A Variational Problem in Conformal Spin Geometry

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Contents

Abstract

Let us fix a conformal class $[g_0]$ and a spin structure σ on a compact manifold M. For any $g \in [g_0]$, let $\lambda_1^+(g)$ be the smallest positive eigenvalue of the Dirac operator D on (M, g, σ). In a previous paper we have shown that

$$\lambda_{\min}^+(M, [g_0], \sigma) := \inf_{g \in [g_0]} \lambda_1^+(g) \mathrm{vol}(M, g)^{1/n} > 0.$$

In this habilitation, we enlarge the conformal class by certain singular metrics. We will show that if $\lambda_{\min}^+(M, [g_0], \sigma) < \lambda_{\min}^+(S^n)$, then the infimum is attained on the enlarged conformal class. By proving this, we will obtain solutions to an elliptic system of semi-linear partial differential equations involving a nonlinearity with critical exponent:

$$D\varphi = \lambda_{\min}^+ |\varphi|^{2/(n-1)} \varphi.$$

The solution of this problem has many analogies to the solution of the Yamabe problem. However, our reasoning is more involved than in the Yamabe problem as the spectrum of the Dirac operator is unbounded from below and unbounded from above. The solution may have a nonempty zero set because a maximum principle is not available.

Using the spinorial Weierstrass representation, the solution of this equation in dimension 2 provides a tool for the construction of periodic constant mean curvature surfaces in R^3 and S^3 with compact fundamental domain.

Chapter 1

Introduction

1.1 Summary

Let (M^n, g, σ), $n \geq 2$, be a Riemannian spin manifold with a fixed spin structure σ. Let D be the classical Dirac operator on (M, g, σ). In this habilitation we will study the functional

$$\mathcal{F}_q^M(\psi) = \frac{\int \langle D^{-1}\psi, \psi\rangle}{\|\psi\|_{L^q}^2}, \tag{1.1.1}$$

defined on the nonzero elements in the H_1^2-closure of the image of D.

If $q \geq 2n/(n+1)$, then the functional is bounded from above. We will investigate whether the supremum is attained. The case $q = 2$ is very simple: $\sup \mathcal{F}_2^M$ is the inverse of the first positive eigenvalue of D, and the supremum is attained exactly by the eigenspinors to this eigenvalue.

If $q > 2n/(n+1)$, $q \neq 2$, then we obtain a nonlinear eigenvalue problem instead of a linear one. The supremum is still attained and the supremum satisfies the Euler-Lagrange equation

$$D^{-1}\psi = \mu \frac{\psi}{|\psi|^{2-q}}. \tag{1.1.2}$$

One sees that ψ is $C^{0,\alpha}$ and even C^∞ away from the zero set.

However, the case $q = 2n/(n+1)$ is much more involved. We have to solve a nonlinear partial differential equation with a nonlinearity which is often called "critical", as the coefficients in the Sobolev embeddings involved are critical. One has to expect that this problem is more involved. In order to get an idea why this is more complicated, Let us have a look at the spheres S^n with constant sectional curvature. Let G be the conformal group of S^n. There is an action of G on the spinor bundle ΣS^n, such that $g \in G$ maps $\Sigma_p S^n$ to $\Sigma_{gp} S^n$ and such that $\mathcal{F}_{2n/(n+1)}^M$ is invariant under this action. Hence, we have a non-compact symmetry group of the problem, and the suplevel sets of $\mathcal{F}^M_{2n/(n+1)}$ are non-compact. As a consequence, an arbitrary maximizing sequence may not converge. In order to achieve convergence we choose a sequence of maximizers ψ_k for $\mathcal{F}^M_{q_k}$, with $q_k \to 2n/(n+1)$, $q_k > 2n/(n+1)$. One shows that this is indeed a maximizing sequence for $\mathcal{F}^M_{2n/(n+1)}$. If the following *spectral condition*

$$\sup \mathcal{F}^M_{2n/(n+1)} > \sup \mathcal{F}^{S^n}_{2n/(n+1)} \tag{1.1.3}$$

holds, then the sequence ψ_k converges to a spinor ψ_∞, and $\mathcal{F}^M_{2n/(n+1)}$ attains its supremum in this spinor.

The solution of our problem has many analogies to Trudinger's and Aubin's solution of the Yamabe problem (see [LP87] for more details). Trudinger [Tru68] and Aubin [Aub76] showed that the infimum of the Yamabe functional is attained if a spectral condition comparable to (1.1.3) is satisfied. Later [Aub76, Sch84, SY79a, SY79b, SY81, SY88, Wit81, PT82], Aubin, Schoen, Yau and Witten proved that the spectral condition is satisfied if M is not conformal to a sphere of constant sectional curvature. It turns out that the difficult cases follow from the positive mass theorem of general relativity. As a consequence any compact Riemannian manifold of dimension ≥ 3 is conformal to one of constant scalar curvature.

However, one should emphasize that several arguments in the Yamabe problem cannot be taken over. One of the difficulties arises from the fact that the spectrum of the Dirac operator accumulates both in $+\infty$ and $-\infty$. Furthermore, we cannot apply the maximum principle as our equations do no longer have scalar values, but values in the spinor bundle.

Sufficient criteria for the spectral condition (1.1.3) are subject of on-going research. The spheres are the only known examples which do not satisfy the spectral condition, and in analogy to the Yamabe problem, one can conjecture, that the spectral condition holds if M is not conformal to a sphere of constant sectional curvature.

The relations of our problem to the Yamabe problem are much closer than on the level described until now. The Hijazi inequality implies that any manifold fulfilling our spectral condition (1.1.3) also fulfills the spectral condition of the Yamabe problem.

The Euler-Lagrange equation satisfied by the maximizers are conformally invariant, nonlinear partial differential equations. We know of two applications for these solutions.

The first application comes from spectral theory. Let us fix a conformal class $[g_0]$ and a spin structure σ on a compact manifold M. For any $g \in [g_0]$, let $\lambda_1^+(g)$ be the smallest positive eigenvalue of the Dirac operator D on (M, g, σ). In Chapter 2 we define

$$\lambda^+_{\min}(M,[g_0],\sigma) := \inf_{g\in[g_0]} \lambda^+_1(g)\mathrm{vol}(M,g)^{1/n}.$$

One sees that $\lambda^+_{\min}(M,[g_0],\sigma) > 0$. We will enlarge the conformal class by certain singular metrics. The spectral condition (1.1.3) translates into $\lambda^+_{\min}(M,[g_0],\sigma) < \lambda^+_{\min}(S^n)$. As a consequence the infimum is attained on the enlarged conformal class if (1.1.3) holds.

Another application (for $\dim M = 2$) lies in surface theory. Using the spinorial version of the Weierstraß representation, the solutions of the Euler-Lagrange equations in dimension 2 provide periodic immersed surfaces of constant mean curvature in $\mathbb{R}^3$ and S^3.

The various dependencies of our statements and of related statements for the Yamabe problem are shown in Figure 1.1.

Let us sketch the structure of the habilitation.

In Chapter 2 *(Classical results in conformal spin geometry)* we summarize several facts that have been known prior to the habilitation.

In Chapter 3 *(Elliptic regularity for Dirac operators)* we develop Global and Interior Schauder- and L^p-regularity-theory for generalized Dirac operators. Most of the results in Sections 3.1 and 3.2 are analogous to results in [GT77]. In order to present a logically complete derivation of the regularity theory we had to reprove many statements. This is partially due to the fact that we need uniform control over our regularity statements. In order to keep the analogy to [GT77] transparent we took over most of the notation in [GT77]. In order to keep our presentation as short as possible, we cited statements of [GT77] and [Ada75] whenever possible. In Section 3.3 we prove various embedding theorems. For proving the different version that we will need, it is convenient to use different methods. E.g. for showing the boundedness of $L^{2n/n+1} \to H^2_{-1/2}$ we study the asymptotics of a Green's function and use the Hardy-Littlewood-Sobolev inequality.

Chapter 4 *(The first Dirac eigenvalue in a conformal class)* is the central part of the habilitation. It can be read independently from Chapter 3 if one believes that the statements about elliptic regularity from Chapter 3 hold. We show that the supremum of $\mathcal{F}^M_q$ is attained as described above. The application to the first positive eigenvalue of D is also contained in this chapter. Nearly all material in this chapter is new. The content of this chapter will be published in [Amm03a].

The principal goal of Chapter 5 *(Spinorial Weierstrass representations of surfaces)* is to describe how to apply our results to surface theory. We start with a short overview. Then, in Section 5.2, we turn to the study of manifolds with Killing spinors as they play a central role in the spinorial Weierstrass representation. We

Logical structure of the habilitation

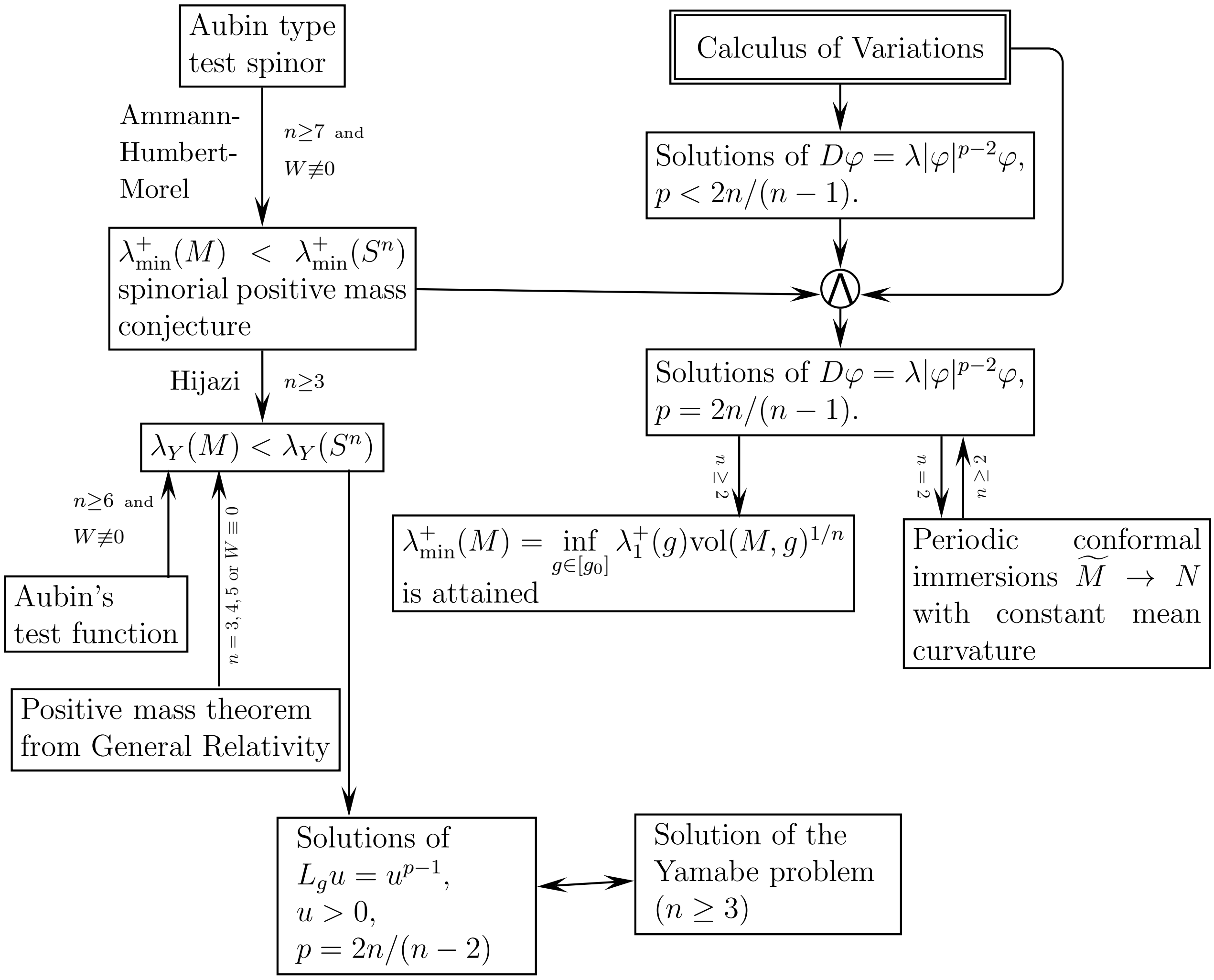

Figure 1.1: Logical dependencies. In this diagram, (M,g_0) is an n-dimensional Riemannian manifold, and N an $n+1$-dimensional Riemannian manifold which carries a fixed real Killing spinor.

investigate Killing spinors on S^3 and $\mathbb{R}^3$. In particular, we give a list of all 3-manifolds carrying a real Killing spinor. After this, in Subsection 5.3.1, we give a short overview over literature related to the spinorial Weierstrass representation. Then, we recall the restriction of Killing spinors to hypersurfaces. This knowledge can then be used to describe how we obtain new cmc-surfaces. We list some open problems that we consider as interesting for future research. The final section is devoted to visualizations of some examples of our construction.

1.2 Acknowledgments

My special thanks go to Prof. Christian Bär. I profited very much from his huge knowledge on Dirac operators, from his beautiful lectures and from many encouraging discussions with him. His generous support gave the necessary freedom to me to work at this habilitation thesis.

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Chapter 2

Classical results in conformal spin geometry

The aim of this chapter is to summarize several classical facts about Dirac operators in conformal geometry, which will play an important role in this paper. These classical results will be enriched by new interpretations and some extra material.

2.1 Some basic definitions

In this section we will recall some important definitions from spin geometry in order to fix notations. For more details we refer to the standard textbooks [LM89, BGV91, Fri00, Roe88] and some lecture notes [Hij01, Bär95, Amm].

Let (M, g) be an n-dimensional Riemannian manifold with a chosen orientation. Let $P_{\mathrm{SO}}(M, g)$ be the set of positively oriented orthonormal frames on (M, g). This is a $\mathrm{SO}(n)$-principal bundle over M. A *spin structure* on (M, g) is a pair $\sigma = (P_{\mathrm{Spin}}(M, g), \vartheta)$ where $P_{\mathrm{Spin}}(M, g)$ is a $\mathrm{Spin}(n)$-principal bundle over M and $\vartheta : P_{\mathrm{Spin}}(M, g) \to P_{\mathrm{SO}}(M, g)$ is a map such that

$$\begin{tikzcd}
P_{\mathrm{Spin}}(M,g) \times \mathrm{Spin}(n) \arrow[r] \arrow[d, "\vartheta \times \Theta"] & P_{\mathrm{Spin}}(M,g) \arrow[d, "\vartheta"] \arrow[dr] & \\
& & M \\
P_{\mathrm{SO}}(M,g) \times \mathrm{SO}(n) \arrow[r] & P_{\mathrm{SO}}(M,g) \arrow[ur] &
\end{tikzcd}$$

commutes where $\Theta : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is the nontrivial double covering of $\mathrm{SO}(n)$.

Definition. A *Riemannian spin manifold* is a Riemannian manifold together with a spin structure.

Note that many authors use the term "spin" in the sense that a spin structure merely exist, whereas we will assume that it is chosen. A spin structure exists if and only if the second Stiefel-Whitney class vanishes.

Now, let $\kappa : \mathrm{Spin}(n) \to \mathrm{End}(\Sigma_n)$ be the n-dimensional complex spinor representation. The space Σ_n is a complex vector space of dimension $2^{[n/2]}$. This representation extends to a representation of the n-dimensional Clifford algebra $\mathbb{C}\mathrm{l}(n)$.

The *spinor bundle* is defined as the associated vector bundle

$$\Sigma M = \Sigma(M, g, \sigma) := P_{\mathrm{Spin}}(M, g) \times_\kappa \Sigma_n.$$

Recall that the spinor bundle carries a natural Clifford multiplication, a natural hermitian metric and a metric connection. This bundle equipped with the Clifford multiplication, this metric and connection satisfies the axioms of a *Dirac bundle.* For the convenience of the reader, let us recall these axioms.

Definition. A *Dirac bundle* S over a Riemannian manifold (M, g) is a complex vector bundle $S \to M$ together with a hermitian metric $\langle \cdot, \cdot \rangle$, a connection ∇^S and a Clifford multiplication $TM \otimes S \to S, X \otimes \varphi \mapsto X \cdot \varphi$ such that

(i) For any $p \in M$, $X, Y \in T_pM$ and $\varphi \in S_pM$

$$X \cdot Y \cdot \varphi + Y \cdot X \cdot \varphi + 2g(X, Y)\,\varphi = 0$$

(ii) $\langle X \cdot \varphi_1, \varphi_2 \rangle = -\langle \varphi_1, X \cdot \varphi_2 \rangle \qquad \forall X \in T_pM \quad \forall \varphi_1, \varphi_2 \in S_p$,

(iii) ∇^S is metric, i.e. $\partial_X \langle \varphi_1, \varphi_2 \rangle = \langle \nabla^S_X \varphi_1, \varphi_2 \rangle + \langle \varphi_1, \nabla^S_X \varphi_2 \rangle$.

(iv) Clifford multiplication is parallel, i.e. for any $X \in \Gamma(TM)$, any $Y \in \Gamma(TM)$ and any $\varphi \in \Gamma(S)$,

$$\nabla^S_X (Y \cdot \varphi) = (\nabla_X Y) \cdot \varphi + X \cdot \nabla^S_X \varphi.$$

On a Dirac bundle S we define the *generalized Dirac-Operator* D^S as the composition

$$\Gamma(S) \xrightarrow{\nabla^S} \Gamma(T^*M \otimes S) \to \Gamma(TM \otimes S) \stackrel{Cliff}{\to} \Gamma(S).$$

In the special case that S is the spinor bundle, the generalized Dirac operator is called the *Atiyah-Singer operator*, the *classical Dirac operator* or if there is no danger of confusion just the *Dirac operator.*

The main results of this habilitation in Chapters 4 and 5 are only valid for classical Dirac operators. The results on elliptic regularity in Chapter 3 also hold for generalized Dirac operators. There, we will use them to obtain shorter proofs. We use generalized Dirac operators to reduce the higher order statements to lower order ones.

2.2 The transformation formula

The classical Dirac operator behaves very nicely under conformal changes.

Proposition 2.2.1 ([Hit74, Hij86b]). *Let g_0 and $g = e^{2u}g_0$ be two conformal metrics on a Riemannian spin manifold. Then, there is an isomorphism of vector bundles $A : \Sigma(M, g_0, \sigma) \to \Sigma(M, g, \sigma)$ which is a fiber-wise isometry such that*

$$D_g(A(\varphi)) = A\left(e^{-\frac{n+1}{2}u} D_{g_0} e^{\frac{n-1}{2}u}\varphi\right).$$

To the knowledge of the author, the first mathematician who calculated this formula was Hitchin [Hit74] when he was studying the dimension of the kernel of the Dirac operator on compact spin manifolds. Hitchin's transformation formula directly implies the following corollary.

Corollary 2.2.2. *Let M be a compact spin manifold. Then the dimension of the kernel of the Dirac operator is invariant under conformal changes of the metric, i.e. it only depends on the conformal class of the metric and of the spin structure.*

However, he also could show that in general the dimension of the kernel of the Dirac operator is not a topological invariant, it depends on the differentiable structure, on the conformal class and on the spin structure [Hit74, Bär97].

2.3 Conformal spinor bundles with weights

In this section we will recall a definition of the spinor bundle in a way which is conformally invariant. This construction depends on a real number $\alpha \in \mathbb{R}$ which is called the *conformal weight* of the spinors. The Dirac operator is then a well-defined operator from spinors of weight $-(n-1)/2$ to spinors of weight $-(n+1)/2$. This section is not needed for a logically complete proof of our main statements, but the point of view presented here is very helpful to understand the idea behind many constructions in this habilitation. We refer to [Feg76], [Hit80] and [Gau91] for further details. We want to remark here that in the literature there is no consensus about the sign of the conformal weights.

