{R}^n; X), g \in L^2(\mathbb{R}^n; X')$ if the products of the functions are understood as duality product pointwise. The proof is the same as in the scalar case or one can proof (4.1) in the vector-valued case by approximating $f \in L^2(\mathbb{R}^n; X), g \in L^2(\mathbb{R}^n; X')$ by simple functions and applying the identity in the scalar case. Furthermore, if $f, g \in L^2(\mathbb{R}^n; H)$ and $H$ is a Hilbert space, then $\mathcal{F}: L^2(\mathbb{R}^n; H) \to L^2(\mathbb{R}^n; H)$ is an isomorphism with inverse $\mathcal{F}^{-1}$.

**THEOREM 4.1 (Mikhlin Multiplier Theorem)**

Let $H_0, H_1$ be Hilbert spaces and let $N = n+2$. Moreover, let $m: \mathbb{R}^n \setminus \{0\} \to \mathcal{L}(H_0, H_1)$ be an $N$-times differentiable function such that

$$\|\partial^\alpha_\xi m(\xi)\|_{\mathcal{L}(H_0, H_1)} \leq C|\xi|^{-|\alpha|}$$

(4.2)

for all $\xi \neq 0$ and $|\alpha| \leq N$ and let

$$m(D_x)f = \mathcal{F}^{-1}\left[m(\xi)\hat{f}(\xi)\right] \quad \text{for all } f \in C^\infty_0(\mathbb{R}^n; H_0).$$

Then $m(D_x)$ extends to a bounded linear operator

$$m(D_x): L^p(\mathbb{R}^n; H_0) \to L^p(\mathbb{R}^n; H_1) \quad \text{for all } 1 < p < \infty.$$  

(4.3)
The theorem is proved in Appendix B. We just note that (4.1) implies that, if $H_0 = H_1 = C$,

$$\|m(D_x)f\|_{L^2(\mathbb{R}^n)} \leq \|m\|_{L^\infty(\mathbb{R}^n)} \|f\|_{L^2(\mathbb{R}^n)} \quad \text{for all } f \in L^2(\mathbb{R}^n).$$

Hence $m(D_x) \in \mathcal{L}(L^2(\mathbb{R}^n))$. In order to prove Theorem 4.1, the main step is to show that $m(D_x) \in \mathcal{L}(L^1(\mathbb{R}^n), L^1_{\text{weak}}(\mathbb{R}^n))$. Once this is proved, $m(D_x) \in \mathcal{L}(L^p(\mathbb{R}^n))$ for all $1 < p \leq 2$ by the Marcinkiewicz interpolation theorem and the case $2 < p < \infty$ will follow by duality. Using that the Plancharel Theorem holds also for $f \in L^2(\mathbb{R}^n; H_j)$, the proof can be generalized to the case of general Hilbert spaces $H_0, H_1$.

### 4.2 A Fourier-Analytic Characterization of Hölder continuity

First of all, we recall that:

1. If $f : \mathbb{R}^n \to \mathbb{C}$ is a continuously differentiable function such that $f \in L^1(\mathbb{R}^n)$ and $\partial_x f \in L^1(\mathbb{R}^n)$, then

   $$\mathcal{F}[\partial_x f] = i\xi_x \mathcal{F}[f] = i\xi_x \hat{f}(\xi). \quad (4.4)$$

2. If $f \in L^1(\mathbb{R}^n)$ such that $x_j f \in L^1(\mathbb{R}^n)$, then $\hat{f}(\xi)$ is continuously partial differentiable with respect to $\xi_j$ and

   $$\partial_{\xi_j} \hat{f}(\xi) = \mathcal{F}[-ix_j f(x)]. \quad (4.5)$$

The latter identities show the duality between differentiability of $f$ and decay of $\hat{f}(\xi)$ as $|\xi| \to \infty$ as well as decay of $f$ for large $x$ and differentiability of $\hat{f}$.

In the following we will use a Littlewood-Paley partition of unity $\varphi_j(\xi)$, $j \in \mathbb{N}_0$ (on $\mathbb{R}^n$). This is a partition of unity $\varphi_j(\xi)$, $j \in \mathbb{N}_0$, on $\mathbb{R}^n$ with $\varphi_j \in C^\infty_0(\mathbb{R}^n)$ such that

$$\text{supp } \varphi_j \subseteq \{ \xi \in \mathbb{R}^n : 2^{j-1} \leq |\xi| \leq 2^{j+1} \} \quad \text{for all } j \geq 1. \quad (4.6)$$

The partition of unity can be constructed such that $\text{supp } \varphi_0 \subset \overline{B_2(0)}$, $\varphi_j(\xi) = \varphi_1(2^{-j+1}\xi)$ for all $j \geq 1$ and (4.6) holds. Then we have

$$|\partial_\xi^\alpha \varphi_j(\xi)| \leq C \|\partial_\xi^\alpha \varphi_1\|_{L^\infty(\mathbb{R}^n)} 2^{-|\alpha|j} \quad \text{for all } \alpha \in \mathbb{N}_0^n, j \geq 1. \quad (4.7)$$
Moreover, we note that
\[ \varphi_j(D_x)f = \mathcal{F}^{-1} \left[ \varphi_j(\xi) \hat{f}(\xi) \right] = \hat{\varphi}_j * f \]
for all \( f \in C^\infty_0(\mathbb{R}^n) \), \( j \in \mathbb{N}_0 \), where \( \hat{\varphi}_j = \mathcal{F}^{-1} [\varphi_j] \) and
\[ \varphi_j(x) = 2^{(j-1)n} \varphi_1(2^{j-1}x) \quad \text{for all } j \in \mathbb{N} \text{ and } x \in \mathbb{R}^n. \]
For \( f \in L^\infty(\mathbb{R}^n) \) we will define \( \varphi_j(D_x)f \) by
\[ \varphi_j(D_x)f = \hat{\varphi}_j * f. \]
Furthermore, since \( \varphi_{j-1} + \varphi_j + \varphi_{j+1} \equiv 1 \) on \( \text{supp } \varphi_j \) (where \( \varphi_{-1} \equiv 0 \), we have
\[ \varphi_j(D_x)f = (\varphi_{j-1}(D_x) + \varphi_j(D_x) + \varphi_{j+1}(D_x)) \varphi_j(D_x)f \quad (4.8) \]
for all \( f \in \mathcal{S}(\mathbb{R}^n), j \in \mathbb{N}_0. \)
Using this decomposition, we obtain the following characterization of Hölder continuous functions.

**Theorem 4.2** Let \( 0 < s < 1 \). Then \( f \in C^s(\mathbb{R}^n) \) if and only if \( f \in L^\infty(\mathbb{R}^n) \) and
\[ \|f\|_{C^s} := \sup_{j \in \mathbb{N}_0} 2^{js} \|\hat{\varphi}_j * f\|_{L^\infty(\mathbb{R}^n)} < \infty. \]
Moreover, \( \| \cdot \|_{C^s} \) is an equivalent norm on \( C^s(\mathbb{R}^n) \).