Let M be an n-dimensional oriented manifold with a fixed conformal class. The conformal class will always be written as $[g_0]$, where g_0 is a Riemannian metric in this conformal class.

Let U be a subset of M. A *conformal frame* on U is an oriented frame $(e_1, \ldots, e_n)$ on U such that there is a smooth function $f : U \to \mathbb{R}^+$ with

$$g_0(e_i, e_j) = f\delta_{ij}.$$

The *conformal frame bundle* $P_{\mathrm{CSO}}(M,[g_0])$ is defined as the bundle over M whose fiber over p consists of all frames on $\{p\}$. The *conformal group* $\mathrm{CSO}(n)$ is $\mathrm{SO}(n)\times\mathbb{R}^+$. We define the right action of $\mathrm{CSO}(n)$ on $P_{\mathrm{CSO}}(M,[g_0])$ such that $(a_{ij})\in\mathrm{SO}(n)$ maps $(e_i)_i$ to $(\sum e_i a_{ij})_j$ and λ maps $(e_i)_i$ to $(\lambda e_i)_i$. The double cover $\Theta:\mathrm{Spin}(n)\to\mathrm{SO}(n)$ extends to a double cover

$$\Theta_C:\mathrm{CSpin}(n):=\mathrm{Spin}(n)\times\mathbb{R}^+\stackrel{\Theta\times\mathrm{id}}{\longrightarrow}\mathrm{CSO}(n).$$

A *conformal spin structure* is given by a $\mathrm{CSpin}(n)$-principal bundle $P_{\mathrm{CSpin}}(M,[g_0])$ together with a map $\vartheta:P_{\mathrm{CSpin}}(M,[g_0])\to P_{\mathrm{CSO}}(M,[g_0])$ such that the diagram

$$\begin{tikzcd}
P_{\mathrm{CSpin}}(M,[g_0])\times\mathrm{CSpin}(n) \arrow[r] \arrow[d, "\vartheta\times\Theta_C"] & P_{\mathrm{CSpin}}(M,[g_0]) \arrow[d, "\vartheta"] \arrow[dr] & \\
P_{\mathrm{CSO}}(M,[g_0])\times\mathrm{CSO}(n) \arrow[r] & P_{\mathrm{CSO}}(M,[g_0]) \arrow[r] & M
\end{tikzcd}$$

commutes.

Note that a conformal spin structure exists iff a spin structure in the sense of Riemannian geometry exists, and choosing a conformal spin structure is equivalent to choosing a spin structure in the sense of Riemannian geometry. We will identify spin structures and conformal spin structures from now on.

Let G be $\mathrm{Spin}(n)$ or $\mathrm{SO}(n)$, and let $\rho:G\to\mathrm{GL}(V)$ be a representation of G. Then the *associated representation $\rho*\alpha$ of conformal weight α*, $\alpha\in\mathbb{R}$ is given by

$$\rho*\alpha:G\times\mathbb{R}^+\to\mathrm{GL}(V),\quad(\rho*\alpha)(A)=\rho(A)\ \forall A\in G,\quad(\rho*\alpha)(\lambda)=\lambda^\alpha\mathrm{Id}_V\ \forall\lambda\in\mathbb{R}^+.$$

The associated vector bundle

$$V_{\rho*\alpha}:=P_{\mathrm{CSpin}}(M,[g_0])\times_{\rho*\alpha}V\text{ or }V_{\rho*\alpha}:=P_{\mathrm{CSO}}(M,[g_0])\times_{\rho*\alpha}V$$

is also said to be *of conformal weight α.*

Examples.

(1) Let $\rho:\mathrm{SO}(n)\to\mathrm{GL}(n,\mathbb{R})$ be the standard inclusion. The map $P_{\mathrm{CSO}}(M)\times\mathbb{R}^n\to TM$, $((e_i)_i,(a_j)_j)\mapsto\sum_i e_i a_i$ is $\mathrm{CSO}(n)$-equivariant and hence induces an isomorphism from $P_{\mathrm{CSO}}(M)\times_{\rho*1}\mathbb{R}^n$ to TM. As a consequence TM has conformal weight 1.

(2) $P_{\mathrm{CSO}}(M)\times_{\rho*0}\mathbb{R}^n$ is a vector bundle which is isomorphic to the tangent bundle. However, there is no "natural" isomorphism which is independent from the choice of metric $g\in[g_0]$.

(3) Λ^pT^*M has conformal weight $-p$.

(4) $\Lambda^p TM$ has conformal weight p.

(5) A Riemannian metric is a symmetric element in $T^*M \otimes T^*M$. Hence, it has conformal weight -2.

(6) We define the conformal spinor bundle of weight α as

$$\Sigma^\alpha M := P_{\mathrm{CSpin}}(M, [g_0]) \times_{\kappa * \alpha} \Sigma_n$$

where κ is the spinor representation. This bundle has conformal weight α. A section of $\Sigma^\alpha M$ is called a *spinor of weight* α.

As M is oriented, $\Lambda^n T^*M$ is a topologically trivial bundle of weight $-n$. A trivializing section is given by the volume form dvol_g where $g \in [g_0]$ is a Riemannian metric. Then tensoring with $(\mathrm{dvol}_g)^{(\alpha-\beta)/n}$ yields a vector bundle isomorphism $V_{\rho*\alpha} \to V_{\rho*\beta}$. A choice of metric also identifies conformal spinor bundles with the spinor bundle that we defined in the previous section.

The following proposition shows that in conformal spin geometry the Dirac operator is only well-defined from spinors of weight $-(n-1)/2$ to spinors of weight $-(n+1)/2$.

PROPOSITION 2.3.1. *Assume that $(M, [g])$ is an oriented conformal manifold which carries a fixed spin structure.*
(a) *If $\alpha = -(n-1)/2$ and $\beta = -(n+1)/2$, then there is a differential operator*

$$D : \Sigma^\alpha M \to \Sigma^\beta M$$

with the following property: If we choose any metric $g \in [g_0]$ and identify as above then D coincides with the classical Dirac operator on (M, g) with respect to the fixed spin structure.
(b) *If $\alpha \neq -(n-1)/2$ or if $\beta \neq -(n+1)/2$, then there is no such D.*

The proof is an immediate consequence of the transformation formula in Proposition 2.2.1.

Similarly as in Riemannian geometry, there is also a scalar product on spinors in conformal geometry. If φ is a spinor of weight α and ψ is a spinor of weight β, then $\langle \varphi, \psi \rangle$ is a complex-valued function of weight $\alpha + \beta$.

In conformal geometry one can integrate a function if and only it has weight $-n$.

We conclude this section with some examples:

Examples.
(1) If φ is a compactly supported smooth spinor of weight $-(n-1)/2$, then

$$\int_M \langle D\varphi, \varphi \rangle$$

is well-defined in conformal geometry.

(2) If φ is a compactly supported continuous spinor of weight $\alpha \in (-n, 0)$, then

$$\int |\varphi|^p$$

is well-defined if and only if $p\alpha = -n$. Obviously, the same holds for the L^p-norm.

These examples will be very important for us. The fact that these integrals are well-defined in the conformal setting means in Riemannian geometry, that if one uses the "good" conformal identification for spinors, these integrals are invariant under conformal changes.

2.4 A spin-conformal lower bound in a conformal class

Let M be a compact manifold, on which we fix a conformal class $[g_0]$ and a spin structure σ. For each metric $g \in [g_0]$ let $\lambda_1^+(g)$ be the smallest positive eigenvalue of the (classical) Dirac operator D on (M, g, σ). Note that the dimension of the kernel of the Dirac operator is a conformal invariant.

We define

$$\lambda^+_{\min}(M, [g_0], \sigma) := \inf_{g\in[g_0]} \lambda_1^+(g) \operatorname{vol}(M, g)^{1/n}. \tag{2.4.1}$$

In the following, we call $\lambda^+_{\min}(M, [g_0], \sigma)$ the *Bär-Hijazi-Lott invariant*.

THEOREM 2.4.2 (J. Lott, [Lot86]). *If D is invertible, then*

$$\lambda^+_{\min}(M, [g_0], \sigma) > 0.$$

Several experts in Dirac operators assumed, that a similar result cannot hold if D is no longer invertible. Amazingly, the author realized that the methods used in [Lot86] could be extended to a more general situation. We obtained

THEOREM 2.4.3 (B. Ammann, [Amm03b]). *For any compact Riemannian spin manifold (M, g, σ) we have*

$$\lambda^+_{\min}(M, [g_0], \sigma) > 0.$$

Proofs of these statements are implicitly contained in Chapter 4.

2.5 The inequalities by Hijazi and Bär

THEOREM 2.5.1 (Hijazi-inequality [Hij86b]). *Let (M, g) be a compact Riemannian spin manifold of dimension $n \geq 3$. Let $\lambda_1(L_g)$ denote the first eigenvalue of the* conformal Laplacian [1]

$$L_g := 4\,\frac{n-1}{n-2}\Delta_g + \mathrm{scal}_g.$$

Then any eigenvalue λ of the Dirac operator satisfies

$$\lambda^2 \geq \frac{n}{4(n-1)}\,\lambda_1(L_g).$$

Clearly, this theorem is non-trivial iff $\lambda_1(L_g) > 0$. In particular, it is non-trivial if the *Yamabe invariant*

$$\lambda_Y(M,[g]) := \inf_{\tilde{g}\in[g]} \lambda_1(L_{\tilde{g}})\,\mathrm{vol}(M,\tilde{g})^{2/n} \tag{2.5.2}$$

is positive.

For a proof of the Hijazi inequality see [Hij91]. Following an idea of Bär, Herzlich and A. Moroianu, it was shown in [CGH00, Sec. 6.2] that the Hijazi inequality can also be deduced from the refined Kato inequality.

In dimension 2 there is an analogue of the Hijazi-inequality due to Bär. It is only non-trivial if $\chi(M) > 0$, i.e. if M is diffeomorphic to S^2.

THEOREM 2.5.3 ([Bär92]). *Let g be any metric on the two-dimensional sphere S^2. Then any eigenvalue λ of the Dirac operator satisfies*

$$\lambda^2 \geq \frac{4\pi}{\mathrm{area}(S^2, g)}.$$

Equality is attained if and only if g is a metric of constant sectional curvature.

In the terminology of the previous section this means that S^2 with its unique conformal spin structure satisfies

$$\lambda^+_{\min}(S^2) = 2\sqrt{\pi}.$$

2.6 Blow up of spheres

It is also very interesting to obtain upper bounds for $\lambda^+_{\min}(M, g, \sigma)$. We proved

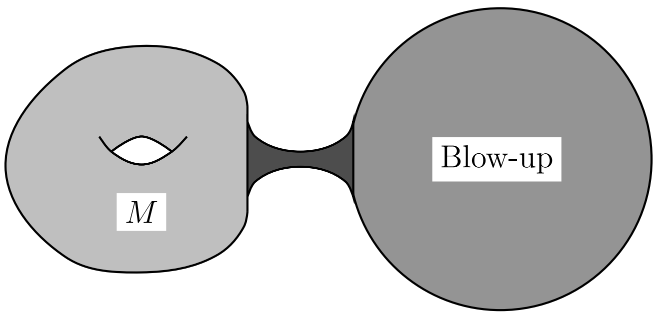
[1] The conformal Laplacian is also called the *Yamabe operator*

THEOREM 2.6.1 ([Amm03b]). *If* $\dim M \geq 3$ *or if* D *is invertible, then*

$$\lambda^+_{\min}(M, [g], \sigma) \leq \lambda^+_{\min}(S^n),$$

where S^n *carries the standard metric and the unique spin structure.*

In order to prove this, one conformally blows up an almost-sphere in a small neighborhood of a given point. On this almost-sphere we take a suitable linear combination of Killing spinors and obtain a test spinor which yields a value arbitrarily close to $\lambda^+_{\min}(S^n)$. If $\dim M \geq 3$ one can show that this test spinor is almost orthogonal to the kernel of D.



For details we refer to [Amm03b].

Whether this statement also holds on compact spin *surfaces* with $\ker D \neq \{0\}$ is still open.

Much stronger upper estimates on the Bär-Hijazi-Lott-invariant will be derived in [AHM03]. In particular, we will see that if (M, g, σ) a Riemannian spin manifolds of dimension ≥ 7 which is not locally conformally flat, then

$$\lambda^+_{\min}(M, [g], \sigma) < \lambda^+_{\min}(S^n).$$

Together with the results of the Chapter 4 this implies that solutions of (4.2.4) exist.

Chapter 3

Elliptic regularity for Dirac operators

3.1 Schauder estimates for Dirac operators

The aim of this section is to adapt the Schauder estimates in [GT77] to Dirac operators. Statements that are identical will be cited. However, many statements have to be modified in order to fit for Dirac operators. Several proofs in elliptic theory are slightly easier for Dirac operator than for the Laplacian. In some estimates we only have to control a function up to the first derivatives instead of up to the second derivatives.

Let us give an overview over how to obtain Schauder estimates: In subsection 3.1.1 we define various norms. Then we begin to study the case of flat manifolds in subsection 3.1.2. In order to apply these results to (non-flat) vector bundles over (non-flat) manifolds, we have to derive some preliminaries on trivializations of vector bundles in subsection 3.1.3. Some needed interpolation inequalities are provided in subsection 3.1.4. Taking all this together, we are ready to prove Interior and Global Schauder estimates in subsection 3.1.5 and 3.1.6.

In all estimates we are not only interested in the embeddings themselves. We will also need certain uniformity statements for the constants.

3.1.1 Definition of Hölder norms

In this subsection we want to define norms that we will need for developing our theory. We have to work with several norms as in the following statements some of these norms are better adapted than others. The definition of these norms is analogous to [GT77, page 53 and 61]. By a *domain* we will always understand an open connected subset of $\mathbb{R}^n$.

Let Ω be domain in $\mathbb{R}^n$ and $K \subset \Omega$ compact, $0 < \alpha \leq 1$. Let $d_x := dist(x, \partial\Omega)$, $d_{x,y} := \min\{d_x, d_y\}$. If $\Omega \neq \mathbb{R}^n$ (what we will assume for simplicity), then $d_x < \infty$.

We define

$$\begin{aligned}
[\varphi]_{k,0;\Omega} &:= [\varphi]_{k;\Omega} := \sup_{x\in\Omega,|\beta|=k} \|\nabla^\beta \varphi(x)\| \\
\|\varphi\|_{k,0;\Omega} &:= \|\varphi\|_{k;\Omega} := \sum_{j=0}^{k} [\varphi]_{j,\Omega}; \\
[\varphi]_{k,\alpha;\Omega} &:= \sup_{x,y\in\Omega,|\beta|=k} \left| \frac{\nabla^\beta\varphi(x) - \nabla^\beta\varphi(y)}{|x-y|^\alpha} \right|; \\
\|\varphi\|_{k,\alpha;\Omega} &:= \|\varphi\|_{k,0;\Omega} + [\varphi]_{k,\alpha;\Omega};
\end{aligned}$$

$$\begin{aligned}
\|\varphi\|'_{k,0;\Omega} &:= \|\varphi\|'_{k;\Omega} := \sum_{j=0}^{k} \mathrm{diam}(\Omega)^j \, [\varphi]_{j,\Omega}; \\
\|\varphi\|'_{k,\alpha;\Omega} &:= \|\varphi\|'_{k,0;\Omega} + \mathrm{diam}(\Omega)^{k+\alpha} \, [\varphi]_{k,\alpha;\Omega};
\end{aligned}$$

$$\begin{aligned}
[\varphi]^*_{k,0;\Omega} &:= [\varphi]^*_{k;\Omega} := \sup_{x\in\Omega,|\beta|=k} d_x^k |\nabla^\beta\varphi(x)| \\
\|\varphi\|^*_{k,0;\Omega} &:= \|\varphi\|^*_{k;\Omega} := \sum_{j=0}^{k} [\varphi]^*_{j,\Omega}; \\
[\varphi]^*_{k,\alpha;\Omega} &:= \sup_{x,y\in\Omega,|\beta|=k} d_{x,y}^{k+\alpha} \left| \frac{\nabla^\beta\varphi(x) - \nabla^\beta\varphi(y)}{|x-y|^\alpha} \right|; \\
\|\varphi\|^*_{k,\alpha;\Omega} &:= \|\varphi\|^*_{k,0;\Omega} + [\varphi]^*_{k,\alpha;\Omega};
\end{aligned}$$

$$\|\varphi\|^{(k)}_{0,\alpha;\Omega} := \sup_{x\in\Omega} d_x^k |\varphi(x)| + \sup_{x,y\in\Omega} d_{x,y}^{k+\alpha} \left| \frac{\varphi(x)-\varphi(y)}{|x-y|^\alpha} \right|$$

Obviously, we have

$$\|\varphi\|^{(k)}_{0,\alpha;\Omega} \leq \max(1, \mathrm{diam}(\Omega)^{k+\alpha}) \|\varphi\|_{0,\alpha;\Omega}$$

If one replaces Ω by K, one obtains completely analogous definitions.

Note that $\|\varphi\|^{(0)}_{0,\alpha;\Omega} = \|\varphi\|^*_{0,\alpha;\Omega}$. If $\operatorname{diam}(\Omega) < \infty$, then

$$\|\varphi\|^*_{k,\alpha;\Omega} \le \max(1, \operatorname{diam}(\Omega)^{k+\alpha})\, \|\varphi\|_{k,\alpha;\Omega},$$

and if $d = d(\Omega', \partial\Omega)$

$$\min(1, d^{k+\alpha})\, \|\varphi\|_{k,\alpha;K} \le \|\varphi\|^*_{k,\alpha;\Omega}.$$

In order to define similar norms for functions on Riemannian manifolds M and for sections of vector bundles over Riemannian manifolds, some modifications have to be done. We replace differences of vectors by differences of parallel transports of vectors along shortest geodesics, and — if the shortest geodesic is not unique — take the supremum over all shortest geodesics. Furthermore, in order to define the $\|.\|'$-norms we assume that the manifold is connected and bounded, which implies the finiteness of diam. Similarly, to define the $\|.\|^*$-norms and $\|.\|^{(k)}$-norms, we assume that each connected component is *non*-complete. Under this condition d_x is defined as the infimum over all lengths of geodesic rays γ_x such that $\gamma_x : [0, d) \to M$ is globally distance minimizing and $\lim_{t\to d} \gamma_x(t)$ does not exist in M. The non-completeness implies that d_x is finite.

The most important norms, the norms $\|\,.\,\|_{k,\alpha;\Omega}$ and $\|\,.\,\|_{k;\Omega}$ will be also denoted by $\|\,.\,\|_{C^{k,\alpha}(\Omega)}$ and $\|\,.\,\|_{C^k(\Omega)}$. When we want to emphasize the bundle $V \to \Omega$ the sections live in, we write $\|\,.\,\|_{C^{k,\alpha}(\Omega;V)}$ and $\|\,.\,\|_{C^k(\Omega;V)}$.

3.1.2 The Dirac-Poisson equation – the flat case

In this subsection we will deal with the case that Ω is a subset of $\mathbb{R}^n$, carrying the euclidean metric, and that the Dirac bundle S over Ω is trivialized by parallel sections. We write $S = \mathbb{C}^m$. This is an important preliminary for the general case. We want to study solutions to the *Dirac-Poisson equation*

$$D\varphi = \psi$$

on Ω where $\varphi \in C^1(\Omega, \mathbb{C}^m)$, $\psi \in C^0(\Omega, \mathbb{C}^m)$. In local coordinates we write $D = \sum_k \sigma_k \partial_k$ where σ_k are endomorphisms satisfying

$$\sigma_i^2 = -\operatorname{id} \qquad \sigma_i\sigma_j = -\sigma_j\sigma_i.$$

Clifford multiplication is then the map $\gamma : \mathbb{R}^n \to \operatorname{End}(\mathbb{C}^m)$, $\gamma(\sum x^i e_i) = \sum x^i \sigma_i$.

LEMMA 3.1.1. *The* $\operatorname{End}(S)$*-valued distribution on* $\mathbb{R}^n$

$$\Gamma(x) = -\operatorname{vol}(S^{n-1})^{-1} \cdot \gamma\left(\frac{x}{|x|^n}\right)$$

satisfies $D\Gamma = \delta \operatorname{Id}_{S_0}$*, where* δ *denotes the* δ*-distribution of* $\mathbb{R}^n$*.*

The function Γ will be called the *Green's function* for the Dirac operator.

We will give two proofs.

Proof # 1. Recall that the Green's function for the Laplacian is

$$\Gamma_\Delta(x) = (n-2)^{-1}\mathrm{vol}(S^{n-1})^{-1}|x|^{2-n}$$

for $n \geq 3$ and

$$\Gamma_\Delta(x) = (2\pi)^{-1}\log|x|$$

for $n = 2$.

Using standard estimates(e.g. [GT77, Lemma 4.1]), one sees that

$$\underbrace{\partial_i \int_\Omega \Gamma_\Delta(x-y)\psi(y)\,dy}_{\in C^1} = \int_\Omega (\partial_i\Gamma_\Delta)(x-y)\psi(y)\,dy.$$

Multiplying σ_i from the left and summing over i yields

$$D\int_\Omega \Gamma_\Delta(x-y)\psi(y)\,dy = \int_\Omega (D\Gamma_\Delta)(x-y)\psi(y)\,dy.$$

Using $D^2 = \Delta$ and using [GT77, Lemma 4.2] and its corollaries, we obtain that the Green's function for D is

$$\Gamma(x) = D\Gamma_\Delta(x).$$

□

Proof # 2. We fix a v in $\mathbb{C}^m$. At first, note that outside 0, we have $D(\Gamma v) = 0$. Let Φ be a compactly supported test spinor. We have to show that

$$(\Gamma v, D\Phi) = \langle v, \Phi(0)\rangle.$$

For this, we integrate by parts and obtain

$$0 = \int_{\mathbb{R}^n\setminus B_\varepsilon(0)} \langle D\Gamma v, \Phi\rangle = \int_{\mathbb{R}^n\setminus B_\varepsilon(0)} \langle \Gamma v, D\Phi\rangle + \int_{S_\varepsilon(0)} \langle\left(-\frac{x}{|x|}\right)\cdot\Gamma v, \Phi\rangle,$$

where $S_\varepsilon(0)$ denotes the boundary of $B_\varepsilon(0)$. For $\varepsilon \to 0$, the first summand converges to $(\Gamma v, D\Phi)$. The boundary term yields

$$\int_{S_\varepsilon(0)} \langle\left(-\frac{x}{|x|}\right)\cdot\Gamma v, \Phi\rangle = -\int_{S_\varepsilon(0)} \langle \mathrm{vol}(S^{n-1})^{-1}\varepsilon^{1-n}v, \Phi\rangle \to -\langle v, \Phi(0)\rangle,$$

and hence the statement follows. □

In analogy to the Newtonian potential for the Laplacian we define the following.

Definition. Let ψ be a continuous spinor on $\mathbb{R}^n$. Let Ω be a domain in $\mathbb{R}^n$. Then the *Dirac-Newton potential* w of ψ with respect to Ω is defined as

$$w(x) = \int_\Omega \Gamma(x-y)\psi(y)\,dy.$$

LEMMA 3.1.2 (anal. to [GT77, Lemma 4.4])**.** *Let $B_1 = B_R(x_0)$, $B_2 = B_{2R}(x_0)$ be concentric balls in $\mathbb{R}^n$. Suppose $\psi \in C^{0,\alpha}(\overline{B}_2)$, $0 < \alpha < 1$, and let w be the Dirac-Newton potential of ψ in B_2. Then $w \in C^{1,\alpha}(\overline{B}_1)$ and*

$$\|\nabla w\|'_{0,\alpha;B_1} \leq C\|\psi\|'_{0,\alpha;B_2},$$

i.e.

$$\|\nabla w\|_{0;B_1} + R^\alpha[\nabla w]_{\alpha;B_1} \leq C\left(\|\psi\|_{0;B_2} + R^\alpha[\psi]_{\alpha;B_2}\right)$$

where $C = C(n,\alpha)$.

Proof. This statement follows directly from [GT77, Lemma 4.4] by performing one Clifford contraction. □

LEMMA 3.1.3 (anal. to [GT77, Theorem 4.6])**.** *Let φ in $C^1(\Omega)$ and $\psi \in C^{0,\alpha}(\Omega)$ and*

$$D\varphi = \psi \qquad \text{on } \Omega.$$

Then $\varphi \in C^{1,\alpha}(\Omega)$ and for concentric balls

$$B_1 \subset B_2 \subset \overline{B_2} \subset \Omega \qquad \text{with } B_k := B_{kR}(x_0)$$

we have

$$\|\varphi\|'_{C^{1,\alpha}(B_1)} \leq C \cdot \left(\|\varphi\|_{C^0(B_2)} + R\|\psi\|'_{C^{0,\alpha}(B_2)}\right)$$

where $C = C(n,\alpha)$.

Proof. Let

$$w(x) := \int_{B_2} \Gamma(x-y)\psi(y)\,dy$$

be the Dirac-Newton potential of ψ on B_2. Then $Dw = \psi$. Hence,

$$h(x) := \varphi(x) - w(x)$$

is D-harmonic, i.e. $Dh = 0$. Because of $\Delta = D^2$, we see that h is Δ-harmonic as well. Thus, [GT77, Theorem 2.10] tells us that

$$\sup_{B_1} |\partial_k h| \leq \frac{n}{R} \sup_{B_2} |h| \qquad \sup_{B_1} |\partial_k \partial_j h| \leq \left(\frac{2n}{R}\right)^2 \sup_{B_2} |h|.$$

Hence,

$$\begin{aligned} R\|\nabla h\|'_{0,\alpha;B_1} &\leq C(n)\sup_{B_2}|h| \leq C(n)\left(\sup_{B_2}|\varphi| + \underbrace{\left|\int_{B_4}\Gamma(x)\right|}_{\leq C(n)\cdot R}\sup_{B_2}|\psi|\right) \\ &\leq C(n)\cdot(\|\varphi\|_{0;B_2} + R\|\psi\|_{0;B_2}). \end{aligned}$$

Lemma 3.1.2 now tells us that

$$R\|\nabla w\|'_{0,\alpha;B_1} \leq C(n)\, R\|\psi\|'_{0,\alpha;B_2}.$$

Adding up the last two inequalities and using

$$\|\varphi\|'_{1,\alpha;B_1} \leq (\|\varphi\|_{0;B_2} + R\|\nabla h\|'_{0,\alpha;B_1} + R\|\nabla w\|'_{0,\alpha;B_1})$$

we obtain the statement. □

THEOREM 3.1.4. *We assume as before that Ω is a subset of $\mathbb{R}^n$, and that the Dirac bundle S is trivialized by parallel sections. Let $\varphi \in C^1(\Omega)$, $\psi \in C^\alpha(\Omega)$ satisfy $D\varphi = \psi$. Then*

$$\|\varphi\|^*_{1,\alpha;\Omega} \leq C\left(\|\varphi\|_{0;\Omega} + \|\psi\|^{(1)}_{0,\alpha;\Omega}\right), \tag{3.1.5}$$

where $C = C(n,\alpha)$.

Proof. For $x \in \Omega$, let $R = (1/3)\, d_x$, $B_1 = B_R(x)$, $B_2 = B_{2R}(x)$. We have for the first derivative $\nabla\varphi$

$$\begin{aligned} d_x|\nabla\varphi(x)| \leq (3R)\|\nabla\varphi\|_{0;B_1} &\leq C\cdot(\|\varphi\|_{C^0(B_2)} + R\|\psi\|'_{C^{0,\alpha}(B_2)}) \\ &\leq C\cdot(\|\varphi\|_{C^0(\Omega)} + \|\psi\|^{(1)}_{C^{0,\alpha}(\Omega)}) \end{aligned}$$

This yields

$$\|\varphi\|^*_{1;\Omega} \leq C\cdot(\|\varphi\|_{C^0(\Omega)} + \|\psi\|^{(1)}_{C^{0,\alpha}(\Omega)}).$$

In order to estimate $[\varphi]^*_{1,\alpha;\Omega}$ we let $x, y \in \Omega$ with $d_x \leq d_y$. We define as above R, B_1 and B_2.

If $y \in B_1 = B_R(x)$, then

$$\begin{aligned} d^{1+\alpha}_{x,y}\frac{|\nabla\varphi(x) - \nabla\varphi(y)|}{|x-y|^\alpha} &\leq (3R)^{1+\alpha}[\nabla\varphi]_{\alpha;B_1} \\ &\leq C\cdot\left(\|\varphi\|_{0;B_2} + R\|\psi\|'_{0,\alpha;B_2}\right) \\ &\leq C\cdot\left(\|\varphi\|_{0;\Omega} + \|\psi\|^{(1)}_{0,\alpha;\Omega}\right). \end{aligned}$$

Otherwise, i.e. if $y \notin B_1 = B_R(x)$, then $3|x-y| \geq d_x = d_{x,y}$ and this implies

$$\begin{aligned} d_{x,y}^{1+\alpha} \frac{|\nabla\varphi(x) - \nabla\varphi(y)|}{|x-y|^\alpha} &\leq 3^\alpha (3R)(|\nabla\varphi(x)| + |\nabla\varphi(y)|) \\ &\leq 6[\varphi]^*_{1;\Omega}. \end{aligned}$$

Hence,

$$[\varphi]^*_{1,\alpha;\Omega} \leq C \cdot \left(\|\varphi\|_{0;\Omega} + \|\psi\|^{(1)}_{0,\alpha;\Omega} \right).$$

□

3.1.3 Trivializations of vector bundles

In order to adapt the theory of flat domains to curved domains and manifolds with curved vector bundles we have to use some results on trivializations.

Lemma 3.1.6 (Trivialization for vector bundles)**.** *Let (M^n, g) be a Riemannian manifold with a (complex) vector bundle S. We assume that the curvature of M is bounded $|R| \leq K$ and the curvature of S is bounded $|R^S| \leq K$. We choose $p \in M$. On a ball of radius $\varepsilon > 0$ around p we choose normal coordinates $(x^1, \dots, x^n)$. Let Γ^k_{ij} be the associated Christoffel symbols. We trivialize the bundle S via parallel transport along radial geodesics. We identify S with $\mathbb{C}^m$, $m = \operatorname{rank} S$. Let* injrad_p *be the injectivity radius in p. If $\varepsilon > 0$ is smaller than $\varepsilon_0 = \varepsilon_0(K, n, \operatorname{injrad}_p) > 0$, then one has for any $q \in B_\varepsilon(p)$*

$$|\Gamma^k_{ij}| \leq C(n)\, K\, d(q,p), \tag{3.1.7}$$

$$\Big| (\nabla_k \varphi)(q) - (\partial_k \varphi)(q) \Big| \leq C(n,m)\, K\, d(q,p)\, |\varphi(q)|, \tag{3.1.8}$$

where $\partial_k = \frac{\partial}{\partial x^k}$ and $\nabla_k = \nabla_{\partial_k}$.

The above trivialization of S is called the *synchronous trivialization.*

Proof. The proof of (3.1.7) is very similar to the proof of (3.1.8). Hence, we only prove the second inequality (3.1.8). In order to prove it, we can assume that $\varepsilon < \operatorname{injrad}_p$, and that φ is parallel along radial geodesics, i.e. $\left(\frac{\partial}{\partial x^k}\varphi\right)(q) = 0$. Let $P_t : S_x \to S_x$ be the parallel transport along the triangle $\triangle_t$ spanned by x, $x + te_k$ and 0, where e_k is the k-th canonical vector, $q = \exp_p x \in B_\varepsilon(p)$, $t \geq 0$. Then

$$(\nabla_k \varphi)(q) = \frac{d}{dt}|_{t=0} P_t \varphi.$$

One easily sees that we can choose ε such that

$$\|(P_t - \mathrm{Id})\varphi\| \leq \operatorname{area}(\triangle_t)\, K\, |\varphi| \leq C\, t\, |x|\, K\, |\varphi|.$$

From this the second inequality is evident. □

In the following subsections, (M^n, g) will be a Riemannian manifold with a Dirac bundle S. This means, that in addition to above we have a compatible Clifford multiplication. We define the sections of $\sigma_k^1 \in \mathrm{End}(S) = \mathrm{End}(\mathbb{C}^m)$ as Clifford multiplication with $\mathrm{grad} x^k$. As a consequence they satisfy

$$\sigma_j^1 \sigma_k^1 + \sigma_k^1 \sigma_j^1 + 2g^{jk} \cdot \mathrm{id} = 0,$$

where g^{jk} is the inverse matrix of g_{jk}. Then the Dirac operator in this localization is

$$D\varphi = \sum_k \sigma_k^1 \partial_k \varphi + \sigma^0 \varphi$$

for a section σ^0 of $\mathrm{End}(S)$.

LEMMA 3.1.9 (Trivialization for Dirac bundles)**.** *If $\varepsilon > 0$ is smaller than $\varepsilon_0 = \varepsilon_0(K, n, \mathrm{injrad}_p) > 0$, then one has for any $q \in B_\varepsilon(p)$*

$$|\sigma^0(q) - \sigma^0(p)| \le C(n)\, K\, d(q,p).$$
$$\left|\sigma_j^1(q) - \sigma_j^1(p)\right| \le C(n)\, K\, d(q,p)^2.$$

Proof. The first inequality follows from (3.1.7). In order to show the second inequality, we deduce from the parallelism of Clifford multiplication and (3.1.7) that

$$\left(\nabla_{\partial/\partial r} \sigma_j^1\right)(q) \le C(n)\, K\, d(p,q)$$

and integrate radially. □

Remark. Similarly derivatives of σ^0 and σ_j^1 can be controlled if one has control over sufficiently many derivatives of the curvature tensor R.

3.1.4 Interpolation inequalities

We will need an interpolation inequality. It is an extension of [GT77, Lemma 6.32] from scalar valued functions to sections in a vector bundle over a manifold.

LEMMA 3.1.10. *Let Ω be a non-complete connected open Riemannian manifold (For example a domain in $\mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$.) Furthermore, let V be a vector bundle over Ω. We assume that the curvature R^V of V satisfies $\|R^V\|_{0;\Omega}^{(2)} \le K$. Suppose that $j+\beta < k+\alpha$, where $j, k \in \{0,1\}$ and $0 \le \alpha, \beta \le 1$. Assume that $\varphi \in C^{k,\alpha}(\Omega, V)$. Then for any $\varepsilon > 0$ and some constant $C = C(\varepsilon, k, j, \alpha, \beta, K)$ we have*

$$[\varphi]^*_{j,\beta;\Omega;V} \le C\|\varphi\|_{0;\Omega;V} + \varepsilon[\varphi]^*_{k,\alpha;\Omega;V},$$
$$\|\varphi\|^*_{j,\beta;\Omega;V} \le C\|\varphi\|_{0;\Omega;V} + \varepsilon\|\varphi\|^*_{k,\alpha;\Omega;V}.$$

Similar estimates also hold for larger values of j and k. We will restrict to these cases, as only these cases are needed, and the proof is case by case.

Proof. Obviously the second estimate follows from the first one. We will prove the first ones. In each of the cases under consideration one can adapt the proofs in [GT77, Lemma 6.32] to our setting. We distinguish different cases.

(i) The case $j=0$, $\beta=0$ is trivial.

(ii) We will now study the case $j=k=1$, $\beta=0$, $\alpha>0$. We choose any $p\in\Omega$. Let μ be a real number $0<\mu\le 1/2$, that we will fix later on. As before we note $d_p=d(p,\partial\Omega)$, $d_{p,q}=\min\{d_p,d_q\}$, $B=B_d(p)$. Let $d(p,q)$ be the distance between the points p and q. We take normal coordinates $(x^1,\dots,x^n)$ in a neighborhood of p, choose a basis of V_p, and we trivialize V via parallel transport along radial geodesics. The components of $\varphi\circ\exp_p$ will be denoted as $\varphi_1,\dots,\varphi_r$, $r=\operatorname{rank}V$. Let $(e_1,\dots,e_n)$ denote the standard orthonormal basis of $\mathbb{R}^n=T_pM$. There is a constant $\mu_0(K)>0$, such that if $\mu\le\mu_0(K)$ then for $|x|<d=\mu d_p$

$$|\partial_i\varphi_j(x)|\le|\nabla_i\varphi(x)|+c_n|R^V|\,|x|\,|\varphi_j(x)|.$$

For any (fixed) $j=1,\dots,r$ and $l=1,\dots,n$, there is $\bar x$ on the line between $-de_l$ and de_l such that

$$|\partial_i\varphi_j(\bar x)|\le\left|\frac{\varphi_j(-de_l)-\varphi_j(de_l)}{2d}\right|\le\frac{1}{d}\,\|\varphi_j\|_{0;B}.$$

We set $q=\exp_p\bar x$. By using $|\bar x|<d$, we obtain

$$\begin{aligned}|\partial_i\varphi_j(0)| &\le |\partial_i\varphi_j(\bar x)|+|\nabla_i\varphi(p)-\nabla_i\varphi(q)|+c_n|R^V|\,|\bar x|\,|\varphi_j(q)|\\ &\le \frac{1}{d}\,\|\varphi_j\|_{0;B}+d^\alpha\sup_{q\in B}d_{p,q}^{-1-\alpha}\sup_{q\in B}d_{p,q}^{1+\alpha}\frac{|\nabla_i\varphi(p)-\nabla_i\varphi(q)|}{d(p,q)^\alpha}\\ &\quad +c_n\sup_{q\in B}|R^V|\,\sup_{q\in B}d(p,q)\,|\varphi(q)|\end{aligned}$$

Using $d_q>d_p/2$, $d_{p,q}>d_p/2$ and $d(p,q)<d\le d_p/2$ we obtain

$$\begin{aligned}|\partial_i\varphi_j(0)| &\le \frac{1}{d}\left[1+4c_n\sup_{q\in B}\left(d_q^2\,|R^V|(q)\right)\right]\,\|\varphi_j\|_{0;B}\\ &\quad +\frac{1}{d}\sup_{q\in B}\left(\frac{d}{d_{p,q}}\right)^{1+\alpha}[\varphi]^*_{1,\alpha;\Omega;V}\\ &\le \frac{1}{d_p}\,\frac{C(n)K}{\mu}\,\|\varphi\|_{0;\Omega;V}+\frac{1}{d_p}\,2^{1+\alpha}\mu^\alpha[\varphi]^*_{1,\alpha;\Omega;V}.\end{aligned}$$

As a consequence, we can choose μ sufficiently small such that

$$d_p|\nabla\varphi(p)|\ \le\ C(\varepsilon,\alpha,n,K)\,\|\varphi\|_{0;\Omega;V}+\varepsilon[\varphi]^*_{1,\alpha;\Omega;V}$$

and hence

$$[\varphi]^*_{1,0;\Omega;V} \leq C(\varepsilon,\alpha,n,K)\,\|\varphi\|_{0;\Omega;V} + \varepsilon[\varphi]^*_{1,\alpha;\Omega;V}$$

(iii) The case $j=0$, $k=1$, $\beta>0$, $\alpha=0$. To obtain a proof in this case one can adapt [GT77, Lemma 6.32 (iii)] to the manifold and vector bundle setting in the same way as we just did in (ii).