**Proof:** First let \( f \in C^s(\mathbb{R}^n) \). Then
\[ \sup_{x \in \mathbb{R}^n} |f(x) - f(y)| \leq \|f\|_{C^s} |y|^s \]
for all \( y \in \mathbb{R}^n \). Since we have chosen \( \varphi_j \) such that \( \varphi_j(\xi) = \varphi_1(2^{1-j}\xi) \) for \( j \geq 1 \), we have \( \hat{\varphi}_j(x) = \psi_{2^{-j}}(x) = 2^{jn} \psi(2^jx) \), where \( \psi(x) = \mathcal{F}^{-1}[\varphi_1(2^{-1}\xi)]. \)
This implies that
\[ \|\hat{\varphi}_j\|_{L^1(\mathbb{R}^n)} \leq C, \quad \|\nabla \hat{\varphi}_j\|_{L^1(\mathbb{R}^n)} \leq C2^j \quad \text{for all } j \in \mathbb{N}_0, \quad (4.9) \]
Moreover,
\[ \int_{\mathbb{R}^n} \hat{\varphi}_j(y)dy = \int_{\mathbb{R}^n} \psi(y)dy = \mathcal{F}[\psi](0) = \varphi_1(0) = 0. \]

28
for all \( j \geq 1 \). Hence
\[
\varphi_j(D_x)f = \int_{\mathbb{R}^n} f(x-y)\psi_{2^{-j}}(y)dy = \int_{\mathbb{R}^n} (f(x-y) - f(x))\psi_{2^{-j}}(y)dy \tag{4.10}
\]
and therefore
\[
\|\varphi_j(D_x)f\|_\infty \leq \|f\|_{C^s} \int_{\mathbb{R}^n} |y|^s\psi_{2^{-j}}(y)dy
\]
\[
= 2^{-js}\|f\|_{C^s} \int_{\mathbb{R}^n} |z|^s\psi(z)dz = C2^{-js}\|f\|_{C^s}
\]
for all \( j \in \mathbb{N} \). The latter inequality implies \( \|f\|_{C^s} \leq C\|f\|_{C^s} \), since also \( \varphi_0(D_x)f\|_\infty \leq C\|f\|_\infty \).

Conversely, let \( f \in L^\infty(\mathbb{R}^n) \) be such that \( \|f\|_{C^s} < \infty \). Now, if \( |y| \leq 1 \),
\[
f(x-y) - f(x) = \sum_{2^j \leq |y|^{-1}} (f_j(x-y) - f_j(x)) + \sum_{2^j > |y|^{-1}} (f_j(x-y) - f_j(x)),
\]
where \( f_j = \varphi_j(D_x)f \). In order to estimate the first sum, we use the mean value theorem to conclude that
\[
|f_j(x-y) - f_j(x)| \leq |y|\|\nabla f_j\|_\infty. \tag{4.11}
\]
Moreover, since
\[
\partial_{x_k}f_j = \partial_{x_k}\varphi_{j-1}(D_x)f_j + \partial_{x_k}\varphi_j(D_x)f_j + \partial_{x_k}\varphi_{j+1}(D_x)f_j,
\]
due to (4.8) and
\[
\|\partial_{x_k}\varphi(D_x)g\|_{L^\infty(\mathbb{R}^n)} \leq \|\partial_{x_k}\tilde{\varphi}(l_{1(\mathbb{R}^n)})\|_{L^1(\mathbb{R}^n)}\|g\|_{L^\infty(\mathbb{R}^n)} \leq C2^l\|g\|_{L^\infty(\mathbb{R}^n)}
\]
for general \( l \in \mathbb{N}_0 \), \( g \in L^\infty(\mathbb{R}^n) \), we obtain
\[
\sum_{2^j \leq |y|^{-1}} |f_j(x-y) - f_j(y)| \leq C \sum_{2^j \leq |y|^{-1}} |y|2^j\|\nabla f_j\|_\infty
\]
\[
\leq C|y| \sum_{2^j \leq |y|^{-1}} 2^j(1-s)\|f\|_{C^s} \leq C|y|^s\|f\|_{C^s}.
\]
The second sum is simply estimated by
\[
\sum_{2^j > |y|^{-1}} |f_j(x-y) - f_j(y)| \leq 2 \sum_{2^j > |y|^{-1}} \|f_j\|_\infty
\]
\[
\leq 2\|f\|_{C^s} \sum_{2^j > |y|^{-1}} 2^{-js} = C|y|^s\|f\|_{C^s}
\]
20
Altogether \( \|f\|_{C^s} \leq C\|f\|_{C^1} \). \( \blacksquare \)

**Remark 4.3** Because of (4.9) we get
\[
\|\varphi_j(D_x)f\|_{L^p(\mathbb{R}^n)} \leq \|\tilde{\varphi}_j\|_{L^1(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}, \tag{4.12}
\]
\[
\|\nabla \varphi_j(D_x)f\|_{L^p(\mathbb{R}^n)} \leq \|\nabla \tilde{\varphi}_j\|_{L^1(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)} \leq C2^j\|f\|_{L^p(\mathbb{R}^n)}, \tag{4.13}
\]
for any \( f \in L^p(\mathbb{R}^n) \), \( 1 \leq p \leq \infty \), \( j \in \mathbb{N}_0 \).

### 4.3 Bessel Potential and Besov Spaces – Definitions and Basic Properties

In the following let \( \langle \xi \rangle := (1 + |\xi|^2)^{\frac{1}{2}} \). We note that for any \( s \in \mathbb{R} \) the function \( \langle \xi \rangle^s \) is a smooth function satisfying
\[
|\partial^\alpha \langle \xi \rangle^s| \leq C_{s,\alpha}(1 + |\xi|)^{s-|\alpha|} \tag{4.14}
\]
for all \( \alpha \in \mathbb{N}_0^n \) and some \( C_{s,\alpha} > 0 \). The latter estimate can be proved by considering \( f : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R} \) defined by \( f_s(\xi, t) = |(t, \xi)|^s \). Since \( f_s \) is smooth and homogeneous of degree \( s \) in \( (t, \xi) \), we have
\[
|\partial^\alpha f_s(\xi, t)| \leq C_{s,\alpha}|t| + |\xi|^{s-|\alpha|}
\]
uniformly in \( (x, t) \neq 0 \) and for all \( \alpha \in \mathbb{N}_0^n \).

Using (4.14) it is easy to show, that \( \langle \xi \rangle^s \hat{f}(\xi) \in S(\mathbb{R}^n) \) for all \( f \in S(\mathbb{R}^n) \). By duality \( \langle \xi \rangle^s \hat{f} \in S'(\mathbb{R}^n) \) for all \( f \in S'(\mathbb{R}^n) \). Therefore we can define \( \langle D_x \rangle^s : S'(\mathbb{R}^n) \to S'(\mathbb{R}^n) \) is well defined as
\[
\langle D_x \rangle^s f = \mathcal{F}^{-1} \left[ \langle \xi \rangle^s \hat{f} \right] \quad \text{for all } f \in S'(\mathbb{R}^n).
\]

**Definition 4.4** Let \( s \in \mathbb{R} \) and let \( 1 < p < \infty \). Then the Bessel potential space \( \mathcal{H}^s_p(\mathbb{R}^n) \) is defined by
\[
\mathcal{H}^s_p(\mathbb{R}^n) = \{ f \in S'(\mathbb{R}^n) : \langle D_x \rangle^s f \in L^p(\mathbb{R}^n) \}
\]
\[
\|f\|_{\mathcal{H}^s_p(\mathbb{R}^n)} = \|\langle D_x \rangle^s f\|_{L^p(\mathbb{R}^n)}.
\]

**Remarks 4.5**

1. If \( p = 2 \), then we have \( f \in \mathcal{H}^s_2(\mathbb{R}^n) \) if and only if \( \langle \xi \rangle^s \hat{f} \in L^2(\mathbb{R}^n) \) by Plancharel's theorem.
2. By definition \( (D_x)^s : \mathcal{H}^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n) \) is an isomorphism with inverse \( (D_x)^{-s} \). Since \( S(\mathbb{R}^n) \) is dense in \( L^p(\mathbb{R}^n) \) and \( (D_x)^{-s} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \), \( \mathcal{S}(\mathbb{R}^n) \) is dense in \( \mathcal{H}^p(\mathbb{R}^n) \) for any \( s \in \mathbb{R}, 1 < p < \infty \).

As a consequence of the Mikhlin multiplier theorem we obtain

**Theorem 4.6** Let \( m \in \mathbb{N}_0 \) and let \( 1 < p < \infty \). Then \( \mathcal{H}^m_p(\mathbb{R}^n) = W^m_p(\mathbb{R}^n) \) with equivalent norms.

**Proof:** We first prove the embedding \( \mathcal{H}^m_p(\mathbb{R}^n) \hookrightarrow W^m_p(\mathbb{R}^n) \). Let \( f \in \mathcal{S}(\mathbb{R}^n) \). Then

\[
\partial_x^\beta f = \mathcal{F}^{-1} \left[ (i\xi)^\beta \hat{f}(\xi) \right] = \mathcal{F}^{-1} \left[ \frac{(i\xi)^\beta}{(\xi)^m} \right] (\xi)^m \hat{f}(\xi)
\]

Hence in order to obtain

\[
\|\partial_x^\beta f\|_{L^p(\mathbb{R}^n)} \leq C_p \| (D_x)^m f \|_{L^p(\mathbb{R}^n)} \equiv C_p \| f \|_{\mathcal{H}^m_p(\mathbb{R}^n)}
\]  

(4.15)

for \( \beta \in \mathbb{N}_0^n \) with \( |\beta| \leq m \) we apply Theorem 4.1 to \( m_\beta(\xi) = \frac{(i\xi)^\beta}{(\xi)^m} \). Therefore we have to verify (4.2) for \( m = m_\beta \). To this end, we use (4.14) and

\[
|\partial_x^{\alpha} (i\xi)^\beta| \leq C_{\alpha,\beta} |\xi|^{|\beta|-|\alpha|}.
\]

(4.16)

Moreover, \( (1 + |\xi|)^{-m-|\alpha|} \leq |\xi|^{-|\beta|-|\alpha|} \) if \( |\beta| \leq m \). Therefore

\[
|\partial_x^{\alpha} m_\beta(\xi)| \leq C_{\alpha,\beta} |\xi|^{-|\alpha|}
\]

(4.17)

follows from (4.16), (4.14), and the following claim:

**Claim:** Let \( s_1, s_2 \in \mathbb{R}, N \in \mathbb{N} \) and let \( p_1, p_2 : \mathbb{R}^n \setminus \{0\} \to \mathbb{C} \) be \( N \)-times continuously differentiable satisfying

\[
|\partial_x^{\alpha} p_j(\xi)| \leq C |\xi|^{s_j-|\alpha|} \quad \text{for all } |\alpha| \leq N, j = 1, 2.
\]

Then

\[
|\partial_x^{\alpha} (p_1(\xi)p_2(\xi))| \leq C |\xi|^{s_1+s_2-|\alpha|}
\]

(4.18)

for all \( |\alpha| \leq N \).

**Proof of Claim:** The claim follows directly from the Leibniz’s formula.

Because of (4.17), the conditions of the Mikhlin Multiplier Theorem 4.1 are satisfied and (4.15) follows for all \( |\beta| \leq m \), which proves \( \mathcal{H}^m_p(\mathbb{R}^n) \hookrightarrow W^m_p(\mathbb{R}^n) \) since \( \mathcal{S}(\mathbb{R}^n) \) is dense in \( \mathcal{H}^m_p(\mathbb{R}^n) \).

31
Hence it remains to prove $W^m_p(\mathbb{R}^n) \hookrightarrow H^m_p(\mathbb{R}^n)$. If $m = 2k$, $k \in \mathbb{N}_0$, is even, then $\langle \xi \rangle^m = (1 + |\xi|^2)^k$ is a polynomial of degree $m$. Therefore $\langle D_x \rangle^m$ is a differential operator of order $m$ and
\[
\| (D_x)^m f \|_{L^p(\mathbb{R}^n; H)} \leq C \sum_{|\alpha| \leq m} \| \partial_x^\alpha f \|_{L^p(\mathbb{R}^n; H)},
\]
which proves the embedding in this case.

If $m = 2k + 1$, $k \in \mathbb{N}_0$, is odd, then
\[
\langle \xi \rangle^m = \langle \xi \rangle^m \left( \frac{1}{\langle \xi \rangle^2} + \sum_{j=1}^n \frac{\xi_j^2}{\langle \xi \rangle^2} \right) = \frac{1}{\langle \xi \rangle^{2k}} + \sum_{j=1}^n \frac{\xi_j}{\langle \xi \rangle} \langle \xi \rangle^{2k} \xi_j,
\]
where $\langle \xi \rangle^{2k}$ and $\langle \xi \rangle^{2k} \xi_j$ are polynomials of degree at most $2k + 1$. Hence
\[
\| (D_x)^m f \|_{L^p(\mathbb{R}^n; H)} \leq C \sum_{|\alpha| \leq m} \sum_{j=0}^n \| m_j(D_x) \partial^\alpha_x f \|_{L^p(\mathbb{R}^n; H)},
\]
where $m_0(\xi) = \langle \xi \rangle^{-1}$ and $m_j(\xi) = \xi_j \langle \xi \rangle^{-1}$, $j = 1, \ldots, n$. Hence it remains to verify the Mikhlin condition (4.2) for $m_j(\xi)$. If $j = 0$, then (4.2) for $m(\xi) = m_0(\xi)$ follows from (4.14) with $s = -1$ because of $\langle \xi \rangle^{-1-|\alpha|} \leq |\xi|^{-|\alpha|}$. If $j = 1, \ldots, n$, then (4.2) follows for $m(\xi) = m_j(\xi)$ from (4.14) with $s = -1$, (4.16) with $\beta = e_j$, and (4.18).

Theorem 4.2 gives a motivation for the following definition of the Besov space $B^s_{pq}(\mathbb{R}^n)$.

**Definition 4.7** Let $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$. Then
\[
B^s_{pq}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \| f \|_{B^s_{pq}(\mathbb{R}^n)} < \infty \right\},
\]
where
\[
\| f \|_{B^s_{pq}(\mathbb{R}^n)} = \left( \sum_{j=0}^\infty 2^{jsq} \| \varphi_j(D_x) f \|_{L^p(\mathbb{R}^n)}^q \right)^{\frac{1}{q}} \quad \text{if } q < \infty,
\]
\[
\| f \|_{B^s_{pq}(\mathbb{R}^n)} = \sup_{j \in \mathbb{N}_0} 2^{js} \| \varphi_j(D_x) f \|_{L^p(\mathbb{R}^n)} \quad \text{if } q = \infty.
\]
Remarks 4.8 1. Because of Theorem 4.2, \( C^s(\mathbb{R}^n) = B^s_{\infty\infty}(\mathbb{R}^n) \) for \( 0 < s < 1 \). More generally, \( C^s(\mathbb{R}^n) := B^s_{\infty\infty}(\mathbb{R}^n), s > 0 \), are called Hölder-Zygmund spaces.