(iv) The case $j=0$, $k=0$, $\alpha>\beta>0$: With the same notation as in (ii), we have for $q \in B$

$$d_p^\beta\,\frac{|\varphi(p)-P_{pq}\varphi(q)|}{d(p,q)^\beta} \leq \mu^{\alpha-\beta}\,d_p^\alpha\,\frac{|\varphi(p)-P_{pq}\varphi(q)|}{d(p,q)^\alpha},$$

where P_{pq} is the parallel transport from q to p along the shortest geodesic. On the other hand if $q \notin B$, then

$$d_p^\beta\,\frac{|\varphi(p)-P_{pq}\varphi(q)|}{d(p,q)^\beta} \leq 2\mu^{-\beta}\|\varphi\|_{0;\Omega;V}.$$

Choosing μ sufficiently small the claimed inequality follows in this case.

(v) The remaining case $j=k=1$ and $\alpha>\beta$ is obtained similarly as in (iv), using results in (ii).

□

3.1.5 Interior Schauder estimates

In the rest of this section, we study Dirac operators over arbitrary manifolds M with arbitrary Dirac bundles. At first, we derive Interior Schauder estimates. The interior estimates will only be needed locally, i.e. for small balls. Hence, for simplicity we want to assume that $\Omega = M$ is a simply connected Riemannian manifold, on which one has sufficient control over the curvature of M and on the curvature of the Dirac bundle.

From now on, we work in this trivialization provided in Subsection 3.1.3. We will adapt the Interior Schauder estimates from [GT77, Theorem 6.2] to what we need.

THEOREM 3.1.11. *Let $S=\mathbb{C}^m$ be a trivial bundle over $\Omega \subset \mathbb{R}^n$ which is an open subset of $\mathbb{R}^n$ and equipped with a Riemannian metric (g_{ij}). Let*

$$D\varphi = \sum_k \sigma^1_k \partial_k \varphi + \sigma^0\varphi$$

where $\sigma_k^1, \sigma^0 \in \Gamma(\mathrm{End}(S))$, $\varphi, \psi \in \Gamma(S)$ *with*

$$\sigma_j^1\sigma_k^1 + \sigma_k^1\sigma_j^1 + 2g^{jk} \cdot \mathrm{id} = 0,$$

$g^{ij}v_iv_j \geq \zeta|v|^2$, *and* $\sum_k \|\sigma_k^1\|_{C^{0,\alpha}(\Omega)} < Z$, $\|\sigma^0\|^{(1)}_{C^{0,\alpha}(\Omega)} < Z$. *Furthermore, let the curvature* R^S *of* S *satisfy*

$$\|R^S\|^{(2)}_{0;\Omega} < Z.$$

Then there is a constant $C = C(n, \zeta, Z, \alpha)$ *such that any pair of spinors* $\psi \in C^{0,\alpha}(\Omega, \mathbb{C}^m)$, $\varphi \in C^{1,\alpha}(\Omega, \mathbb{C}^m)$ *with* $D\varphi = \psi$ *satisfies*

$$\|\varphi\|^*_{C^{1,\alpha}(\Omega)} \leq C \cdot \left(\|\varphi\|_{C^0(\Omega)} + \|\psi\|^{(1)}_{C^{0,\alpha}(\Omega)}\right). \tag{3.1.12}$$

Note that all distances used in this theorem are not with respect to the euclidean metric on $\mathbb{R}^n$ but with respect to the distance on (Ω, g) in the sense of Riemannian geometry.

Proof of the theorem. At first, we show that we can assume that Ω is relatively compact in $\mathbb{R}^n$. In order to show that this is sufficient, let Ω be an arbitrary domain in $\mathbb{R}^n$, which is compactly exhausted by a sequence of compact subsets Ω_i. If we know that inequality (3.1.12) holds on every compact set Ω_i, then the uniform upper bound on C implies that (3.1.12) also holds on Ω. Hence, it is sufficient to prove the statement for relatively compact Ω.

Let x_0 and y_0 be distinct points in Ω and suppose $d_{x_0} = d_{x_0,y_0} = \min(d_{x_0}, d_{y_0})$. Let $\mu \leq 1/2$ be a positive constant to be specified later, and set $d = \mu\, d_{x_0}$, $B = B_d(x_0)$. In the following C is a constant depending on n, m, α, ζ and Z, and $C(\mu)$ is a constant depending on n, m, α, ζ, Z and μ, whereas C_n only depends on n. We set

$$F(x) := \sum_k \left(\sigma_k^1(x_0) - \sigma_k^1(x)\right)\partial_k\varphi - \sigma^0\varphi + \psi = \sum_k \sigma_k^1(x_0)\partial_k\varphi.$$

After changing to Riemann normal coordinates based in x_0, we can assume that $g_{ij}(x_0) = \delta_{ij}$.

Now, we can apply Theorem 3.1.4 to the flat approximation $D^{\mathrm{flat}} := \sum_k \sigma_k^1(x_0)\partial_k$ of the Dirac operator. The theorem implies for $y_0 \in B_{d/2}(x_0)$, that any first derivative satisfies for any k

$$\left(\frac{d}{2}\right)^{1+\alpha} \frac{|\partial_k\varphi(x_0) - \partial_k\varphi(y_0)|}{d(x_0, y_0)^\alpha} \leq C(\|\varphi\|_{0;B} + \|F\|^{(1)}_{0,\alpha;B}).$$

In x_0 we have $\nabla\varphi(x_0) = \sum_k dx^k \otimes \partial_k\varphi(x_0)$. Via the trivialization results of Subsection 3.1.3 one obtains

$$\left(\frac{d}{2}\right)^{1+\alpha} \frac{|\nabla\varphi(y_0) - dx^k \otimes \partial_k\varphi(y_0)|}{d(x_0, y_0)^\alpha} \leq C_n \underbrace{\left(d^2 \sup_{y\in B}|R^S(y)|\right)}_{\leq Z} |\varphi(y_0)|$$

and thus

$$d_{x_0}^{1+\alpha}\frac{|\nabla\varphi(x_0)-\nabla\varphi(y_0)|}{d(x_0,y_0)^\alpha}\le\frac{C}{\mu^{1+\alpha}}(\|\varphi\|_{0;B}+\|F\|_{0,\alpha;B}^{(1)}).$$

On the other hand for $|x_0-y_0|\ge d/2$,

$$d_{x_0}^{1+\alpha}\frac{|\nabla\varphi(x_0)-\nabla\varphi(y_0)|}{d(x_0,y_0)^\alpha}\le\left(\frac{2}{\mu}\right)^\alpha\left\{d_{x_0}\,|\nabla\varphi(x_0)|+d_{y_0}\,|\nabla\varphi(y_0)|\right\}\le\frac{4}{\mu^\alpha}[\varphi]_{1;\Omega}^*,$$

so that, combining these two inequalities, we obtain

$$d_{x_0}^{1+\alpha}\frac{|\nabla\varphi(x_0)-\nabla\varphi(y_0)|}{d(x_0,y_0)^\alpha}\le\frac{C}{\mu^{1+\alpha}}(\|\varphi\|_{0;B}+\|F\|_{0,\alpha;B}^{(1)})+\frac{4}{\mu^\alpha}[\varphi]_{1;\Omega}^*. \tag{3.1.13}$$

Now, $\|F\|_{0,\alpha;B}^{(1)}$ will be estimated in terms of $\|\varphi\|_{0;\Omega}$ and $[\varphi]_{1,\alpha;\Omega}^*$. Note that any $C^{0,\alpha}$-function h on M satisfies

$$\begin{aligned}\|h\|_{0,\alpha;B}^{(1)} &\le d\|h\|_{0;B}+d^{1+\alpha}\|h\|_{0,\alpha;B}\\ &\le \frac{\mu}{1-\mu}\,[h]_{0;\Omega}^{(1)}+\frac{\mu^{1+\alpha}}{(1-\mu)^{1+\alpha}}\,[h]_{0,\alpha;\Omega}^{(1)}\\ &\le 2\mu\,[h]_{0;\Omega}^{(1)}+4\mu^{1+\alpha}\,[h]_{0,\alpha;\Omega}^{(1)}\end{aligned}$$

For the principal term we obtain

$$\sum_k\left\|\left(\sigma_k^1(x_0)-\sigma_k^1(x)\right)\partial_k\varphi\right\|_{0,\alpha;B}^{(1)}\le\left(\sum_k\left\|(\sigma_k^1(x_0)-\sigma_k^1(x)\right\|_{0,\alpha;B}^{(0)}\right)\|(\nabla\varphi)\|_{0,\alpha;B}^{(1)}.$$

The first factor is estimated as

$$\begin{aligned}\left(\sum_k\left\|\sigma_k^1(x_0)-\sigma_k^1(x)\right\|_{0,\alpha;B}^{(0)}\right) &\le \sum_k\left\{\sup_{x\in B}|\sigma_k^1(x_0)-\sigma_k^1(x)|+d^\alpha[\sigma_k^1]_{0,\alpha;B}\right\}\\ &\le 2d^\alpha\left(\sum_k[\sigma_k^1]_{0,\alpha;B}\right)\le 2^{1+\alpha}\mu^\alpha[\sigma_k^1]_{0,\alpha;\Omega}^*\\ &\le 4Z\mu^\alpha,\end{aligned}$$

whereas the second one is bounded from above

$$\|(\nabla\varphi)\|_{0,\alpha;B}^{(1)}\le\left(2\mu\|\varphi\|_{1;\Omega}^*+4\mu^{1+\alpha}\|\varphi\|_{1,\alpha;\Omega}^*\right).$$

Hence,

$$\begin{aligned}\sum_k\left\|\left(\sigma_k^1(x_0)-\sigma_k^1(x)\right)\partial_k\varphi\right\|_{0,\alpha;B}^{(1)} &\le 16Z\mu^{1+\alpha}\left([\varphi]_{1;\Omega}^*+\mu^\alpha[\varphi]_{1,\alpha;\Omega}^*\right)\\ &\le 16Z\mu^{1+\alpha}\left(C(\mu)\|\varphi\|_{0;\Omega}+2\mu^\alpha[\varphi]_{1,\alpha;\Omega}^*\right),\end{aligned}$$

where we have used interpolation inequalities (Lemma 3.1.10 in the last inequality with given $\varepsilon = \mu^\alpha$).

Furthermore, we get

$$\|\sigma^0\varphi\|^{(1)}_{0,\alpha;B} \leq 4\mu\|\sigma^0\|^{(1)}_{0,\alpha;B}\|\varphi\|^{(0)}_{0,\alpha;B} \leq 4Z\mu\left(C(\mu)\,\|\varphi\|_{0,\Omega} + \mu^{2\alpha}\|\varphi\|^*_{1,\alpha;B}\right)$$

where we have used interpolation inequalities (Lemma 3.1.10) in the last inequality with given $\varepsilon = \mu^{2\alpha}$.

The last term of F is

$$\|\psi\|^{(1)}_{0,\alpha;B} \leq 4\mu\|\psi\|^{(1)}_{0,\alpha;\Omega}.$$

By adding up, we obtain

$$\|F\|^{(1)}_{0,\alpha;B} \leq C\mu^{1+2\alpha}[\varphi]^*_{1,\alpha;\Omega} + C(\mu)\left(\|\varphi\|_{0;\Omega} + \|\psi\|^{(1)}_{0,\alpha;\Omega}\right).$$

Together with (3.1.13) and using once again interpolation inequalities with $\varepsilon = \mu^{2\alpha}$, we obtain

$$[\varphi]^*_{1,\alpha;\Omega} \leq C\mu^\alpha[\varphi]^*_{1,\alpha;\Omega} + C(\mu)\left(\|\varphi\|_{0;\Omega} + \|\psi\|^{(1)}_{0,\alpha;\Omega}\right).$$

Finally, by choosing μ with $C\mu^\alpha < 1/2$, we obtain

$$[\varphi]^*_{1,\alpha;\Omega} \leq C(\|\varphi\|_{0;\Omega} + \|\psi\|^{(1)}_{0,\alpha;\Omega}).$$

□

Corollary 3.1.14 (Interior Schauder estimates)**.** *Let Ω be a bounded open subset of $\mathbb{R}^n$, equipped with a Riemannian metric. Let $K \subset \Omega$ be compact. Let S be a Dirac bundle over $\overline{\Omega}$. We assume that the components of the inverse metric g^{ij} satisfy $g^{ij}v_iv_j \geq \zeta|v|^2$ for all $v \in T\Omega$ and $\|g^{ij}\|_{C^{1,\alpha}(\Omega)} \leq Z$. We further assume that S is trivialized by orthonormal sections $\beta_1, \dots, \beta_r$ such that $\|\beta_j\|_{C^{1,\alpha}(\Omega)} \leq Z$. Let $A \in \mathrm{End}(S)$, $\|A\|_{C^{0,\alpha}(\Omega)} \leq Z$. Let ψ be a $C^{0,\alpha}$-spinor on Ω. Then for any C^1-solution φ of*

$$(D + A)\varphi = \psi$$

we have $\varphi \in C^{1,\alpha}(K)$ and

$$\|\varphi\|_{C^{1,\alpha}(K;S)} \leq C \cdot (\|\psi\|_{C^{0,\alpha}(\Omega;S)} + \|\varphi\|_{C^0(\Omega;S)})$$

where C only depends on n, α, $\mathrm{diam}(\Omega)$, $\mathrm{dist}(K, \partial\Omega)$, ζ and Z.

Remark. Locally the condition $\|g^{ij}\|_{C^{1,\alpha}(\Omega)} \leq Z(K, \zeta)$ follows in normal coordinates from $\|R_g\|_{C^0(\Omega)} \leq K$. Similarly, if Ω is simply connected, then the condition on the β_j follows from a bound on $\|R^S\|_{C^0(\Omega)}$.

Proof. The corollary follows directly from the previous theorem and the previous lemma. □

In order to control higher derivatives, we recall the following identity

LEMMA 3.1.15 (Twist with the tangential bundle)**.** *Let $S \to M$ be a Dirac bundle and D^S the associated Dirac operator. Then the twisted bundle $T^*M \otimes S$ is again a Dirac bundle, with associated Dirac operator $D^{T^*M \otimes S}$. Then, one has*

$$D^{S \otimes T^*M} \circ \nabla \varphi - \nabla \circ D^S \varphi = \sum_{i,j=1}^{n} e_j^b \otimes e_i \cdot R^S(e_i, e_j)\varphi$$

for a local orthonormal frame $e_1, \ldots, e_n$. In particular, this expression is a differential operator of order 0.

The proof is a straightforward calculation and can be found e.g. in [Bär95] or [Amm].

THEOREM 3.1.16 (Interior Schauder estimates of higher order)**.** *Let Ω be a bounded open subset of $\mathbb{R}^n$, equipped with a Riemannian metric g. Let $K \subset \Omega$ be compact. Let S be a Dirac bundle over $\overline{\Omega}$. We assume that the components of the inverse metric g^{ij} satisfy $g^{ij} v_i v_j \geq \zeta |v|^2$ for all $v \in T\Omega$ and $\|g^{ij}\|_{C^{k+1,\alpha}(\Omega)} \leq Z$. We further assume that S is trivialized by orthonormal sections $\beta_1, \ldots, \beta_r$ such that $\|\beta_j\|_{C^{k+1,\alpha}(\Omega)} \leq Z$. Let $A \in \mathrm{End}(S)$, $\|A\|_{C^{k,\alpha}(\Omega)} \leq Z$. Let ψ be a $C^{k,\alpha}$-spinor on Ω. Then if φ is a C^1-solution of*

$$(D + A)\varphi = \psi,$$

then φ is a $C^{k+1,\alpha}$ spinor on K and

$$\|\varphi\|_{C^{k+1,\alpha}(K;S)} \leq C \cdot \left(\|\psi\|_{C^{k,\alpha}(\Omega;S)} + \|\varphi\|_{C^0(\Omega;S)} \right)$$

where C only depends on n, α, $\mathrm{diam}(\Omega)$, k, $\mathrm{dist}(K, \partial\Omega)$, ζ and Z.

Remark. For $k \geq 1$, locally the condition $\|g^{ij}\|_{C^{k+1,\alpha}(\Omega)} \leq Z(K, \zeta)$ follows in normal coordinates from $\|R_g\|_{C^{k-1,\alpha}(\Omega)} \leq K$. Similarly, if Ω is simply connected, then the condition on the β_j follows from a bound on $\|R^S\|_{C^{k-1,\alpha}(\Omega)}$.

Proof. The theorem is proven over induction in k. We will show how to derive $k = 1$ from $k = 0$. The general induction step from k to $k + 1$ runs similarly. We define $\mathcal{R}\varphi = \sum_{i,j}^{n} e_i^b \cdot R^S(e_i, e_j)\varphi \otimes e_j^b$. Then

$$(D + \mathrm{id} \otimes A)\nabla\varphi = \nabla\psi + \mathcal{R}\varphi - (\nabla A)\,\varphi.$$

We chose a compact K' contained in Ω and whose interior contains K. Then

$$\begin{aligned} \|\nabla\varphi\|_{1,\alpha;K} &\leq C\ \left(\|\nabla\psi\|_{0,\alpha;\Omega} + \|\varphi\|_{0;\Omega} + \|\mathcal{R}\varphi - (\nabla A)\varphi\|_{0,\alpha;K'} \right) \\ &\leq C\ \left(\|\nabla\psi\|_{0,\alpha;\Omega} + \|\varphi\|_{0;\Omega} + 2\,Z\|\varphi\|_{0,\alpha;K'} \right) \\ &\leq C\ \left(\|\nabla\psi\|_{0,\alpha;\Omega} + \|\varphi\|_{0;\Omega} + C\,Z\|\psi\|_{0,\alpha;\Omega} \right). \end{aligned}$$

Hence,

$$\|\varphi\|_{2,\alpha;K} \leq C\ \left(\|\nabla\psi\|_{1,\alpha;\Omega} + \|\varphi\|_{0;\Omega} \right). \qquad \square$$

3.1.6 Global Schauder estimates

THEOREM 3.1.17 (Global Schauder estimates). *Let (M, g, σ) be a compact Riemannian spin manifold and $\psi \in C^{k,\alpha}(\Sigma M)$. Then any solution of*

$$D\varphi = \psi$$

satisfies $\varphi \in C^{k+1,\alpha}(\Sigma M)$, and there is a constant $C = C(M, g, \sigma)$ such that

$$\|\varphi\|_{C^{k+1,\alpha}(M)} \leq C \left(\|\psi\|_{C^{k,\alpha}(M)} + \|\varphi\|_{C^0(M)} \right).$$

Proof. For the proof of the Global Schauder estimates, we cover the manifold by finitely many small balls Ω_i, we choose compact $K_i \subset \Omega_i$ such that $\bigcup_i K_i = M$. Then we apply the interior Schauder estimates to (Ω_i, K_i) and sum it over i. This yields the claimed inequality. □

3.2 L^p-theory for Dirac operators

3.2.1 Sobolev norms of integer order

Let us introduce the function spaces we need. Let Ω be any Riemannian manifold. We assume that a metric is chosen on Ω. Let S be a vector bundle over Ω. For any smooth section ψ on Ω, and $p \in [1, \infty]$, $k \in \mathbb{N}$, we define the H_k^p-norm of ψ as

$$\|\psi\|_{H_k^p} = \sum_{l=0}^{k} \|\underbrace{\nabla \ldots \nabla}_{l-\text{times}} \psi\|_{L^p}.$$

Obviously, on compact manifolds, the H_p^k-norms for different connections are equivalent.