2. Note that \( f \in B^s_{pq}(\mathbb{R}^n) \) if and only if
\[
(\varphi_j(D_x)f)_{j \in \mathbb{N}_0} \in \ell^s_q(\mathbb{N}_0; L^p(\mathbb{R}^n)),
\]
Here \( \ell^s_q(\mathbb{M}; X), \mathbb{M} \subseteq \mathbb{Z} \), is the space of all \( X \)-valued sequences \( x = (x_j)_{j \in \mathbb{M}} \) such that
\[
\|x\|_{\ell^s_q(\mathbb{M}; X)} = \left\{ \begin{array}{ll}
\left( \sum_{j \in \mathbb{M}} (2^{js}\|x\|_X)^q \right)^{\frac{1}{q}} & \text{if } q < \infty, \\
\sup_{j \in \mathbb{M}} 2^{js}\|x\|_X & \text{if } q = \infty.
\end{array} \right.
\]
Moreover, we set \( \ell^q(\mathbb{M}; X) = \ell^0_q(\mathbb{M}; X) \). Of course \( (x_j)_{j \in \mathbb{M}} \in \ell^s_q(\mathbb{M}; X) \) if and only if \( (2^{js}x_j)_{j \in \mathbb{M}} \in \ell^q(\mathbb{M}; X) \).

3. Using Plancharel’s theorem, it is not difficult to show that
\[
B^s_{22}(\mathbb{R}^n) = H^s_2(\mathbb{R}^n).
\]
The proof is left to the reader as an exercise. – But the statement will also follow from Corollary 4.15 below.

Some simple relations between the Besov spaces are summarized in the following.

Lemma 4.9 Let \( s \in \mathbb{R}, 1 \leq p, q_1, q_2 \leq \infty \), and let \( \varepsilon > 0 \). Then
\[
B^s_{pq_1}(\mathbb{R}^n) \hookrightarrow B^s_{pq_2}(\mathbb{R}^n) \text{ if } q_1 \leq q_2, \quad B^s_{p\infty}(\mathbb{R}^n) \hookrightarrow B^{s+\varepsilon}_{p1}(\mathbb{R}^n).
\]

Proof: The first embedding follows from
\[
\ell^0_q(\mathbb{N}_0; X) \hookrightarrow \ell^p_q(\mathbb{N}_0; X), \ell^s_q(\mathbb{N}_0; X) \hookrightarrow \ell^{s+\varepsilon}_{q_2}(\mathbb{N}_0; X) \text{ if } q_1 \leq q_2. \quad (4.19)
\]
The second embedding follows \( \ell^{s+\varepsilon}_{\infty}(\mathbb{N}_0; X) \hookrightarrow \ell^1_q(\mathbb{N}_0; X) \) because of
\[
\|(a_j)\|_{\ell^s_q(\mathbb{N}_0; X)} = \sum_{j=0}^{\infty} 2^{sj}\|a_j\|_X \\
\leq \left( \sum_{j=0}^{\infty} 2^{-\varepsilon j} \sup_{j \in \mathbb{N}_0} 2^{(s+\varepsilon)j}\|a_j\|_X \right) \leq C_\varepsilon \|(a_j)\|_{\ell^{s+\varepsilon}_{\infty}(\mathbb{N}_0; X)}.
\]

33
Remark 4.10 The latter lemma shows that \( q \) measures regularity of \( f \) on a finer scale than \( s \), meaning, if \( s > s' \), then \( B^s_{pq1}(\mathbb{R}^n) \hookrightarrow B^{s'}_{pq2}(\mathbb{R}^n) \) with arbitrary \( 1 \leq q_1, q_2 \leq \infty \).

Exercise 1 Show that

\[
B^s_{pq1}(\mathbb{R}^n) \hookrightarrow C^0(\mathbb{R}^n).
\]

Hint: Use that \( \varphi_j(D_x)f = \hat{\varphi}_j \ast f \), where \( \|\hat{\varphi}_j\|_{L^s(\mathbb{R}^n)} \leq C2^{js} \) uniformly in \( j \in \mathbb{Z} \).

In order to get a sharp comparision of Besov and Bessel potential spaces we prove:

**Theorem 4.11** Let \( s \in \mathbb{R}, 1 < p < \infty \). Then there are constants \( c, C > 0 \) such that

\[
c\|f\|_{H^s_p(\mathbb{R}^n)} \leq \left( \sum_{j=0}^{\infty} 2^{js} |\varphi_j(D_x)f(x)|^2 \right)^{\frac{1}{2}} \leq C\|f\|_{H^s_p(\mathbb{R}^n)}
\]

for all \( f \in H^s_p(\mathbb{R}^n) \).

Remark 4.12 Because of the latter equivalent norm on \( H^s_p(\mathbb{R}^n) \), one defines more generally the Triebel-Lizorkin space \( F^s_{pq}(\mathbb{R}^n) \), \( s \in \mathbb{R}, 1 < p, q < \infty \), as

\[
F^s_{pq}(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \|f\|_{F^s_{pq}(\mathbb{R}^n)} < \infty \right\},
\]

\[
\|f\|_{F^s_{pq}(\mathbb{R}^n)} = \left\| \left( \sum_{j=0}^{\infty} 2^{js} |\varphi_j(D_x)f(x)|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)}.
\]

Hence the latter theorem shows that \( H^s_p(\mathbb{R}^n) = F^s_{pq}(\mathbb{R}^n) \). Finally, note that

\[
\|f\|_{F^s_{pq}(\mathbb{R}^n)} = \|\varphi_j(D_x)f\|_{L^p(\mathbb{R}^n; \ell^q(\mathbb{N}))}.
\]

**Proof of Theorem 4.11:** First we will show that \( \|f\|_{F^s_{pq}(\mathbb{R}^n)} \leq C\|f\|_{H^s_p(\mathbb{R}^n)} \). To this end we define a mapping

\[
Q : S(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n; \ell^q(\mathbb{N}))
\]
by
\[(Qg)(x) = (2^j \varphi_j(D_x)(D_x)^{-s}g(x))_{j \in \mathbb{N}_0} \in \ell^2(\mathbb{N}_0) \quad \text{for all } x \in \mathbb{R}^n.\]

Then
\[(Qg)(x) = \mathcal{F}_{\xi \rightarrow x}^{-1} \lfloor m(\xi) \hat{g}(\xi) \rfloor\]
where \(m(\xi) \in \mathcal{L}(\mathbb{C}, \ell^2(\mathbb{N}_0))\) is defined by
\[m(\xi)a = (2^j \varphi_j(\xi)\langle \xi \rangle^{-s})_{j \in \mathbb{N}_0}a \quad \text{for all } a \in \mathbb{C}, \xi \in \mathbb{R}^n.\]

In order to show that \(Q\) extends to a bounded operator
\[Q: L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n; \ell^2(\mathbb{N}_0)) \quad \text{for all } 1 < p < \infty, \quad (4.20)\]
we verify the condition for the Mikhlin multiplier theorem 4.1:

\[
\|\partial^\alpha \xi m(\xi)\|_{L^2(\mathbb{C}, \ell^2(\mathbb{N}_0))} = \sum_{j=0}^{\infty} 2^{2js} \|\partial^\alpha \xi (\varphi_j(\xi)\langle \xi \rangle^{-s})\|^2 \\
\leq C_{\alpha,s} 2^{2js} \langle \xi \rangle^{-2s-2|\alpha|} \chi_{\text{supp } \varphi_j}(\xi) \leq C_{\alpha,s} \langle \xi \rangle^{-2|\alpha|}
\]
for all \(\alpha \in \mathbb{N}_0^n\), where we have used that \(2^{-1} \leq |\xi| \leq 2^{j+1}\) on \(\text{supp } \varphi_j\) if \(j \geq 1\) and
\[|\partial^\alpha \xi (\varphi_j(\xi)\langle \xi \rangle^{-s})| \leq C_{\alpha,s} \langle \xi \rangle^{-s-|\alpha|}\]
uniformly in \(j \in \mathbb{N}_0\), which follows from (4.7), (4.14), and the product rule. Hence (4.20) follows and therefore
\[\|f\|_{F_{p,2}(\mathbb{R}^n)} = \|Q(D_x)^s f\|_{L^p(\mathbb{R}^n; \ell^2(\mathbb{N}_0))} \leq C\|\langle D_x \rangle^s f\|_{L^p(\mathbb{R}^n)} \equiv C\|f\|_{H^p_2(\mathbb{R}^n)}.
\]

Note that we have shown that
\[\tilde{Q}: H^s_p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n; \ell^2_q(\mathbb{N}_0))\]
is bounded, where
\[\tilde{Q}f = (2^{-js}Q(D_x)^s f)_{j \in \mathbb{N}_0} = (\varphi_j(D_x)f)_{j \in \mathbb{N}_0}. \quad (4.21)\]

Conversely, we define a mapping
\[R: S(\mathbb{R}^n; \ell^2(\mathbb{N}_0)) \subset L^p(\mathbb{R}^n; \ell^2(\mathbb{N}_0)) \rightarrow L^p(\mathbb{R}^n)\]
by

$$(Ra)(x) = \sum_{j=0}^{\infty} 2^{-js} \tilde{\varphi}_j(D_x)(D_x)^s a_j(x) \quad \text{for all } x \in \mathbb{R}^n, a \in \mathcal{S}(\mathbb{R}^n; \ell^2(\mathbb{N}_0)).$$

Here $\tilde{\varphi}_j(\xi) = \varphi_{j-1}(\xi) + \varphi_j(\xi) + \varphi_{j+1}(\xi)$, $j \in \mathbb{N}_0$, and $\varphi_{-1}(\xi) = 0$. – Note that $\tilde{\varphi}_j(\xi)\varphi_j(\xi) = \tilde{\varphi}_j(\xi)$ since $\tilde{\varphi}_j(\xi) = 1$ on supp $\varphi_j$. Then

$$(Ra)(x) = \mathcal{F}_{\xi \to x}^{-1} [m(\xi) \hat{a}_j(\xi)]$$

where $m(\xi) \in \mathcal{L}(\ell^2(\mathbb{N}_0), \mathbb{C})$ is defined by

$$m(\xi)a = \sum_{j=0}^{\infty} 2^{-js} \tilde{\varphi}_j(\xi)\varphi_j(\xi)^s a_j \quad \text{for all } (a_j)_{j \in \mathbb{N}_0} \in \ell^2(\mathbb{N}_0).$$

Similarly, as before

$$\|\partial^\alpha_{\xi} m(\xi)\|^2_{\mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathbb{C})} \leq \sum_{j=0}^{\infty} 2^{-2js} |\partial^\alpha_{\xi} (\tilde{\varphi}_j(\xi)\varphi_j(\xi)^s)|^2$$

$$\leq C_q \sum_{j=0}^{\infty} 2^{-2js} |\xi|^{2s-2|\alpha|} \chi\{2^{j-2} \leq |\xi| \leq 2^{j+2}\} \leq C_q |\xi|^{-2|\alpha|},$$

where we have used that for each $\xi \in \mathbb{R}^n$ at most 5 terms in the sum above are non-zero and that $2^{-2js} \leq C |\xi|^{-s}$ on supp $\tilde{\varphi}_j \subset \{2^{j-2} \leq |\xi| \leq 2^{j+2}\}$. Hence, applying Theorem 4.1 once more, we obtain that $R$ extends to a bounded operator

$$R: L^p(\mathbb{R}^n; \ell^2(\mathbb{N}_0)) \to L^p(\mathbb{R}^n) \quad \text{for all } 1 < p < \infty.$$

Now we apply $R$ to $a_j = 2^{js} \varphi_j(D_x)f$, $j \in \mathbb{N}_0$. Then

$$Ra = \sum_{j=0}^{\infty} 2^{-js} \tilde{\varphi}_j(D_x)(D_x)^s 2^{js} \varphi_j(D_x)f = (D_x)^s f$$

since $\sum_{j=0}^{\infty} \tilde{\varphi}_j(D_x)\varphi_j(D_x) = \sum_{j=0}^{\infty} \varphi_j(D_x) = I$. Thus

$$\|f\|_{H^s_0(\mathbb{R}^n)} = \|\langle D_x \rangle^s f\|_{L^p(\mathbb{R}^n)} = \|Ra\|_{L^p(\mathbb{R}^n)} \leq C \|\langle 2^{js} \varphi_j(D_x)f \rangle\|_{\ell^2(\mathbb{N}_0)} \leq C \|f\|_{F^s_{p,2}(\mathbb{R}^n)},$$

36
which proves the lemma. Finally, we note that the previous estimates imply that
\[ \tilde{R}: L^p(\mathbb{R}^n; \ell^s_2(N_0)) \to H^s_p(\mathbb{R}^n) \]
is bounded, where
\[ \tilde{R}(a_j)_{j \in N_0} := \langle D_x \rangle^s R(2^j sa_j)_{j \in N_0} = \sum_{j=0}^{\infty} \tilde{\varphi}_j(D_x)a_j \] (4.22)
and therefore \( \tilde{R} \tilde{Q} = I \) on \( H^s_p(\mathbb{R}^n) \).

**Remark 4.13** Note that the latter proof shows that \( H^s_p(\mathbb{R}^n) \) is a retract of \( L^p(\mathbb{R}^n; \ell^s_2(N_0)) \). In general:

**Definition 4.14** A Banach space \( X \) is called a retract of a Banach space \( Y \) if there are bounded, linear operators \( R: Y \to X \) and \( Q: X \to Y \) such that \( RQ = \text{id}_X \).

If \( \tilde{R}, \tilde{Q} \) are defined by (4.22) and (4.21), then \( \tilde{R} \tilde{Q} = I \) on \( H^s_p(\mathbb{R}^n) \). Note that the mappings are independent of \( p \) and \( s \). Moreover, using the same mappings \( R \) and \( Q \) as in the previous proof it is easy to show that \( B^s_p(\mathbb{R}^n) \) is a retract of \( \ell^q(N_0; L^p(\mathbb{R}^n)) \).

**Corollary 4.15** Let \( 1 < p < \infty, s \in \mathbb{R} \). Then
\[ B^s_p(\mathbb{R}^n) \hookrightarrow H^s_p(\mathbb{R}^n) \hookrightarrow B^s_{p2}(\mathbb{R}^n) \quad \text{if } 1 < p \leq 2, \] (4.23)
\[ B^s_{p2}(\mathbb{R}^n) \hookrightarrow H^s_p(\mathbb{R}^n) \hookrightarrow B^s_{p\infty}(\mathbb{R}^n) \quad \text{if } 2 \leq p < \infty. \] (4.24)

In particular, \( H^2_s(\mathbb{R}^n) = B^s_{p2}(\mathbb{R}^n) \) for all \( s \in \mathbb{R} \).