3.2.2 Interior L^p-estimates

In this subsection, let Ω be a simply connected open domain in $\mathbb{R}^n$ equipped with a metric that extends smoothly to $\overline{\Omega}$. We assume that S is a Dirac bundle over $\overline{\Omega}$.

THEOREM 3.2.1. *Suppose Ω is a simply connected open set in $\mathbb{R}^n$ with an arbitrary $C^{k+1,\alpha}$-metric (g_{ij}). Suppose that S is a Dirac bundle over $\overline{\Omega}$ with $\|R^S\|_{C^k} < Z$. Let D be the associated Dirac operator. We assume that the components of the inverse* *metric g^{ij} satisfy $g^{ij}v_iv_j \geq \zeta|v|^2$ for all $v \in T\Omega$ and $\|g^{ij}\|_{C^{k+1,\alpha}(\Omega)} \leq Z$. Furthermore, $A \in H_k^\infty$, $\|A\|_{H_k^\infty} \leq Z$. Let K be a compact subset of Ω. Let $\varphi \in L^1_{loc}(\Omega; S)$ be a solution of $(D+A)\varphi = \psi$. If $\psi \in H_k^p(\Omega; S)$, then $\varphi \in H_{k+1}^p(\Omega; S)$ and*

$$\|\varphi\|_{H^p_{k+1}(K;S)} \leq C\left(\|\psi\|_{H^p_k(\Omega;S)} + \|\varphi\|_{L^p(\Omega;S)}\right)$$

where C only depends on k, n, p, ζ, Z, $\mathrm{diam}(\Omega)$, $d = \mathrm{dist}(K, \partial\Omega)$.

Remark. A similar inequality still holds if g is only a C^{k+1}-metric, and $\|g^{ij}\|_{C^{k+1}} < Z$. The constant then also depends on the modulus of continuity of $\partial^{k+1}g^{ij}$. As before, we could replace the condition on the g^{ij} by a condition on the g_{ij}.

In the proof we will need the following Calderon-Zygmund inequality for the Dirac operator.

LEMMA 3.2.2. *Let S be a Dirac bundle over a bounded open subset Ω of $\mathbb{R}^n$, equipped with the euclidean metric. Let $\varphi \in L^p(\Omega; S)$, $1 < p < \infty$, and let w be the Dirac-Newton potential of φ. Then $w \in H_1^p(\Omega)$, $Dw = \varphi$ almost everywhere and*

$$\|\nabla w\|_{L^p(\Omega)} \leq C\|\varphi\|_{L^p(\Omega)}$$

where C depends only on n and p.

This lemma directly follows from the Calderon-Zygmund inequality [GT77, Theorem 9.9]. □

Proof of Theorem 3.2.1. We write

$$D = \sum_j \sigma_j(x)\nabla_j.$$

We fix $x_0 \in K$. Let D^{flat} be the linear differential operator obtained by freezing the coefficients in x_0, i.e.

$$D^{\mathrm{flat}} = \sum_j \sigma_j(x_0)\nabla_j.$$

It follows from the previous lemma that any spinor $\tilde{\varphi} \in H^p_{1,\mathrm{comp}}(\Omega)$ satisfies

$$\|\nabla\tilde{\varphi}\|_{p;\Omega} \leq \frac{C}{\zeta}\,\|D^{\mathrm{flat}}\tilde{\varphi}\|_{p;\Omega},$$

with $C = C(n, p)$.

There is a $\delta = \delta(k, \alpha, Z, \zeta)$ such that

$$\sum_j \|\sigma_j(x) - \sigma_j(x_0)\| \leq \zeta/(2C)$$

if $d(x, x_0) < \delta$. In particular, δ can be chosen independent from x_0.

Let $R = \frac{1}{2} \min\{\delta, \mathrm{dist}(K, \partial\Omega)\}$. Hence, if $\tilde{\varphi}$ has support in the open ball $B_R(x_0)$, then

$$\|\nabla\tilde{\varphi}\|_{L^p;B_R(x_0)} \leq \frac{C}{2\lambda}\|D\tilde{\varphi}\|_{L^p;B_R(x_0)}, \qquad C = C(n,p)$$

We apply this for $\tilde{\varphi} := \chi\varphi$, where φ is the given spinor and where χ is a cut-off function such that

(1) $\chi \equiv 1$ on $B_{R/2}(x_0)$,

(2) $\mathrm{supp}\,\chi \subset B_R(x_0)$, and

(3) $|\mathrm{grad}\chi| \leq 3/R$.

Then

$$\begin{aligned}\|\nabla\varphi\|_{L^p;B_{R/2}(x_0)} &\leq \|\nabla(\chi\varphi)\|_{L^p;B_R} \\ &\leq \frac{2C}{\zeta}\| \underbrace{D(\chi\varphi)}_{=\chi D\varphi + \mathrm{grad}\chi\cdot\varphi} \|_{L^p;B_R(x_0)} \\ &\leq \frac{2C}{\zeta}\|\psi\|_{L^p;B_R(x_0)} + \frac{2C\cdot 3}{\zeta R}\|\varphi\|_{L^p;B_R(x_0)}\end{aligned}$$

Now, we cover K by finitely many balls of radius $R/2$. By summing up the last inequality, we obtain the desired result. □

3.2.3 Global L^p-estimates on compact manifolds

Now let $\Omega = M$ be a compact manifold. We cover M by finitely many small open balls Ω_i, and compact $K_i \subset \Omega_i$. One applies the interior estimates to each (Ω_i, K_i) and by summing over i and an induction over k, one obtains the global estimate that we will only need with the trivial potential $A \equiv 0$.

THEOREM 3.2.3 (Global L^p estimates)**.** *Let $k \in \mathbb{N}$. Let (M, g) be a compact Riemannian manifold with a Dirac bundle S, and $\psi \in H^p_k(S)$. Then any solution of*

$$D\varphi = \psi$$

satisfies $\varphi \in H^p_{k+1}(S)$*, and there is a constant* $C = C(M, g, S)$ *such that*

$$\|\varphi\|_{H^p_{k+1}} \le C \left(\|\psi\|_{H^p_k} + \|\varphi\|_{L^p} \right).$$

From this we deduce:

COROLLARY 3.2.4. *Let* M *be compact. For any* $k \in \mathbb{N}$ *the norm*

$$\varphi \mapsto \sum_{\ell=0}^{k} \|D^\ell \varphi\|_{L^p}$$

and the H^p_k*-norm are equivalent norms on* $\Gamma(M; S)$*.*

We define the Sobolev space $H^p_k(M; S) = H^p_k$ as the completion of the smooth spinors with respect to this norm.

3.2.4 $H^2_{-1/2}$-spinors

In this section let M be a compact manifold with a Dirac bundle S. The $H^2_{-1/2}$-norm on spinors is defined as

$$\|\psi\|_{H^2_{-1/2}} = \left\| \, |D|^{-1/2}\psi \right\|_{L^2} + \|\pi\psi\|_1,$$

where π is the L^2-orthogonal projection to the kernel of D and where $\| \cdot \|_1$ is an arbitrary norm on the kernel. The fractional power $|D|^{-1/2}$ here is defined in the spectral sense. We will give more details in the next section.

Again, we define the Sobolev space $H^2_{-1/2}(\Sigma M) = H^2_{-1/2}$ as the completion of the smooth spinors with respect to this norm.

LEMMA 3.2.5. *Let* M *be a compact manifold with Dirac bundle* S*. If* $\varphi \in H^2_1$*, then*

$$\|D\varphi\|_{H^2_{-1/2}} \le C \|\varphi\|_{L^2} \, \|\varphi\|_{H^2_1}$$

where C *only depends on* (M, g) *and* S*.*

Here, by saying that C only depends on S we mean that C depends on S and its fixed connection and Clifford multiplication.

The proof follows immediately if one writes φ as a sum of eigenspinors, and if one uses the Cauchy-Schwarz inequality.

3.2.5 Sobolev norms of fractional order on compact M.

In the preceding subsection we gave two different definitions of Sobolev norms. The first one only applies for integer derivation order. The second applies for $k = -1/2$, but only on compact manifolds and for $p = 2$. It is a priori unclear how these two norms fit into a common framework, the so-called fractional Sobolev spaces. What we would need to unify these two definitions into a unified approach is a theory of H^k_p-spaces with arbitrary $p \in (1, \infty)$, $k \in \mathbb{R}$ on arbitrary manifolds (complete or not-complete). This is a very complicated subject, which has many partial results, but also many open problems.

The aim of the present subsection is to describe how these different norms fit into a common theory if the underlying manifold is **compact**.

However, we want to emphasize, that this section will not be needed in order to derive a logically complete proof of the results in Chapter 4.

We restrict to the case that M is a compact Riemannian manifold with a Dirac bundle S. Our aim in this section is to extend the definition of Sobolev norms to orders $k \in \mathbb{R} \setminus \mathbb{N}$ on M. Such orders k are called *negative orders* if $k < 0$ and *fractional order* if $k \in \mathbb{R} \setminus \mathbb{Z}$. To define them, we use fractional powers for D^2. These powers have been studied in [See67] and many other papers. These operators are pseudo-differential operators, and many regularity results still hold if we replace differential operators by pseudo-differential operators of positive order. However, many proofs in L^p-theory are much more involved.

The spectrum of the Dirac operator D is discrete, and there are L^2-orthonormal smooth spinors $(\beta_j \,|\, j \in \mathbb{Z})$ and real numbers $(\lambda_j \,|\, j \in \mathbb{Z})$ such that the span of $(\beta_j \,|\, j \in \mathbb{Z})$ is L^2-dense in the smooth sections $\Gamma(M; S)$ and such that $D\beta_j = \lambda_j \beta_j$.

For finite sums $\sum a_j \beta_j \in \Gamma(M; S)$ and $m \in \mathbb{R}$ we define

$$|D|^m\left(\sum a_j \beta_j\right) = \sum a_j |\lambda_j|^m \beta_j \tag{3.2.6}$$

where the sum on the right hand side runs over all j with $\lambda_j \neq 0$.

THEOREM 3.2.7. *The operator $|D|^m$ extends to a continuous operator*

$$|D|^m : C^\infty(M; S) \to C^\infty(M; S).$$

Furthermore for $m \in \mathbb{Z}$, $k \in \mathbb{N}$, $k > m$, $1 < p < \infty$ the operator $|D|^m$ extends to a continuous operator

$$|D|^m : H^p_k(M; S) \to H^p_{k-m}(M; S).$$

Obviously, these statement hold if $m \in 2\mathbb{N}$, as then $|D|^m = (D^2)^{m/2}$. The global Schauder theory and the Global L^p-estimates also show that they hold if $m \in -2\mathbb{N}$.

However, the other cases of this theorem are quite involved. We omit the proofs and refer to Chapter XI of [Tay81], in particular Theorem 2.5. An alternative reference is §6 of [NS79].

Definition. We define for any $k \in \mathbb{R}$, $1 < p < \infty$

$$\|\psi\|_{\widetilde{H}_k^p} = \left\| |D|^k \psi \right\|_{L^p} + \|\pi\psi\|_1, \tag{3.2.8}$$

where π is the L^2-orthogonal projection to the kernel of D and where $\|\cdot\|_1$ is an arbitrary norm on the kernel. The space that is obtained as the closure of the smooth functions with respect to this norm is denoted by $\widetilde{H}_p^k(M;S)$.

We obtain from the previous Theorem:

COROLLARY 3.2.9. *Let M be compact with a Dirac bundle S. For any $k \in \mathbb{N}$ the $\widetilde{H}_p^k$-norms and H_p^k-norms are equivalent norms on $\Gamma(M;S)$.*

As we are only interested in norms up to equivalence we will write H_p^k for both norms from now on.

Again, we define the Sobolev space $H_k^p(M;S) = H_k^p$ as the completion of the smooth spinors with respect to this norm.

3.3 Sobolev embeddings

In Chapter 4 we will need several Sobolev embeddings. We begin with some preliminaries on the Green's function for $|D|$ which will be helpful to prove the Sobolev embeddings.

3.3.1 The Green's function of $|D|$

THEOREM 3.3.1. *Let (M,g) be a compact Riemannian manifold with a Dirac bundle S. Then there is a unique function G, the Green's function for the absolute value of the Dirac operator, with the following properties.*

(1) *$G : (M \times M) \setminus \delta_M \to \bigcup_{x,y} \mathrm{End}(S_x, S_y)$ is a smooth function and $G(x,y) \in \mathrm{End}(S_x, S_y)$. Here δ_M denotes the diagonal $\delta(M) := \{(m,m) \,|\, m \in M\}$ in $M \times M$.*

(2) *For any smooth spinor* φ *the integral*

$$A(\varphi)(x) = \int_M G(x,y)\varphi(y)\, dy$$

exists, $A(\varphi)$ *is smooth.*

(3) $|D|\, A(\varphi) = \varphi$ *if* φ *is* L^2 *orthogonal to the kernel of* D*.*

(4) $A(\varphi) = 0$ *if* φ *is in the kernel of* D*.*

Moreover G *satisfies the asymptotic bound*

$$|G(x,y)| \le C(M,g,\sigma)\, d(x,y)^{1-n}.$$

An elegant way to prove this theorem is with heat kernel methods and the Mellin transform. We will sketch the ideas here. For more details in [AF].

Sketch of Proof.

Let $(\varphi_i)_{i\in J}$ be an L^2-basis of $L^2(M;S)$, where φ_i is an eigenvalue of D to the eigenvalue λ_i. Let J_1 be the index set of all i with $\lambda_i \neq 0$.

For $x, y \in M$, $x \neq y$ one defines the series

$$G(x,y) := \sum_i |\lambda_i|^{-1} \varphi_i(x) \otimes \varphi_i(y)$$

and shows that it converges locally in the C^∞-topology. One then uses the relation (for $\lambda > 0$)

$$\lambda^{-1} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-t\lambda^2} t^{-1/2}\, dt,$$

and one obtains

$$G(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty k_t(x,y) t^{-1/2}\, dt,$$

where $k_t(x,y)$ is the integral kernel to the heat operator e^{-tD^2}. The kernel k_t is well-studied in the literature. In particular, (see e.g. [Dav87, DP89]) one knows the asymptotic estimate

$$|k_t(x,y)| \le C(M,g,W)\, t^{-n/2} e^{-d^2(x,y)/5t} \qquad \text{for} \quad t \le \operatorname{diam}(M)^2.$$

This gives an estimate

$$\left| \int_0^{\operatorname{diam}(M,g)^2} k_t(x,y) t^{-1/2}\, dt \right| \le C(M,g,W)\ d(x,y)^{1-n}.$$

On the other hand one can use Weyl's asymptotic formula and the Sobolev embedding theorem in order to obtain an upper bound for

$$\left| \int_{\mathrm{diam}(M,g)^2}^{\infty} k_t(x,y) t^{-1/2}\, dt \right|$$

and $|G(x,y)| \leq C(M,g,\sigma)\, d(x,y)^{1-n}$ follows. □

3.3.2 Sobolev embedding theorems

Let S be a Dirac bundle over a Riemannian manifold M.

THEOREM 3.3.2 (Sobolev embedding theorem I). *Let $k, s \in \mathbb{R}$, $k \geq s$ and $q, r \in (1,\infty)$ with*

$$\frac{1}{r} - \frac{s}{n} \geq \frac{1}{q} - \frac{k}{n}, \tag{3.3.3}$$

then $H_k^q(M;S)$ is continuously embedded into $H_s^r(M;S)$.

THEOREM 3.3.4 (Rellich-Kondrakov theorem). *Under the same conditions as in Theorem 3.3.2, but with strict inequality* (3.3.3) *and $k > s$ the inclusion $H_k^q(M;S) \hookrightarrow H_s^r(M;S)$ is a compact operator.*

THEOREM 3.3.5 (Sobolev embedding theorem II). *Suppose* $0 < \alpha < 1$, $m \in \{0,1\}$ *and*

$$\frac{1}{q} \leq \frac{k - m - \alpha}{n}. \tag{3.3.6}$$

Then $H_k^q(M;S)$ is continuously embedded into $C^{m,\alpha}(M;S)$.

The proofs of Theorems 3.3.2, 3.3.4 and 3.3.5 in the general case are not easy to prove. As we will only need some special cases, which will be easier to prove, we will only give a sketch of the proof here. In order to get the results in full generality, one covers the manifolds by finitely many normal coordinate neighborhoods $B_\varepsilon(p_i) \cong B_\varepsilon(0) \subset \mathbb{R}^n$ and trivializes the bundle with parallel transport along radial geodesics. One chooses a suitable partition of unity χ_i. One then has to show that:

$$\varphi \in H_k^q(M;S) \text{ in the sense of (3.2.8)}$$

$$\Longleftrightarrow$$

$$\forall i: \quad \chi_i \varphi|_{B_\varepsilon(p_i)} \in L^{k,q}(\mathbb{R}^n; S)$$

where $L^{k,q}(\mathbb{R}^n; S)$ are fractional Sobolev spaces as defined in Chapter VII of [Ada75]. Then [Ada75, Theorem 7.63 (d) and (e)] proves the local Sobolev inequalities

$$\|\chi_i\varphi\|_{L^{s,r}} \leq C \cdot \|\chi_i\varphi\|_{L^{k,q}},$$
$$\|\chi_i\varphi\|_{C^{m,\alpha}} \leq C \cdot \|\chi_i\varphi\|_{L^{k,q}}.$$

The global embedding theorems then follows by summing over i, and taking care of the extra terms caused by the derivatives of the cutoff functions. One obtains Theorems 3.3.2 and 3.3.5. With similar arguments one obtains the compactness statement in Theorem 3.3.4.

Now, we will present detailed proofs for the cases that we actually need. In the proofs we will apply the following inequality. See [LL96, Theorem 4.3] or [Lie83] for a proof.

THEOREM 3.3.7 (Hardy-Littlewood-Sobolev inequality)**.** *Let $p, r > 1$ and $0 < \alpha < n$ with $1/p + \alpha/n + 1/r = 2$. Let $f \in L^p(\mathbb{R}^n)$ and $h \in L^r(\mathbb{R}^n)$. Then there exists a constant $C(n, \alpha, p)$ such that*

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)|x-y|^{-\alpha} h(y)\, dx\, dy \right| \leq C(n, \alpha, p)\, \|f\|_{L^p} \|h\|_{L^r}.$$

This theorem can be applied to compact manifolds.