**Proof:** The statements follow from Theorem 4.11 and the embeddings:
\[ \ell^q(N_0; L^p(\mathbb{R}^n)) \hookrightarrow L^p(\mathbb{R}^n; \ell^q(N_0)) \quad \text{if } 1 \leq q \leq p \leq \infty \] (4.25)
\[ L^p(N_0; \ell^q(\mathbb{R}^n)) \hookrightarrow \ell^q(\mathbb{R}^n; L^p(N_0)) \quad \text{if } 1 \leq p \leq q \leq \infty. \] (4.26)
where $X$ is a general Banach space, as well as from (4.19). Here (4.25) follows from
\[
\|(f_j)_{j \in N_0}\|_{L^p(\mathbb{R}^n; \ell^q(N))} \\
= \left( \int_{\mathbb{R}^n} \left( \sum_{j=0}^{\infty} |f_j(x)|^q \right)^{\frac{p}{q}} \, dx \right)^{\frac{1}{p}} = \left( \sum_{j=0}^{\infty} \|f_j(\cdot)|^q \right)^{\frac{1}{p}}_{L^p(\mathbb{R}^n)} \\
\leq \left( \sum_{j=0}^{\infty} \|f_j(\cdot)|^q \right)^{\frac{1}{p}}_{L^p(\mathbb{R}^n)} = \left( \sum_{j=0}^{\infty} \|f_j(\cdot)|^q \right)^{\frac{1}{p}}_{L^p(\mathbb{R}^n)},
\]
where we have used Minkowski’s inequality. The inequality (4.26) is proved analogously.

\[\]

4.4 Interpolation of Vector-Valued $L^p$-Spaces, Bessel Potential, and Besov Spaces

Recall that $H_p^s(\mathbb{R}^n)$ and $B_{pq}^s(\mathbb{R}^n)$ are retracts of $L^p(\mathbb{R}^n; \ell^2(N_0))$ and $\ell^s_q(N_0; L^p(\mathbb{R}^n))$ with same retraction and co-retractions, which are independent of $p, s$. More precisely,
\[
\tilde{Q}: H_p^s(\mathbb{R}^n) \to L^p(\mathbb{R}^n; \ell^2(N_0)), \\
\tilde{Q}: B_{pq}^s(\mathbb{R}^n) \to \ell^s_q(N_0; L^p(\mathbb{R}^n))
\]
and
\[
\tilde{R}: L^p(\mathbb{R}^n; \ell^2(N_0)) \to H_p^s(\mathbb{R}^n), \\
\tilde{R}: \ell^s_q(N_0; L^p(\mathbb{R}^n)) \to B_{pq}^s(\mathbb{R}^n)
\]
are bounded linear operators satisfying $\tilde{R}\tilde{Q} = I$ where
\[
\tilde{Q}f = (\varphi_j(Dx)f)_{j \in N_0}, \quad f \in H_p^s(\mathbb{R}^n) \cup B_{pq}^s(\mathbb{R}^n),
\]
and
\[
\tilde{R}(a_j)_{j \in N_0} := \sum_{j=0}^{\infty} \varphi_j(Dx)a_j, \quad (a_j)_{j \in N_0} \in L^p(\mathbb{R}^n; \ell^2(N_0)) \cup \ell^s_q(N_0; L^p(\mathbb{R}^n)).
\]
Generally we have:

\[\]
Proposition 4.16 Let $(X_0, X_1)$ and $(Y_0, Y_1)$ be admissible Banach spaces and let $Q: Y_0 + Y_1 \to X_0 + X_1$, $R: X_0 + X_1 \to Y_0 + Y_1$ be linear mappings such that $Q \in \mathcal{L}(Y_j, X_j)$, $R \in \mathcal{L}(X_j, Y_j)$ and $RQx = x$ for all $x \in X_j$ and $j = 0, 1$. Then

$$R(X_0, X_1)_{\theta,p} = (Y_0, Y_1)_{\theta,p} \quad R(X_0, X_1)_{[\theta]} = (Y_0, Y_1)_{[\theta]}$$

with equivalent norms for all $\theta \in (0, 1), 1 \leq p \leq \infty$, where $R(X_0, X_1)_{\theta,p}$ is equipped with the quotient norm

$$\|y\|_{R(X_0, X_1)_{\theta,p}} = \inf_{x \in (X_0, X_1)_{\theta,p}: Rx = y} \|x\|_{\theta,p}.$$

Proof: The proof of the proposition is similar to the proof of Example 3.10. ■

Because of the latter proposition, it is sufficient to obtain interpolation results for $L^p(\mathbb{R}^n; \ell^2_q(\mathbb{N}))$ and $\ell^q_s(\mathbb{N}_0; L^p(\mathbb{R}^n))$ to characterize the real and complex interpolation spaces of $H^s_p(\mathbb{R}^n)$ and $B^s_{pq}(\mathbb{R}^n)$.

We start with a result for the real interpolation method:

THEOREM 4.17 Let $X$ be a Banach spaces, let $s_0 \neq s_1 \in \mathbb{R}$, $0 < \theta < 1$, and let $1 \leq q_0, q_1 \leq \infty$. Then for every $1 \leq q \leq \infty$

$$(\ell^s_{q_0}(\mathbb{Z}; X), \ell^s_{q_1}(\mathbb{Z}; X))_{\theta,q} = \ell^s_q(\mathbb{Z}; X),
\ell^s_s(q_0(\mathbb{N}_0; X), \ell^s_{q_1}(\mathbb{N}_0; X))_{\theta,q} = \ell^s_q(\mathbb{N}_0; X)$$

with equivalent norms, where $s = (1 - \theta)s_0 + \theta s_1$.

As a consequence we obtain

THEOREM 4.18 Let $s_0 \neq s_1 \in \mathbb{R}$, $0 < \theta < 1$ and let $s = (1 - \theta)s_0 + \theta s_1$. Then for all $1 \leq p, q_0, q_1, q \leq \infty$

$$(B^s_{\theta q_0}(\mathbb{R}^n), B^s_{\theta q_1}(\mathbb{R}^n))_{\theta,q} = B^s_{pq}(\mathbb{R}^n)$$

with equivalent norms. Moreover, for any $1 \leq p, q \leq \infty$ we have

$$(H^s_p(\mathbb{R}^n), H^s_q(\mathbb{R}^n))_{\theta,q} = B^s_{pq}(\mathbb{R}^n)$$

with equivalent norms.
**Remark 4.19** Because of Example 3.10, we know that

\[(L^p(\mathbb{R}^n), W^{1/p}_p(\mathbb{R}^n))_{\theta,p} = W^p(\mathbb{R}^n)\]

for all \(1 \leq p < \infty\). On the other hand, we have by the previous theorem:

\[(L^p(\mathbb{R}^n), W^{1/p}_p(\mathbb{R}^n))_{\theta,p} = (H^0_p(\mathbb{R}^n), H^1_p(\mathbb{R}^n))_{\theta,p} = B^{\theta}_p(\mathbb{R}^n)\]

if \(1 < p < \infty\), \(\theta \in (0, 1)\). Hence \(B^{\theta}_p(\mathbb{R}^n) = W^s(\mathbb{R}^n)\) for all \(s \in (0, 1)\), \(1 < p < \infty\) and

\[\|f\|_{W^s_p(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)} + \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^\theta p + n} d(x, y) \right)^{\frac{1}{p}}\]

is an equivalent norm on \(B^{\theta}_p(\mathbb{R}^n)\). A more general statement of this kind will be given below.

For the complex interpolation method we have:

**Theorem 4.20** Let \(1 \leq p_0, p_1, q_0, q_1 < \infty\), \(0 < \theta < 1\), and let \((U, \mu), (V, \nu)\) be two measure spaces. Moreover, let \(\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}\), \(\frac{1}{q} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}\). Then

\[(L^{p_0}(U; L^{q_0}(V)), L^{p_1}(U; L^{q_1}(V)))_{\theta,p} = L^{p}(U; L^{q}(V))\]

with equal norms.