COROLLARY 3.3.8. *Let $p, r > 1$ and $0 < \alpha < n$ with $1/p + \alpha/n + 1/r = 2$, Let W be a vector bundle over the compact manifold (M, g). Let $f \in \Gamma_{L^p}(M; W)$, $h \in \Gamma_{L^r}(M; W)$. Let G be a continuous* $\mathrm{End}(W)$*-valued function on $(M \times M) \setminus \delta_M$ such that $|G(x, y)| \leq d(x, y)^{-\alpha}$. Then there exists a constant $C = C(\alpha, p, M, g, W)$ such that*

$$\left| \int_M \int_M f(x) G(x, y) h(y)\, dx\, dy \right| \leq C\, \|f\|_{L^p(M)} \|h\|_{L^r(M)}.$$

Proof of Theorem 3.3.2 for $k = 0$, $q \geq 2n/(n+1)$, $s = -1/2$, $r = 2$. We use the Sobolev norm as defined in subsection 3.2.4. Let φ be an L^q-spinor. We write $\varphi = \varphi_0 + \varphi_\perp$, where $\varphi_0 \in \ker D$ and $\varphi_\perp \perp \ker D$. As $\ker D$ is finite-dimensional

$$\|\varphi_0\|_{H^2_{-1/2}} \leq C\|\varphi\|_{L^q}.$$

On the other, the orthogonal complement satisfies

$$\begin{aligned}
\|\varphi_\perp\|^2_{H^2_{-1/2}} &= \big\|\, |D|^{-1/2}(\varphi_\perp)\, \big\|^2_{L^2} \\
&= (\, |D|^{-1}\varphi_\perp, \varphi_\perp\,) = \left| \int \int G^{|D|}(x, y)\varphi(x)\varphi(y)\, dx\, dy \right| \\
&\leq C \int \int d(x, y)^{1-n} |\varphi(x)|\, |\varphi(y)|\, dx\, dy \\
&\leq C\|\varphi\|^2_{L^q}
\end{aligned}$$

with $q = 2n/(n+1)$. □

Proof of Theorem 3.3.2 for $k = 1$, $q \geq 2n/(n+1)$, $s = 0$. We use the Sobolev norm as defined in subsection 3.2.3. At first, we cover M by finitely many open balls $B_\delta(p_i)$ of radius $\delta = \text{injrad}(M)/2$. We choose compact $K_i \subset B_\delta(p_i)$ such that $\bigcup K_i = M$ and a smooth partition of unity η_i such that $\operatorname{supp} \eta_i \subset B_\delta(p_i)$ and $\eta_i|_{K_i} \equiv 1$. Let φ be an H_1^q-spinor. Now, [GT77, Theorem 7.1] yields that any H_1^q-function supported in K_i satisfies

$$\|u\|_{L^r} \leq C\|du\|_{L^p}. \tag{3.3.9}$$

On each $B_\delta(p_i)$ we trivialize the spinor bundle via parallel transport along radial geodesics. It is then an easy calculation to prove that the above inequality on functions implies

$$\|\eta_i\varphi\|_{L^r} \leq C\|\nabla(\eta_i\varphi)\|_{L^p}.$$

Using $\sup_i \sup_{x \in M}(\text{grad}\eta_i)(x) < \infty$ this implies the desired inequality

$$\|\varphi\|_{L^r} \leq \|\varphi\|_{H_1^q}.$$

□

The compactness statements in Theorem 3.3.4 are then obtained with standard methods.

Proof of Theorem 3.3.5 for $k = 1$, $q > n$, $m = 0$. This proof runs in a manner analogous to the previous proof, but one has to replace inequality (3.3.9) by the embedding

$$H_{1,\text{comp}}^q \to C^{0,\alpha}$$

proved in [GT77, Section 7.8]. Here $H_{1,\text{comp}}^q$ denotes the compactly supported $H_1^q(B_\delta(p_i))$-functions. □

Chapter 4

The first Dirac eigenvalue in a conformal class

4.1 Preliminaries

Let M be a compact manifold, on which we fix a conformal class $[g_0]$ and a spin structure σ. For each metric $g \in [g_0]$ let $\lambda_1^+(g)$ be the smallest positive eigenvalue of the (classical) Dirac operator D on (M, g, σ). As in Chapter 2 we set

$$\lambda^+_{\min}(M, [g_0], \sigma) := \inf_{g\in[g_0]} \lambda_1^+(g) \operatorname{vol}(M, g)^{1/n}. \tag{4.1.1}$$

If the dimension is ≥ 3 or if D is invertible, then as explained in Theorem 2.6.1

$$\lambda^+_{\min}(M, [g_0], \sigma) \leq \lambda^+_{\min}(S^n), \tag{4.1.2}$$

where S^n carries the standard metric. The main result (Theorem 4.2.2) of this habilitation states that if strict inequality holds in (4.1.2), then the nonlinear partial differential equation

$$D\varphi = \lambda^+_{\min}|\varphi|^{2/(n-1)}\varphi \tag{4.1.3}$$

has a solution. The nonlinearity is critical in the sense that the exponents of the Sobolev embeddings involved are critical. The solution of this equation has two major applications. The first application tells us that the infimum in (2.4.1) is actually attained if we slightly enlarge the conformal class. In the following chapter we present a second application. If $\dim M = 2$, then we obtain an existence result for surfaces with constant mean curvature.

Remark. Similar critical nonlinearities have been the subject of intensive research. We will give two examples:

The first one is the Yamabe problem [LP87]. Yamabe, Trudinger and Aubin have shown that if the Yamabe invariant $\lambda_Y = \lambda_Y(M,[g])$ is smaller than the Yamabe invariant of a sphere with constant sectional curvature, then the Yamabe equation

$$\left(4\frac{n-1}{n-2}\Delta + \mathrm{scal}\right) u = \lambda_Y u^{p-1}$$

admits a positive smooth solution u. By deep results, explained e.g. in [LP87], it has been shown that any manifold M which is not conformal to the standard sphere actually satisfies

$$\lambda_Y(M,[g]) < \lambda_Y(S^n). \tag{4.1.4}$$

As a consequence, there is a metric with constant scalar curvature on M, conformal to g. If M is a spin manifold, Witten [Wit81, PT82] realized that one could simplify the arguments by using spinors, in particular the Schrödinger-Lichnerowicz formula and a Bochner-type argument. Witten's proof used analysis on asymptotically flat manifolds. In [AH05] we found a modification of Witten's arguments which yields a very short proof for inequality 4.1.4 which only needs analysis on compact manifolds.

The other problem that we want to mention is the Yamabe problem on CR-manifolds, studied by Jerison and Lee [JL89]. For any compact, orientable, strictly pseudo-convex $(2n+1)$-dimensional CR-manifold M, they construct an invariant $\lambda^{CR}(M)$ satisfying

$$\lambda^{CR}(M) \le \lambda^{CR}(S^{2n+1}). \tag{4.1.5}$$

Jerison and Lee prove that if strict inequality holds in (4.1.5), then M admits a pseudo-hermitian structure with constant (Webster or Fefferman) scalar curvature.

4.2 Singular metrics and main results

Let $D^{-1}(\psi)$ denote the preimage of ψ under D which is orthogonal to the kernel of D.

In order to formulate our main result, we have to introduce certain types of singularities.

Definition. Let $\mathcal{M}(g_0)$ be the set of metrics in $[g_0]$ with unit volume. We define the set of *generalized conformal metrics* $\overline{\mathcal{M}(g_0)}$ by

$$\overline{\mathcal{M}(g_0)} := \{g = f^{2/(n-1)}\cdot g_0 \mid \begin{array}{l} f \ge 0,\ f \in C^{1,\alpha}(M) \cap C^\infty(M \setminus f^{-1}(\{0\}))\ \forall \alpha \in (0,1), \\ \operatorname{supp} f = M,\ \mathrm{vol}(M,g) := \int f^{n/(n-1)} = 1\}. \end{array}$$

We say that g is *regular*, if $f^{-1}(\{0\}) = \emptyset$. Otherwise g is *singular* and $\mathcal{S}_g := f^{-1}(\{0\})$ is called the *singular set*.

Note that singular metrics are not complete.

For example, let $M \to S^2$ be a branched conformal covering of a Riemann surface M to S^2. Then the pullback of the standard metric on S^2 is a generalized conformal metric in the above sense.

For any generalized conformal metric g, we define the *smallest positive Dirac eigenvalue* on (M, g, σ) as

$$\lambda_1^+(g) := \inf \left\{ \frac{(\psi, \psi)_g}{(\psi, D_g^{-1}\psi)_g} \,\middle|\, \psi \in \mathrm{im}_{C^\infty} D_g, \quad \|\psi\|_{L^\infty} < \infty, \quad (\psi, D_g^{-1}\psi)_g > 0 \right\}. \tag{4.2.1}$$

Here $\mathrm{im}_{C^\infty} D_g$ is the image of the Dirac operator on $(M \setminus \mathcal{S}_g, g)$ acting on smooth spinors. The scalar product $(.,.)_g$ is the L^2-scalar product on spinors on $M \setminus \mathcal{S}_g$. Obviously, this definition coincides with the smallest positive Dirac eigenvalue in the ordinary sense if g is regular. In Lemma 4.6.1 we show that

$$\inf_{g \in \overline{\mathcal{M}(g_0)}} \lambda_1^+(g) = \inf_{g \in \mathcal{M}(g_0)} \lambda_1^+(g).$$

THEOREM 4.2.2. *Let M be a compact manifold of dimension ≥ 2 with a fixed conformal class $[g_0]$ and a spin structure σ. Assume that $\lambda_{\min}^+ = \lambda_{\min}^+(M, [g_0], \sigma) := \inf\{\lambda_1^+(g)\mathrm{vol}(M, g)^{1/n} \,|\, g \in \mathcal{M}(g_0)\}$ satisfies*

$$\lambda_{\min}^+(M, [g_0], \sigma) < \lambda_{\min}^+(S^n) = \frac{n}{2}\,\omega_n^{1/n}. \tag{4.2.3}$$

Then

(A) *there is a spinor field $\varphi \in C^{1,\alpha}(\Sigma M) \cap C^\infty(\Sigma(M \setminus \varphi^{-1}(0)))$ on (M, g_0) such that*

$$D_{g_0}\varphi = \lambda_{\min}^+ |\varphi|^{2/(n-1)}\varphi, \qquad \|\varphi\|_{2n/(n-1)} = 1, \tag{4.2.4}$$

(B) *there is a generalized conformal metric $g \in \overline{\mathcal{M}(g_0)}$ such that*

$$\lambda_1^+(g) = \lambda_{\min}^+.$$

It is not difficult to see that statement (B) follows directly from statement (A). If we have a solution as in (A), then we set $g_1 := f^{2/(n-1)} g_0$ with $f = \langle \varphi, \varphi \rangle$. Note that $\mathrm{vol}(M, g_1) = \int |\varphi|^{2n/(n-1)} = 1$, hence $g_1 \in \overline{\mathcal{M}(g_0)}$.

The transformation formula for the Dirac operator under conformal changes (Proposition 2.2.1) implies that there is a spinor φ_1 on (M, g_1, σ) such that

$$D_{g_1}\varphi_1 = \lambda_{\min}^+ \varphi_1, \qquad |\varphi_1|_{g_1} \equiv 1$$

on $M \setminus \mathcal{S}_{g_1}$. Then obviously, $\lambda_1^+(g_1) = \lambda_{\min}^+$.

Proof of (A)**.** For any $q \geq q_D := 2n/(n+1)$ we define

$$\mu_q := \sup \frac{(\psi, D^{-1}\psi)}{\|\psi\|^2_{L^q}}$$

where the supremum ranges over all smooth non-zero spinors on (M, g_0, σ) in the image of D. For $q > q_D$, the problem is "sub-critical" and we will see in Proposition 4.4.4 that there is a weak solution ψ_q of

$$D^{-1}\psi_q - \mu_q|\psi_q|^{q-2}\psi_q \in \ker D, \qquad \psi_q \in L^q(\mathcal{D}), \qquad \|\psi_q\|_{L^q} = 1. \tag{4.2.5}$$

Our Regularity Theorem (Theorem 4.4.3) will imply that the solution is $C^{0,\alpha}$.

By a straightforward calculation we see that the following duality principle holds.

LEMMA 4.2.6 (Duality principle)**.**
Let $p, q > 1$, $\lambda, \mu \in \mathbb{R}^+$ with $p^{-1} + q^{-1} = 1$ and $\lambda\mu = 1$.

(i) *If ψ satisfies* (4.2.5)*, then $\varphi := |\psi|^{q-2}\psi$ satisfies*

$$D\varphi = \lambda\, |\varphi|^{p-2}\varphi, \qquad \varphi \in L^p(\mathcal{D}), \qquad \|\varphi\|_{L^p} = 1 \tag{4.2.7}$$

(ii) *If φ satisfies* (4.2.7)*, then $\psi := \mu\, D\varphi$ satisfies* (4.2.5)*.*

We will study the behavior of the solutions $\varphi_q := D^{-1}\psi_q$ for $q \to q_D$. Theorem 4.4.5 shows that the solutions φ_q are uniformly bounded in L^∞. Applying the Regularity Theorem (Theorem 4.4.3) once again, we see that ψ_q is even uniformly bounded in $C^{0,\alpha}$, hence φ_q is uniformly bounded in $C^{1,\alpha}$. After taking a subsequence $q_i \to q_D$, the spinor fields φ_{q_i} converge to a $C^{1,\alpha}$-solution of (4.2.4). □

The structure of the chapter is as follows. In Section 4.3 we introduce a functional $\mathcal{F}^M_q$ whose critical points are solutions to (4.2.5). It remains to find conditions under which the supremum $\mathcal{F}^M_q$ is attained. Section 4.4 proves the regularity theorem. We obtain $C^{1,\alpha}$-solutions to the sub-critical problem and we see that (4.2.3) implies the existence of a solution to the critical problem. Section 4.5 deals with the singularities that may appear.

Remark. *The largest negative Dirac eigenvalue.* Denote the largest negative eigenvalue of the Dirac operator by $-\lambda_1^-(g)$, i.e. $-\lambda_1^-(g)$ is the negative eigenvalue closest to 0. All statements of the present article hold analogously if we replace the smallest positive eigenvalue $\lambda_1^+(g)$ by $\lambda_1^-(g)$. We omit the proofs as they are the same up to some sign changes.

4.3 The functional

In the present section we will work with a conformally invariant functional. We will not explicitly use results of section 2.3, but it useful to *think* about spinors in this conformally invariant way. We will keep to the convention that any spinor of conformal weight $-(n+1)/2$ is denoted by ψ. All other spinors will be denoted by φ.

For any $q \in [q_D, \infty)$ we define the functional

$$\mathcal{F}_q^M : (\mathrm{im}_{C^\infty} D_{g_0}) \setminus \{0\} \to \mathbb{R}, \quad \psi \mapsto \frac{(\psi, D^{-1}\psi)}{\|\psi\|_{L^q}^2}.$$

Here we choose the preimage $D^{-1}\psi$ of ψ orthogonal to the kernel of $\ker D$.

Then

$$\mu_q = \mu_q(M, g_0, \sigma) = \sup_{\psi \in (\mathrm{im}_{C^\infty} D_{g_0}) \setminus \{0\}} \mathcal{F}_q^M(\psi).$$

Because D_{g_0} has positive eigenvalues, we see that $\mu_q \in (0, \infty]$.

LEMMA 4.3.1. *Let $q \in [q_D, \infty)$. Then*

(1) $\mu_q < \infty$,

(2) $\mathcal{F}_q^M$ *extends to a differentiable functional on* $L^q(\mathrm{im}_{C^\infty} D_{g_0}) \setminus \{0\}$ *and the derivation is given by*

$$d\mathcal{F}_q^M(\psi)(\varphi) = \frac{2}{\|\psi\|_{L^q}^2} \int \langle D^{-1}\psi - \rho_{q,\psi} |\psi|^{q-2}\psi, \varphi \rangle, \tag{4.3.2}$$

where $\rho_{q,\psi} = \mathcal{F}_q^M(\psi) \|\psi\|_{L^q}^{2-q}$.

Proof. The Embedding Theorem 3.3.2 implies that for any q there is a constant C_q such that

$$\|\psi\|_{H^2_{-1/2}} \le C_q \, \| \, \psi \, \|_{L^q} \qquad \forall \psi \in \mathrm{im}_{C^\infty} D_{g_0}.$$

Hence

$$\frac{(\psi, |D|^{-1}\psi)}{\|\psi\|_{L^q}^2} \le C'$$

and hence (1). Similarly one proves that $\mathcal{F}_q^M$ extends to $L^q(\mathrm{im}_{C^\infty} D_{g_0}) \setminus \{0\}$.

Now take p with $p^{-1} + q^{-1} = 1$. If ψ is in L^q, $q \ge \frac{2n}{n+1}$, then $D^{-1}\psi$ is in H_1^q and hence in L^p. Also $|\psi|^{q-2}\psi \in L^p$. Hence the right hand side of (4.3.2) defines a continuous functional on L^q which we denote by $\varphi \mapsto \mathrm{RHS}_\psi(\varphi)$. Similarly one sees that

$$\mathcal{F}_q^M(\psi + \varphi) - \mathcal{F}_q^M(\psi) - \mathrm{RHS}_\psi(\varphi) \le o(\|\varphi\|_{L^q}),$$

hence $\mathcal{F}_q^M$ is Fréchet differentiable with derivative RHS_ψ. $\square$

PROPOSITION 4.3.3 (Properties of μ_q)**.** *The function* $[q_D, \infty) \to (0, \infty), \quad q \mapsto \mu_q$ *is*

(1) *non-increasing in* q*,*

(2) *is continuous from the right,*

(3) $\mu_2 = (\lambda_1^+(g_0))^{-1}$.

(4) μ_{q_D} *is conformally invariant, i. e. for* $g_1 \in [g_0]$

$$\mu_{q_D}(M, g_0, \sigma) = \mu_{q_D}(M, g_1, \sigma).$$

(5) $\mu_{q_D}(M, g, \sigma) \geq \mu_{q_D}(S^n, g_{\mathrm{can}})$ *if* D *is invertible or if* $n \geq 3$.

Here g_{can} is the metric on S^n of constant sectional curvature and volume 1.

Proof.
(1) This is evident as $q \mapsto \|\psi\|_{L^q}$ is nondecreasing,
(2) For a given $q \geq q_D$, we take a smooth spinor field ψ such that $\mathcal{F}_q^M(\psi) \geq \mu_q - \varepsilon$. Observe that for $q' \geq q$

$$\mathcal{F}_{q'}^M(\psi) = \frac{\|\psi\|_{L^q}}{\|\psi\|_{L^{q'}}} \mathcal{F}_q^M(\psi).$$

The function $q' \mapsto \|\psi\|_{L^{q'}}$ is continuous, hence if q' is sufficiently close to q, then

$$\mu_{q'} \geq \mathcal{F}_{q'}^M(\psi) \geq \mathcal{F}_q^M(\psi) - \varepsilon \geq \mu_q - 2\varepsilon.$$

Because $q \mapsto \mu_q$ is non-increasing, the statement follows.
(3) follows directly, by decomposing $L^2(\mathrm{im}_{C^\infty} D_{g_0})$ into eigenspaces for D.
(4) This follows from Proposition 2.2.1.
(5) This has been already explained in Section 2.6. □

In Proposition 4.4.4 we will see that the supremum defining μ_q is attained for $q > q_D$ by a $C^{0,\alpha}$-function which in turn implies that the function

$$[q_D, \infty) \to (0, \infty], \quad q \mapsto \mu_q$$

is also continuous from the left.

COROLLARY 4.3.4. *The smallest positive eigenvalue of the Dirac operator is bounded from below by* $\mu_{q_D}^{-1}$*:*

$$\lambda_1^+ \geq \mu_{q_D}^{-1}.$$

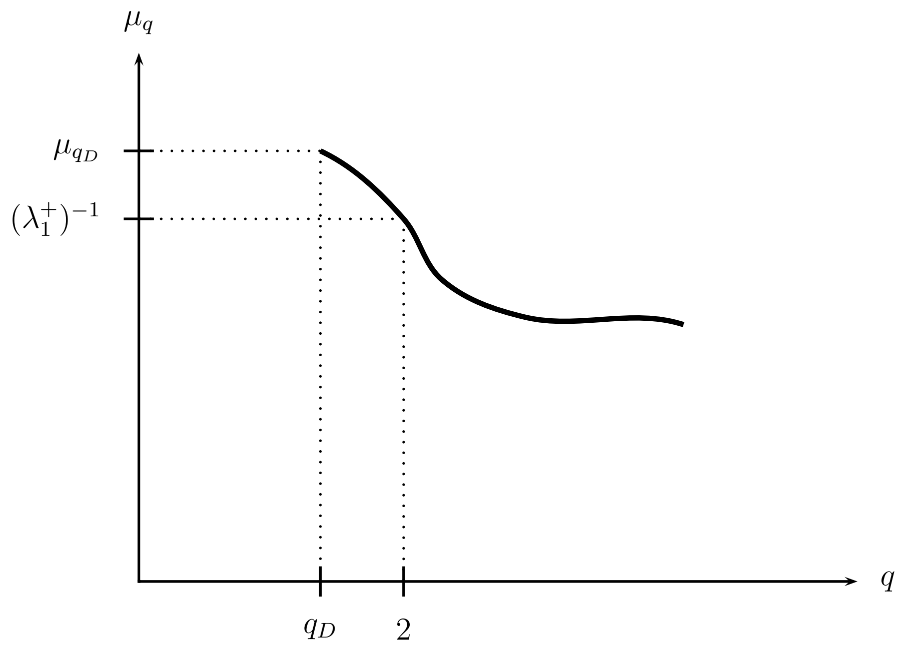


Figure 4.1: μ_q as a function of q

4.4 Solution of the equation

If the supremum of $\mathcal{F}_q^M$ is actually attained by a function ψ_q with $\|\psi_q\|_{L^q} = 1$, then because of (4.3.2) ψ_q is a solution of

$$D^{-1}\psi - \|\psi\|_{L^q}^{2-q}\mu_q \quad |\psi|^{q-2}\psi \in \ker D. \tag{4.4.1}$$

Obviously, $\mathcal{F}_q^M(r\psi) = \mathcal{F}_q^M(\psi)$ for all $r \in \mathbb{R}^+$. As a consequence, any solution can be rescaled to one with $\|\psi\|_{L^q} = 1$. Hence, we study

$$D^{-1}\psi - \mu_q|\psi|^{q-2}\psi \in \ker D, \qquad \|\psi\|_{L^q} = 1, \qquad \psi \in L^q(\mathcal{D}). \tag{4.4.2}$$

THEOREM 4.4.3 (Regularity theorem)**.** *Suppose that $\psi \in L^q$, $q \geq q_D$ is a solution of equations* (4.2.5)*. Suppose that there is an $r > q_D$ such that $\|\psi\|_{L^r} < \infty$. We choose $k, K > 0$ such that $\|\psi\|_{L^r} < k$ and $\mu_q \geq K$. Then for any $\alpha \in (0,1)$ there is a constant C depending only on (M, g, σ), q, r, K, k and α with*

$$\|\psi\|_{C^{0,\alpha}} \leq C \qquad \textit{and} \qquad \|D^{-1}\psi\|_{C^{1,\alpha}} \leq C$$

Proof. Without loss of generality we can assume $r < n$. We apply the Global L^p-estimates 3.2.3 to $\psi \in L^r$, and then the Sobolev embedding 3.3.2 and obtain

$$D^{-1}(\psi) \in H_1^r \hookrightarrow L^s$$

with $1-(n/r)=-n/s$, or equivalently $s=rn/(n-r)$. Hence, by equation 4.2.5, we know $\psi\in L^{s(q-1)}\hookrightarrow L^{s(q_D-1)}$. We set

$$r' := s(q_D-1) = \frac{n-1}{n+1}\frac{rn}{(n-r)}.$$

The inequality $r>q_D$ implies $r'>r$. By iterating this bootstrap argument we obtain higher and higher regularity for ψ. The function

$$r\mapsto \frac{n-1}{n+1}\frac{rn}{(n-r)}$$

tends to ∞ if r converges to n from below. This shows that $\psi\in L^r$ for all $r\in(0,\infty)$. Thus $\psi\in H_1^r$ for all $r\in(0,\infty)$ and by the Sobolev embedding theorem 3.3.5 one obtains $\psi\in C^{0,\alpha}$ for any $\alpha>0$. And finally by the Schauder estimates 3.1.17 $D^{-1}\psi\in C^{1,\alpha}$. The uniform upper bound C of the norms is now clear from the construction. □

Remark. Outside the zero locus $\psi^{-1}(0)$ we can continue the bootstrap argument and apply inductively the interior Schauder estimates Theorem 3.1.16. We conclude that ψ is smooth on $M\setminus\psi^{-1}(0)$. Similarly, if $p:=(q-1)^{-1}+1$ is an even integer, then $\varphi\mapsto|\varphi|^{p-2}\varphi$ is also smooth in 0, and hence ψ is smooth on M.

PROPOSITION 4.4.4. *For any $q>q_D$ the supremum μ_q is attained by a spinor field $\psi_q\in C^{0,\alpha}$ which is a solution of* (4.2.5).

Proof. Let ψ_i be a maximizing sequence for $\mathcal{F}_q^M$, i. e. $\mathcal{F}_q^M(\psi_i)\to\mu_q$. We may assume $\|\psi_i\|_{L^q}=1$. After taking a subsequence there is a $\psi_\infty\in L^q$ such that ψ_i converges weakly to ψ_∞ in L^q. Because the embedding $L^q\hookrightarrow H^2_{-1/2}$ is compact (Rellich-Kondrakov Theorem 3.3.4), we can again choose a subsequence, and we obtain, in addition to the weak convergence in L^q, strong convergence to ψ_∞ in $H^2_{-1/2}$. Hence,

$$\mu_q\le\limsup\mathcal{F}_q^M(\psi_i)\le\limsup\frac{\|\psi_i\|^2_{H^2_{-1/2}}}{\|\psi_i\|^2_{L^q}}\le\mathcal{F}_q^M(\psi_\infty)\le\mu_q.$$

As a consequence, we have equality in all inequalities, in particular $\|\psi_\infty\|_{L^q}=1$. By the variational formula ψ_∞ is a solution of

$$D^{-1}\psi_\infty-\mu_q|\psi_\infty|^{q-2}\psi_\infty\in\ker D.$$

Now the regularity theorem tells us that ψ_∞ is actually $C^{0,\alpha}$. □

THEOREM 4.4.5. *Let ψ be a solution of* (4.2.5) *with* $q \in (q_D, 2]$ *and* $\mu_q \geq \mu_{q_D}^{S^n} + \varepsilon$, $\varepsilon > 0$. *Then there is a constant* $C = C(M, g, \sigma, \varepsilon)$ *such that*

$$\|\psi\|_{C^0} < C.$$

Proof. Assume that such a constant does not exist. Then we find a sequence of solutions ψ_k to (4.2.5) with $q = q_k$, $\mu_q = \mu_k \geq \mu_{q_D}^{S^n} + \varepsilon$ and

$$\|\psi_k\|_{C^0} \to \infty. \tag{4.4.6}$$

Let us assume for a moment that $q_\infty := \limsup q_k > q_D$. In this case, we can choose a subsequence with $q_k \to q_\infty$. Then the regularity theorem (Theorem 4.4.3) applied to a real number $r \in (q_D, q_\infty)$, says that $\|\psi_k\|_{C^0}$ is bounded, in contradiction to (4.4.6). Hence, $\lim q_k = q_D$.

We now study

$$\varphi_k := \frac{1}{\mu_k} D^{-1}\psi_k,$$

which are solutions of the dual equation (4.2.7).

There is a sequence of points $s_k \in M$ with

$$m_k := |\varphi_k(s_k)| = \max\{\varphi_k(x) \,|\, x \in M\} \to \infty.$$

Since M is compact, we can assume, after passing to a further subsequence, that s_k converges to $p \in M$.

Now, we define rescaled geodesic normal coordinates $(\sigma_k)^{-1}$ via the formula

$$\sigma_k(x) = \exp_p\Big(\delta_k x + \exp_p^{-1}(z_k)\Big),$$

where $\delta_k = m_k^{2-p_k} \to 0$.

Then a straightforward calculation shows that

$$\widetilde{\varphi}_k(x) := m_k^{-1}\varphi_k \circ \sigma_k(x)$$

is a solution of

$$D_k\widetilde{\varphi}_k = \frac{1}{\mu_k}\,|\widetilde{\varphi}_k|^{p_k-2}\widetilde{\varphi}_k,$$

where D_k is the Dirac operator associated to the metric $g_k := \delta_k^{-2}\sigma_k^*(g)$. Writing the metric g on M in geodesic normal coordinates centered in p one immediately sees that on any closed ball the sequence g_k converges to the Euclidean metric in the C^∞-topology.

Note that with respect to g_k

$$\|\widetilde{\varphi}_k\|_{C^0} \le |\widetilde{\varphi}_k(0)| = 1.$$

Hence, we may apply the interior L^p- and Schauder-estimates 3.2.1 and 3.1.16 to conclude that

$$\|\widetilde{\varphi}_k\|_{C^{1,\alpha}} \le C(R), \qquad \forall k > k(R),$$

with constants $C(R)$ and $k(R)$.

Compare D with the Dirac operator D^{flat} on Euclidean $\mathbb{R}^n$ (see e.g. [Pfä02] or [AHM03] for explicit computations). We have

$$\|(D^{\mathrm{flat}} - D_k)\widetilde{\varphi}_k\|_{C^{0,\alpha}(B_R(0))} \le \tau_k \|\widetilde{\varphi}_k\|_{C^{1,\alpha}(B_R(0))},$$

with $\tau_k \to 0$.

We choose a sequence of radii $R_m \to \infty$. After passing to a diagonal sequence, we see that there is a spinor $\widetilde{\varphi}_\infty$ on $\mathbb{R}^n$, such that $\widetilde{\varphi}_k|_{B_R(0)}$ converges to $\widetilde{\varphi}_\infty|_{B_R(0)} \in C^{1,\alpha}(B_R(0))$.

For any $\varepsilon > 0$ and $R > 0$ there is a $k_0(R,\varepsilon)$ such that

$$\|\widetilde{\varphi}_k\|_{L^{p_k}(B_R(0))} \le 1 + \varepsilon$$

for all $k \ge k_0$. Because of the C^1-convergence $\tilde{\varphi}_k \to \tilde{\varphi}_\infty$, Fatou's lemma yields

$$\|\widetilde{\varphi}_\infty\|_{L^{q_D}(B_R(0))} \le 1$$

for any R, and finally for $R = \infty$. Then $\widetilde{\varphi}_\infty$ is a solution of

$$D^{\mathrm{flat}}\widetilde{\varphi}_\infty = \mu_{q_D}\, |\widetilde{\varphi}_\infty|^{p_D - 2}\widetilde{\varphi}_\infty,$$

and then $\widetilde{\psi}_\infty := \mu_{q_D}\, D^{\mathrm{flat}}\widetilde{\varphi}_\infty$ is a solution of

$$(D^{\mathrm{flat}})^{-1}\widetilde{\psi}_\infty - \mu_{q_D}|\widetilde{\psi}_\infty|^{q_D-2}\widetilde{\psi}_\infty \in \ker D.$$

We identify $\widetilde{\psi}_\infty$ via stereographic projection with an L^{q_D}-spinor $\widehat{\psi}_\infty$ on S^n with the identification as a spinor of weight $-(n+1)/2$. See section 2.3. Note that

$$\begin{aligned}
\|\widehat{\psi}_\infty\|_{L^{q_D}(S^n)} &= \|\widetilde{\psi}_\infty\|_{L^{q_D}(\mathbb{R}^n)} \le 1 \\
(D^{-1}\widehat{\psi}_\infty, \widehat{\psi}_\infty)_{S^n} &= (D^{-1}\widetilde{\psi}_\infty, \widetilde{\psi}_\infty)_{\mathbb{R}^n} = \mu_{q_D}\|\widetilde{\psi}_\infty\|^{q_D}_{L^{q_D}(\mathbb{R}^n)}, \\
\mu^{S^n}_{q_D} \ge \mathcal{F}^{S^n}_{q_D}(\widehat{\psi}_\infty) &= \mu_{q_D}\|\widetilde{\psi}_\infty\|^{q_D-2}_{L^{q_D}(\mathbb{R}^n)} \ge \mu_{q_D}
\end{aligned}$$

which is obviously a contradiction to our assumption $\mu_{q_D} \ge \mu^{S^n}_{q_D} + \varepsilon$. □

Proposition 4.4.7. *If there is a $q_0 > q_D$ and an $r > q_D$ such that for all $t \in (q_D, q_0)$ there is a solution ψ_t of equation* (4.2.5) *with $q = t$ such that $\|\psi_q\|_{L^r}$ is bounded by a constant C independent from t, then there is a sequence $t_i \to q_D$ such that ψ_{t_i} converges in the C^1-topology to a solution of equation* (4.2.5) *with $q = q_D$.*

Proof. For q sufficiently close to q_D, we know because of Proposition 4.3.3 that μ_q is bounded from below by a positive constant. Thus, we can apply the regularity theorem (Theorem 4.4.3) which tells us that (ψ_t) is uniformly bounded in $C^{0,\alpha}$. By elliptic regularity, $D^{-1}(\psi_t)$ is uniformly bounded in $C^{1,\alpha}$. Hence, for a sequence (t_i) with $t_i > q_D$, converging to q_D, the spinor fields $D^{-1}(\psi_{t_i})$ converge in the C^1-topology to a C^1-spinor field φ_{q_D}. Then $D\varphi_{q_D}$ is a solution of equation (4.2.5) with $q = q_D$. □

4.5 The size of the singular set

In this section we recall the weak Unique Continuation Property. We obtain directly that the set on which a solution of (4.4.1) vanishes does not contain any non-empty open set.

Theorem 4.5.1 (Weak Unique Continuation Property [BBMW, Theorem 2.1])**.** *Let P be a locally bounded function on a connected Riemannian manifold M and let φ be a solution of*

$$D\varphi = P \cdot \varphi$$

that vanishes on a nonempty open set. Then φ is identically 0*.*

Corollary 4.5.2. *The singular set of C^1-solutions to equation* (4.2.7) *does not contain any nonempty open set.*

In case that the function P is smooth, there is a stronger result by Christian Bär.

Theorem 4.5.3 (Nodal sets of Dirac Operators [Bär97])**.** *Let M^n be compact and connected and let φ be a solution of*

$$D\varphi = P\varphi$$

where P is a smooth endomorphism. Then the zero set of φ has at most Hausdorff dimension $n-2$. If $n = 2$, then the zero set is discrete.

In the case $n = 2$ this theorem implies that the zero set of a solution to (4.2.4) is discrete.

4.6 The smallest positive Dirac eigenvalue on singular spaces

As before we fix a spin structure σ and a conformal structure $[g_0]$ on a compact manifold M. Let $g = f^{4/(n-1)} \cdot g_0$ be a generalized conformal metric. We define the smallest positive Dirac eigenvalue on (M, g, σ) by (4.2.1). It is evident, that if g is regular (i. e. $f > 0$), then $\lambda_1^+(g)$ is the smallest positive eigenvalue of the Dirac operator.

LEMMA 4.6.1. *The following identities hold*

$$\inf_{g\in\mathcal{M}(g_0)} \lambda_1^+(g) = \inf_{g\in\overline{\mathcal{M}(g_0)}} \lambda_1^+(g) = \mu_{q_D}^{-1}.$$

Proof. Obviously,

$$\inf_{g\in\mathcal{M}(g_0)} \lambda_1^+(g) \geq \inf_{g\in\overline{\mathcal{M}(g_0)}} \lambda_1^+(g).$$

From the previous section we already know that $\lambda_1^+(g) \geq \mu_{q_D}^{-1}$ for all $\mathcal{M}(g_0)$. Let us show that the argument even holds on $\overline{\mathcal{M}(g_0)}$. For this we write $g = h^2 g_0$ on $M \setminus \mathcal{S}_g$, $u \in C^\infty(M \setminus \mathcal{S}_g)$. Using the conformal change formula we see that

$$\lambda_1^+(g) = \inf \left\{ \frac{(h^{-1}\psi, \psi)_{g_0}}{(\psi, D_{g_0}^{-1}\psi)_{g_0}} \,\middle|\, \psi \in \mathrm{im}_{C^\infty} D_{g_0}, \quad \|h^{-(n+1)/2}\psi\|_{L^\infty} < \infty, \quad (\psi, D_{g_0}^{-1}\psi)_{g_0} > 0 \right\}.$$

In particular all test spinors have finite $\|\psi\|_{L^\infty}$. Applying Hölder's inequality and using $\int h^n = 1$, we see that $(h^{-1}\psi, \psi)_{g_0} \geq \|\psi\|_{L^{q_D}}^2$. Hence,

$$\lambda_1^+(g) \geq \mu_{q_D}^{-1} \qquad \forall g \in \overline{\mathcal{M}(g_0)}.$$

On the other hand, let ψ be a spinor field such that $\mathcal{F}_{q_D}^M(\psi) \geq \mu_{q_D} - \varepsilon$ with arbitrary small $\varepsilon > 0$. We can assume that ψ is a smooth spinor field without zeros. Then for $g := |\psi|^{4/(n+1)} g_0$ we obtain $\lambda_1^+(g) \leq \mathcal{F}_{q_D}^M(\psi)^{-1}$. Hence

$$\inf_{g\in\mathcal{M}(g_0)} \lambda_1^+(g) \leq \mu_{q_D}^{-1}.$$

□

Remark. Much recent research deals with Laplacians acting on functions on singular spaces. The function Laplacian can be defined on a much larger category of spaces, i. e. on metric spaces carrying a measure with certain compatibility conditions. (See e.g. [Gro99],[KMS01], [KS01] and the references therein.)

Chapter 5

Spinorial Weierstrass representations of surfaces

5.1 Overview

The solution of equation (4.2.7) provides a strong tool for showing the existence of a new class of periodic constant mean curvature (=cmc) surfaces. Special cases of such surfaces have been studied before with the help of completely different techniques, e.g. [GB98]. The class of surfaces we obtain is much larger than those particular cases.

A complete description of all periodic cmc surfaces is also important for solving the periodic isoperimetric problem, a problem which is still unsolved until today [Ros01, Ros].

We will explain in this chapter that there is a natural one-to-one relationship between periodic cmc-surfaces into $\mathbb{R}^3$ and S^3 and critical points of $\mathcal{F}_q$, $q = 4/3$, $\dim M = 2$. Thus, we are interested in studying the critical points of $\mathcal{F}_q$. This is a quite involved problem. In this habilitation, we will only do the first step and study the suprema of $\mathcal{F}_q$. Applying the results of the previous chapter, we will see that any surface on which the spinorial positive mass conjecture holds, i.e. $(\lambda^+_{\min}) = (\sup \mathcal{F}_q)^{-1} < 2\sqrt{\pi}$ admits a conformal periodic branched immersion of constant mean curvature of its universal covering into $\mathbb{R}^3$ and S^3.

We will prove:

THEOREM 5.4.1. *If M is a compact spin manifold such that $\lambda^+_{\min}(M, [g_0], \sigma) < 2\sqrt{\pi}$, then there exists a branched conformal immersion $F : \widetilde{M} \to \mathbb{R}^3$ and a group homomorphism $h : \Gamma \to \mathbb{R}^3$ such that*

(1) $F(\gamma \cdot p) = F(p) + h(\gamma)$

(2) *If* M_1 *is a fundamental domain of* $\widetilde{M} \to M$, *then* $\operatorname{area}(F(M_1)) = 1$, *or more exactly* $\operatorname{area}(M_1, F^* g_{\mathrm{eucl}}) = 1$.