**Remark 4.21** If \((X_0, X_1)\) is an admissible pair of Banach spaces and \(1 \leq p_0, p_1 < \infty\), \(0 < \theta < 1\), then

\[(L^{p_0}(U; X_0), L^{p_1}(U; X_1))_{\theta,p} = L^{p}(U; (X_0, X_1)_{\theta,p})\]

with equal norms, where \(\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}\). cf. [1, Theorem 5.1.2].

**Corollary 4.22** Let \(s_0, s_1 \in \mathbb{R}\), \(0 < \theta < 1\), and let \(1 \leq q_0, q_1, p_0, p_1 < \infty\). Moreover, let \(\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}\), \(\frac{1}{q} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}\). Then

\[\ell^{\theta}_{q_0}(N_0; L^{p_0}(\mathbb{R}^n)), \ell^{s_1}_{q_1}(N_0; L^{p_1}(\mathbb{R}^n)))_{\theta,q} = \ell^{s}_q(N_0; L^{p}(\mathbb{R}^n))\]

\[L^{p_0}(\mathbb{R}^n; \ell^{q_0}_{q_0}(N_0)), L^{p_1}(\mathbb{R}^n; \ell^{q_1}_{q_1}(N_0)))_{\theta,q} = L^{p}(\mathbb{R}^n; \ell^{s}_q(N_0))\]

with equal norms, where \(s = (1 - \theta)s_0 + \theta s_1\).

As an application we obtain:

40
THEOREM 4.23 Let \( s_0, s_1 \in \mathbb{R}, \) \( 0 < \theta < 1 \) and let \( s = (1 - \theta)s_0 + \theta s_1. \) Then
\[
(H^{s_0}_{p_0}(\mathbb{R}^n), H^{s_1}_{p_1}(\mathbb{R}^n))_{[\theta]} = H^{s}_{p}(\mathbb{R}^n)
\]
with equal norms for any \( 1 < p_0, p_1 < \infty \) and \( \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \) Moreover, for any \( 1 \leq q_0, q_1, p_0, p_1 < \infty \) we have
\[
(B^{s_0}_{p_0q_0}(\mathbb{R}^n), B^{s_1}_{p_1q_1}(\mathbb{R}^n))_{[\theta]} = B^{s}_{pq}(\mathbb{R}^n)
\]
with equal norms where \( \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \) and \( \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \)

4.5 Sobolev Embeddings and Traces

We start with a Sobolev-type embedding theorem for Besov and Bessel potential spaces.

THEOREM 4.24 Let \( s, s_1 \in \mathbb{R} \) with \( s \leq s_1 \) and \( 1 \leq p_1 \leq p \leq \infty \) such that
\[
s - \frac{n}{p} \leq s_1 - \frac{n}{p_1}.
\]
Then
\[
B^{s_0}_{p_0q_1}(\mathbb{R}^n) \hookrightarrow B^{s}_q(\mathbb{R}^n) \quad \text{for all} \quad 1 \leq q_1 \leq q \leq \infty, \quad (4.27)
\]
\[
H^{s_0}_{p_1}(\mathbb{R}^n) \hookrightarrow H^{s}_p(\mathbb{R}^n). \quad \text{if} \quad 1 < p_1 \leq p < \infty \quad (4.28)
\]

Proof: See [1, Theorem 6.51].

Remarks on the proof: The embedding (4.27) follows from:
\[
\varphi_k(D_x)f = \tilde{\varphi}_k(D_x)\varphi_k(D_x)f
\]
with \( \tilde{\varphi}_k(D_x) = \varphi_{k-1}(D_x) + \varphi_k(D_x) + \varphi_{k+1}(D_x), \varphi_{-1}(D_x) := 0 \) as well as
\[
\varphi_k(D_x)g = \psi_{2^{-k}} * g,
\]
which implies
\[
\|\varphi_k(D_x)g\|_{L^p(\mathbb{R}^n)} \leq \|\psi_{2^{-k}}\|_{L^q(\mathbb{R}^n)}\|g\|_{L^{p_1}(\mathbb{R}^n)} \leq C2^{-kn} \|g\|_{L^{p_1}(\mathbb{R}^n)},
\]
where \( \frac{1}{p} = 1 - \frac{1}{q} = \frac{1}{p_1} - \frac{1}{p}. \) Then (4.28) follows from (4.27) by a clever interpolation.
THEOREM 4.25 Let $1 \leq p < \infty$, $1 \leq q \leq \infty$, $s > \frac{1}{p}$ and let $\text{Tr} \ f = f|_{x_n=0}$ for any continuous $f \in H^s_p(\mathbb{R}^n) \cup B^s_{pq}(\mathbb{R}^n)$. Then $\text{Tr}$ can be extended to a bounded linear operator

\[ \text{Tr}: B^s_{pq}(\mathbb{R}^n) \to B^{s-\frac{1}{p}}_{pq}(\mathbb{R}^n), \]
\[ \text{Tr}: H^s_p(\mathbb{R}^n) \to B^{s-\frac{1}{p}}_{pp}(\mathbb{R}^n). \]

**Proof:** See [1].

**Remarks on the proof:** If $m = 1$, then

\[ \text{Tr}: H^m_p(\mathbb{R}^n) = W^m_p(\mathbb{R}^n) \to W^{m-\frac{1}{p}}(\mathbb{R}^n-1) = B^{m-\frac{1}{p}}_{pq}(\mathbb{R}^n-1) \]

follows from the trace method as before. Using $a \in B^{m-\frac{1}{p}}_{pq}(\mathbb{R}^n-1)$ if and only if $\partial_{\alpha} a \in B^{1-\frac{1}{p}}_{pq}(\mathbb{R}^n-1)$ for all $|\alpha| \leq m-1$, the same follows for general $m \in \mathbb{N}$. Then the statement follows for $s \geq 1$ by interpolation.

In the case $\frac{1}{p} < s \leq 1$ one uses that $B^{\frac{1}{p}}_{p1}(\mathbb{R}) \hookrightarrow C^0(\mathbb{R})$. Therefore

\[ |\text{Tr} \ f(x')| \leq C\|f(x',.)\|_{B^{\frac{1}{p}}_{p1}(\mathbb{R})}, \]
\[ \Rightarrow \|\text{Tr} \ f\|_{LP(\mathbb{R}^{n-1})} \leq C\|f\|_{Lp(\mathbb{R}^{n-1};B^{\frac{1}{p}}_{p1}(\mathbb{R}))} \leq C\|f\|_{B^{\frac{1}{p}}_{p1}(\mathbb{R}^{n})}. \]

An iterated interpolation then yields the statement of the theorem for $\frac{1}{p} < s < 1$.

4.6 Equivalent Norms

The following theorem is the direct generalization of Theorem 4.2 for general Besov spaces $B^s_{pq}(\mathbb{R}^n)$ with $0 < s < 1$.

**THEOREM 4.26** Let $0 < s < 1$ and let $1 \leq p, q \leq \infty$. Then there are constants $c, C$ (depending on $s, p, q$) such that

\[ c\|f\|_{B^s_{pq}(\mathbb{R}^n)} \leq \|f\|_{LP(\mathbb{R}^n)} + \left( \int_0^\infty \frac{\omega_p(t;f)^q}{t^{sq}} \frac{dt}{t} \right)^{\frac{1}{q}} \leq C\|f\|_{B^s_{pq}(\mathbb{R}^n)} \quad (4.29) \]

if $q < \infty$ and

\[ c\|f\|_{B^s_{pq}(\mathbb{R}^n)} \leq \|f\|_{LP(\mathbb{R}^n)} + \sup_{t>0} \frac{\omega_p(t;f)^q}{t^{sq}} \frac{dt}{t} \leq C\|f\|_{B^s_{pq}(\mathbb{R}^n)} \quad (4.30) \]
if \( q = \infty \), where

\[
\omega_p(t; f) = \sup_{|h| \leq t} \| f(\cdot + h) - f \|_{L^p(\mathbb{R}^n)}
\]

is the \( L^p \)-modulus of continuity of \( f \).

**Remark 4.27** We refer to [1, Theorem 6.2.5] for a more general statement in the case \( s > 0 \).

**Proof of Theorem 4.26:** We will only prove the case \( q < \infty \) since the proof in the case \( q = \infty \) is a simple variant of the proof of Theorem 4.2.

First of all, since \( t \mapsto \omega_p(t; f) \) is a monotone increasing function and \( t \) is proportional to \( 2^{-j} \) on \([2^{-j+1}, 2^{-j}]\),

\[
\int_{0}^{1} \frac{\omega_p(t; f)^q}{t^q} \frac{dt}{t} \leq C \sum_{j=0}^{\infty} 2^{sjq} \omega_p(2^{-j}; f)^q
\]

and

\[
\sum_{j=0}^{\infty} 2^{sjq} \omega_p(2^{-j}; f)^q \leq C \int_{0}^{2} \frac{\omega_p(t; f)^q}{t^q} \frac{dt}{t}.
\]

Moreover,

\[
\int_{1}^{\infty} \frac{\omega_p(t; f)^q}{t^q} \frac{dt}{t} \leq 2^q \| f \|_{L^p(\mathbb{R}^n)}^q \int_{1}^{\infty} t^{-s-1} \frac{dt}{t} = C \| f \|_{L^p(\mathbb{R}^n)}^q
\]

since \( \omega_p(t; f) \leq 2 \| f \|_{L^p(\mathbb{R}^n)} \). Hence we can replace the middle term in (4.29) by

\[
\| f \|_{L^p(\mathbb{R}^n)} + \left( \sum_{j=0}^{\infty} 2^{sjq} \omega_p(2^{-j}; f) \right)^{\frac{1}{q}}.
\]

First we prove the second inequality in (4.29). For \( f \in B_{pq}^s(\mathbb{R}^n) \) we denote \( f_k = \varphi_k(D_x) f \). Then

\[
\| f_k(\cdot + h) - f_k \|_{L^p(\mathbb{R}^n)} \leq |h| \| \nabla f_k \|_{L^p(\mathbb{R}^n)}
\]

due to (4.11) and therefore

\[
\omega_p(t; f_k) \leq t \| \nabla f_k \|_{L^p(\mathbb{R}^n)} = t \| \nabla \varphi_k(D_x) \tilde{\varphi}_k(D_x) f_k \|_{L^p(\mathbb{R}^n)} \leq Ct^{2^k} \| \tilde{\varphi}_k(D_x) f_k \|_{L^p(\mathbb{R}^n)}
\]

where
because of (4.13), where \( \tilde{\varphi}_k(D_x) = \varphi_{k-1}(D_x) + \varphi_k(D_x) + \varphi_{k+1}(D_x), \) \( k \in \mathbb{N}_0, \) and \( \varphi_{-1}(D_x) = 0. \) On the other hand, \( \omega_p(t, f_k) \leq 2\|f_k\|_{L^p(\mathbb{R}^n)} \) and \( f = \sum_{k=0}^\infty f_k. \) Therefore

\[
2^{j} \omega_p(2^{-j}; f) \leq C \left( \sum_{j=0}^\infty 2^{sj} \min(1, 2^{-j+k}) \|\tilde{\varphi}_k(D_x) f_k\|_{L^p(\mathbb{R}^n)} \right)
\]

\[
\leq C \left( \sum_{j=0}^\infty 2^{(j-k)} \min(1, 2^{-j+k}) 2^{sk} \|\varphi_k(D_x) f_k\|_{L^p(\mathbb{R}^n)} \right).
\]

Now, defining \( a_j = C 2^{sj} \min(1, 2^{-j}), \) \( j \in \mathbb{Z}, b_j = 2^{sj} \|\varphi_k(D_x) f_j\|_{L^p(\mathbb{R}^n)} \) if \( j \geq 0 \) and \( b_j = 0 \) else, we see that \( 2^{sj} \omega_p(2^{-j}; f) \) \( \leq (a * b)_j, \) where

\[
(a * b)_j = \sum_{k \in \mathbb{Z}} a_{j-k} b_k
\]

is the convolution of two sequences. Hence

\[
\left( \sum_{j=0}^\infty 2^{sj} \omega_p(2^{-j}; f) \right)^\frac{1}{q} \leq \|a * b\|_{\ell^q(\mathbb{Z})} \leq \|a\|_{\ell^1(\mathbb{Z})}\|b\|_{\ell^q(\mathbb{Z})} \leq C \|f\|_{B_{p,q}(\mathbb{R}^n)},
\]

where \( a \in \ell^1(\mathbb{Z}) \) since \( s \in (0,1). \) Here we have used the discrete version of Young’s inequality \( \|a * b\|_{\ell^r} \leq \|a\|_{\ell^t}\|b\|_{\ell^r}, \) which can be proved in the same way as for the usual convolution using Hölder’s inequality.

In order to prove the first inequality in (4.29), we use that

\[
\varphi_j(D_x)f = \int_{\mathbb{R}^n} (f(x - 2^{-j}z) - f(x)) \psi(z) \, dz,
\]

cf. (4.10). Therefore

\[
\|\varphi_j(D_x)f\|_{L^p(\mathbb{R}^n)} \leq \int_{\mathbb{R}^n} |f(. - 2^{-j}z) - f|_{L^p(\mathbb{R}^n)} |\psi(z)| \, dz
\]

\[
\leq \int_{\mathbb{R}^n} \omega_p(2^{-j}|z|; f) |\psi(z)| \, dz.
\]
\[
\left( \sum_{j=1}^{\infty} 2^{sjq} \| \varphi_j (D_x) f \|_{L^p(\mathbb{R}^n)}^q \right)^{\frac{1}{q}} \\
\leq \int_{\mathbb{R}^n} \left( \sum_{j=1}^{\infty} 2^{sjq} \omega_p(2^{-j} |z|; f)^q \right)^{\frac{1}{q}} |\psi(z)| \, dz \\
\leq C \int_{\mathbb{R}^n} \left( \int_{0}^{\infty} \frac{\omega_p(t|z|; f)^q \, dt}{t^{sq} / t} \right)^{\frac{1}{q}} |z|^s |\psi(z)| \, dz \\
= C \left( \int_{0}^{\infty} \frac{\omega_p(t; f)^q \, dt}{t^{sq} / t} \right)^{\frac{1}{q}} \int_{\mathbb{R}^n} |z|^s |\psi(z)| \, dz,
\]

where we can estimate \( \sum_{j=0}^{\infty} 2^{sjq} \omega_p(2^{-j} |z|; f)^q \) by the corresponding integral by the same arguments as in the beginning of the proof. Finally, \( \| \varphi_0 (D_x) f \|_{L^p(\mathbb{R}^n)} \leq C \| f \|_{L^p(\mathbb{R}^n)} \), which finishes the proof. \[\blacksquare\]