(3) *The mean curvature of* $F(M)$ *is constant and equals to* $\lambda^+_{\min}(M, [g_0], \sigma)$.

Similarly, we obtain immersions into S^3. We will identify S^3 with $\mathrm{SU}(2)$.

THEOREM 5.4.2. *If* $\lambda^+_{\min}(M, [g_0], \sigma) < 2\sqrt{\pi}$, *then there exists a conformal branched immersion* $F : \widetilde{M} \to \mathrm{SU}(2)$ *and a group homomorphism* $h : \Gamma \to \mathrm{SU}(2)$ *such that*

(1) $F(\gamma \cdot p) = h(\gamma) \cdot F(p)$.

(2) *If* M_1 *is a fundamental domain of* $\widetilde{M} \to M$, *then* $\operatorname{area}(F(M_1)) = 1$, *or more exactly* $\operatorname{area}(M_1, F^*(\kappa^{-1} g_{\mathrm{can}})) = 1$.

(3) *The mean curvature of* $F(M)$ *is constant and equals to* H *with*

$$H^2 + 1 = \lambda^+_{\min}(M, [g_0], \sigma)^2.$$

Definition. Branched conformal immersions $F : \widetilde{M} \to \mathbb{R}^3$ or S^3 satisfying (1) will be called *periodic conformal branched immersions*.

5.2 Killing spinors

C. Bär proved in [Bär98] that if N is a Riemannian manifold carrying a Killing spinor φ_K, and if N is an oriented hypersurface in M, then the restriction of φ_K to N satisfies an equation close to (4.2.7). This construction is central to the spinorial Weierstrass representation. Hence, before we turn to this representation itself, we want to study manifolds carrying Killing spinors.

5.2.1 Preliminaries on Killing spinors

Definition. A *Killing spinor* to the *Killing constant* α is a (classical) spinor φ such that

$$\nabla_X \varphi = \alpha X \cdot \varphi.$$

PROPOSITION 5.2.1 ([CGLS86, Prop. 5])**.** *If the* n*-dimensional manifold* (N, g) *carries a non-trivial Killing spinor to the Killing constant* α, *then* (N, g) *is an Einstein manifold with Ricci curvature*

$$\mathrm{Ric} = 4(n-1)\alpha^2 g.$$

In particular, α is real or purely imaginary. We will restrict to the case that α is real. Such Killing spinors are called *real Killing spinors*. As a consequence $\mathrm{Ric} \geq 0$. Then any real Killing spinor ψ satisfies

$$\partial_X |\psi|^2 = 2\mathrm{Re}\langle \nabla_X \psi, \psi \rangle = 2\alpha \mathrm{Re}\langle X \cdot \psi, \psi \rangle = 0.$$

Thus, its length is constant.

A description of complete manifolds of dimension $n \geq 4$ which carry a real Killing spinor is given in [Bär93]. One of the amazing facts in there is that in even dimension n, $n \neq 6$, only rescaled spheres carry Killing spinors with $\alpha \in \mathbb{R} \setminus \{0\}$. For more details on Killing spinors on quotients of spheres we refer to [Bär96]. However, note that [Bär96, Theorem 4] needs the additional assumption $n \neq 3$.

5.2.2 Killing spinors on 3-manifolds

The most important case for our application, $n = 3$, is special. We will now completely describe all complete 3-manifolds carrying a real Killing spinor. To the knowledge of the author, this habilitation gives the first complete classification in dimension 3.

As the Weyl tensor of any 3-manifold vanishes, we know that any 3-manifold carrying a Killing spinor φ to the Killing constant α has constant sectional curvature $4\alpha^2$. If $\alpha = 0$, i.e. if φ is parallel, then one directly sees that $N = \mathbb{R}^3/\Gamma$, where Γ is a discrete subgroup of translations. If $\alpha \in \mathbb{R} \setminus \{0\}$, then by rescaling we can achieve that $\alpha = \pm 1/2$. Then, obviously, N is a quotient of S^3.

We study the Lie group $\mathrm{SU}(2)$ which is diffeomorphic to S^3.

LEMMA 5.2.2. *Let G be a Lie group with a bi-invariant Riemannian metric. Then the Levi-Civita-connection satisfies for any left invariant vector fields X and Y*

$$\nabla_X Y = \frac{1}{2}[X, Y]. \tag{5.2.3}$$

Proof. There is a unique connection ∇ such that equation 5.2.3 holds. One easily shows that this connection is metric and torsion free. □

Now, let $G = \mathrm{SU}(2)$, equipped with the bi-invariant metric and orientation such that

$$e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

is a positively oriented, orthonormal basis of the Lie algebra. We have

$$[e_1, e_2] = 2e_3, \quad [e_2, e_3] = 2e_1, \quad [e_3, e_1] = 2e_2.$$

There is a unique spin structure. We choose the spinor representation such that Clifford multiplication with $e_1 \cdot e_2 \cdot e_3$ is the identity on spinors. Let φ be a left invariant spinor, and we extend e_i to left invariant vector fields. Then, the local formula for the connection on the spinor bundle implies

$$\nabla_{e_i}\varphi = \frac{1}{4}\sum_{j,k}\Gamma_{ij}^k e_j \cdot e_k \cdot \varphi.$$

Because of the previous lemma

$$\Gamma_{ij}^k = \langle \nabla_{e_i} e_j, e_k\rangle = \varepsilon_{ijk} := \begin{cases} +1 & \text{if } (ijk) \text{ is an even permutation} \\ -1 & \text{if } (ijk) \text{ is an odd permutation} \\ 0 & \text{if } i,\ j \text{ and } k \text{ are not pairwise different} \end{cases}.$$

Hence

$$\nabla_{e_i}\varphi = -\frac{1}{2}\, e_i \cdot \varphi,$$

i.e. any left invariant spinor on SU(2) is a spinor to the Killing constant $-1/2$. In particular, SU(2) with the above metric is isometric to S^3 with the standard metric. On the other hand, a Killing spinor to the constant $\alpha = -1/2$ on a connected manifold is uniquely characterized by its values at one single point. Hence, the space of Killing spinors to the constant $\alpha = -1/2$ has complex dimension 2. Thus, the Killing spinors on SU(2) to the Killing constant $-1/2$ are exactly the left invariant spinors.

In a completely analogous way one proves that right invariant spinors coincide with Killing spinors to the Killing constant $1/2$. Hence, we have proven the following theorem.

THEOREM 5.2.4. *We identify $S^3 = \mathrm{SU}(2)$ as above. Then, left invariant spinors are exactly the Killing spinors to the constant $-1/2$. Right invariant spinors are exactly the Killing spinors to the constant $+1/2$.*

We obtain the following classification of 3-manifolds carrying a Killing spinor to the constant $\alpha \in \mathbb{R} \setminus \{0\}$. As before, it suffices to deal with $\alpha = \pm 1/2$.

Recall that $(A_1, A_2) \in \mathrm{SU}(2) \times \mathrm{SU}(2)$ acts on $\mathrm{SU}(2) = S^3$ via $B \mapsto A_1 B A_2^{-1}$. Note that if Γ is a freely and properly discontinuously acting subgroup of $\mathrm{Isom}^+(S^3)$, then spin structures on S^3/Γ correspond to homomorphisms $\Gamma \to \mathrm{Spin}(4) = P_{\mathrm{Spin}}(S^3)$ such that

$$\begin{array}{ccl} & & P_{\mathrm{Spin}}(S^3) = \mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2) \\ & \nearrow & \downarrow \\ \Gamma & \to & P_{\mathrm{SO}}(S^3) = \mathrm{SO}(4) \end{array}.$$

COROLLARY 5.2.5 (Classification of 3-mfds with Killings spinors, $\alpha \in \mathbb{R} \setminus \{0\}$).
(1a) *If a complete Riemannian spin 3-manifold N carries a Killing spinor φ to the constant $\alpha = -1/2$, then there is a discrete subgroup Γ such that there is an orientation preserving isometry $N \to \Gamma\backslash\mathrm{SU}(2)$. The spin structure is given by the homomorphism $\Gamma \to \mathrm{SU}(2) \times \mathrm{SU}(2)$, $\gamma \mapsto (\gamma, 1)$.*
(1b) *Let Γ be a discrete subgroup of $\mathrm{SU}(2)$. We equip $N := \Gamma\backslash\mathrm{SU}(2)$ with the spin structure given by the homomorphism $\Gamma \to \mathrm{SU}(2) \times \mathrm{SU}(2)$, $\gamma \mapsto (\gamma, 1)$. Then on N there is a complex 2-dimensional space of Killing spinors to the Killing constant $-1/2$.*
(2a) *If a complete Riemannian spin 3-manifold N carries a Killing spinor φ to the constant $\alpha = 1/2$, then there is a discrete subgroup Γ such that there is an orientation preserving isometry $N \to \mathrm{SU}(2)/\Gamma$. The spin structure is given by the homomorphism $\Gamma \to \mathrm{SU}(2) \times \mathrm{SU}(2)$, $\gamma \mapsto (1, \gamma)$.*
(2b) *Let Γ be a discrete subgroup of $\mathrm{SU}(2)$. We equip $N := \mathrm{SU}(2)/\Gamma$ with the spin structure given by the homomorphism $\Gamma \to \mathrm{SU}(2) \times \mathrm{SU}(2)$, $\gamma \mapsto (1, \gamma)$. Then on N there is a complex 2-dimensional space of Killing spinors to the Killing constant $1/2$.*

Note that this corollary only holds for the choice of orientation and spinor representation described above. For other choices, some signs have to be adjusted.

Remark. The discrete subgroups of SU(2) are classified in [Wol67, Page 87–88, Theorem 2.6.7] [1]. They are conjugated to one of the groups in the following list.

(1) The cyclic group $\mathbb{Z}_m$ of order m, i.e. the subgroup generated by
$$\begin{pmatrix} \exp(2\pi i/m) & 0 \\ 0 & \exp(-2\pi i/m) \end{pmatrix}.$$

(2) The binary isometry group D^*_m of an m-gon: Let D^+_m be the group of orientation preserving isometries of a regular m-gon in $\mathbb{R}^3$, then one defines D^*_m as $D^*_m := \Theta^{-1}(D^+_m)$, where $\Theta : \mathrm{SU}(2) \to \mathrm{SO}(3)$ is the universal covering map.

(3) The binary tetrahedral group T^* defined as $T^* := \Theta^{-1}(T^+)$ where T^+ is the orientation preserving isometry group of a tetrahedron.

(4) The binary octahedral group O^* defined as $O^* := \Theta^{-1}(O^+)$ where O^+ is the orientation preserving isometry group of an octahedron.

(5) The binary icosahedral group I^* defined as $I^* := \Theta^{-1}(I^+)$ where I^+ is the orientation preserving isometry group of an icosahedron.

[1] Thanks to T. Friedrich [Fri80]

COROLLARY 5.2.6 ([Hij86a]). *If a complete Riemannian spin 3-manifold N carries both a Killing spinor to the Killing constant $1/2$ and to the Killing constant $-1/2$, then N is isometric to S^3.*

Note that according to Friedrich's estimate, we know the following. Let N be a compact Riemannian spin 3-manifold with scal ≥ 6. Then, all eigenvalues of D^2 are greater than or equal to $9/4$.

COROLLARY 5.2.7 ([Hij86a]). *Let N be a compact Riemannian spin 3-manifold with* scal ≥ 6. *If the smallest eigenvalue of D^2 is $9/4$ and has multiplicity at least 3, then N is isometric to S^3. If the smallest eigenvalue is $9/4$ with multiplicity at least 1, then N is necessarily of the form described in Theorem 5.2.4 (1a/b) or (2a/b).*

5.3 The spinorial Weierstrass representation

5.3.1 Historical background

In this section we want to explain, how a branched conformal immersion $M \to N$ induces a spinor on M. Most of the results in this section are well-known, however it is unclear to the author, where those results appeared for the first time.

The fact that a cmc-surface in $\mathbb{R}^3$ yields a spinor satisfying equation (4.2.7) is a modification of results by Weierstrass and has already been known for a long time. Some ideas seem to go back to the work of Eisenhart (1909). Abresch presented it in a talk in Luminy, which was cited by many people, but — unfortunately — was never published. Other references are [Tai97a, Tai97b, Ric97] and works by Pinkall, Kamberov, Bobenko and their collaborators.

The possible target manifold N was extended to arbitrary manifolds carrying a Killing spinor by Christian Bär in [Bär98]. In particular, this is very helpful for studying cmc-immersions into S^3. The main emphasis in Bär's article was the improvement of extrinsic eigenvalue estimates. We will recall this construction in a slightly modified version in Subsection 5.3.2. If the hypersurface has constant mean curvature, then one can transform the induced spinor into a spinor that satisfies even the simpler equation (5.3.4) (see Theorem 5.3.3, Part (b) and Theorem 5.3.6, Part (b).)

For our application, it is important that any solution of equation 4.2.7 is obtained from a cmc-immersion of the universal covering into $\mathbb{R}^3$. For this result, we refer to [Fri98], but a similar statement is contained in [KS96]. Similarly, L. Voss [Vos99] and independently B. Morel [Mor02] have worked out theorems that tell us that analogous statement holds for immersions into S^3 and H^3.

Our presentations follows closely to [Bär98] and [Fri98].

The spinorial Weierstrass representation also plays a central role in M.U. Schmitt's preprint [Sch02] where progress towards the Willmore conjecture was achieved. His work is a very rich source of results on Dirac operators on compact surfaces, in particular on 2-dimensional tori.

We also want to add that Hitchin [Hit90] has developped an SU(2)-gauge-theoretical approach for harmonic maps from a compact Riemann surface M into $S^3 = \mathrm{SU}(2)$. In particular, he proves the existence of new minimal surfaces $T^2 \to S^3$ with the help of the technique of spectral curves which he develops in there. Another relation of our result to Hitchin's paper is that the Gauss map of any period cmc-immersion $M \to \mathbb{R}^3$ is a harmonic map into S^2. An extension of Hitchin's methods could be used by E. Carberry for the construction of minimal immersions of T^2 into S^3 of arbitrary spectral genus [Car02]. Similar results for cmc-immersions into $\mathbb{R}^3$ were obtained in [EKT].

5.3.2 Isometric immersions

And this and the following subsection we want to describe how any conformally immersed hypersurface in a manifold carrying a non-trivial Killing spinor induces a solution to the equation

$$D\varphi = H|\varphi|^{p_D-1}\varphi \qquad p_D = 2m/(m-1).$$

The converse of this for $m = 2$ will then be the subject of Subsection 5.3.4.

Let N be an $m+1$-dimensional spin manifold which carries a Killing spinor with Killing constant $\alpha \in \mathbb{R}$. Let M be an oriented m-dimensional manifold, and let $F : M \to N$ be an isometric immersion.

For simplicity we restrict to the case m even, but the statement is similar for m odd.

The normal vector of M as a submanifold of N will be denoted by $\mathbf{n}$, and the second fundamental form (with values in the normal bundle) by II. Then $H \cdot \mathbf{n} := \frac{1}{m} \operatorname{tr} \mathrm{II}$ is the mean curvature vector field and H is the mean curvature function.

We will now explain, that the immersion F induces a spin structure on M. The chosen spin structure on N is given by a pair $(P_{\mathrm{Spin}}N, \varphi)$ where $P_{\mathrm{Spin}}N$ is a $\mathrm{Spin}(m+1)$-principal bundle and $\varphi : P_{\mathrm{Spin}}N \to P_{\mathrm{SO}}N$ is a $\mathrm{Spin}(m+1) \to \mathrm{SO}(m+1)$ equivariant map.

The map F induces a mapping

$$\begin{aligned} F_* : P_{\mathrm{SO}}M &\to P_{\mathrm{SO}}N \\ (e_1, \dots, e_m) &\mapsto (\mathbf{n}, F_*e_1, \dots, F_*e_m). \end{aligned}$$

We define a Spin$(m+1)$-principal bundle over M by

$$P_{\mathrm{Spin}}M := \{(q,A) \in P_{\mathrm{Spin}}N \times P_{\mathrm{SO}}M \mid \varphi(q) = F_*(A)\}$$

and with the natural maps the following diagram commutes

$$\begin{tikzcd}
P_{\mathrm{Spin}}M \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & P_{\mathrm{Spin}}N \arrow[d, "\varphi"] \\
P_{\mathrm{SO}}M \arrow[r, "F_*"] & P_{\mathrm{SO}}N
\end{tikzcd}$$

where π_1 and π_2 are the projections to the first and second component. Hence, $(P_{\mathrm{Spin}}M, \pi_2)$ defines a spin structure on M, the *pulled-back spin structure on M*.

Remark. Two immersions $F_1, F_2 : M \to N$ are called *regularly homotopic* if there is a homotopy $H : M \times [0,1] \to N$ from $F_1 = H(\,.\,,0)$ to $F_2 = H(\,.\,,1)$ such that $H(\,.\,,t)$ is an immersion or all t. If F_1 and F_2 are regularly homotopic, then the pulled-back spin structures coincide. Vice versa, in many cases, including the case $m = 2$, $N = \mathbb{R}^3$ and the case $m = 2$, $N = S^3$ the induced spin structures are equivalent if and only if F_1 and F_2 are regularly homotopic. (See [JT66]. Alternatively it can be derived from Gromov's h-principle, see e.g. [EM02].)

We will now describe how to pull-back spinors. Let $\sigma_{m+1} : \mathrm{Spin}(m+1) \to \mathrm{End}(\Sigma_{m+1})$ be the spinor representation. For simplicity, we will now restrict to the case that m is even. Then σ_{m+1} restrict to the spinor representation $\sigma_m : \mathrm{Spin}(m) \to \mathrm{End}(\Sigma_m)$, $\Sigma_m = \Sigma_{m+1}$. For odd m several small modifications have to be done.

As usual we have

$$\Sigma M := P_{\mathrm{Spin}}M \times_{\sigma_m} \Sigma_m, \qquad \Sigma N := P_{\mathrm{Spin}}N \times_{\sigma_{m+1}} \Sigma_{m+1}.$$

The map $P_{\mathrm{Spin}}M \to P_{\mathrm{Spin}}N$ induces an isomorphism

$$A : \Sigma M \to \Sigma N|_M.$$

We will usually identify via A.

Because of $\Sigma N = P_{\mathrm{Spin}}N \times_{\mathrm{Spin}(m+1)} \Sigma_{m+1}$ we can write a spinor on N locally in the form $[q_N, \sigma]$, where q_N is a local section of $P_{\mathrm{Spin}}N$ and σ is a local Σ_{m+1}-valued function. We can even choose q_N and σ such that $q_N|_{\mathrm{image}F}$ lies in $F_*(P_{\mathrm{Spin}}M)$ i.e.

there is a local section q_M of $P_{\mathrm{Spin}}M$ with $F_* \circ q_M = q_N \circ F$. We now define the pullback of Ψ to be

$$F^*\Psi := [q_M, \sigma \circ F]\,.$$

This definition is independent from the choice of the neighborhood and from the choice of q_N. Hence, it is globally well-defined.

PROPOSITION 5.3.1. *For any $i = 1, \ldots, m$ we have*

$$F^*\left(\nabla^{\Sigma N}_{e_i}\Psi\right) = \nabla^{\Sigma M}_{e_i}F^*\Psi + \frac{1}{2}\sum_{j=1}^{m} e_j \cdot (\mathrm{II}(e_i, e_j)) \cdot F^*\Psi.$$

Proof. We choose a local section q_N of $P_{\mathrm{Spin}}N$, and a local section q_M with $F_* \circ q_M = q_N \circ F$. Again, we write $(e_0, e_1, \ldots, e_m) = \varphi \circ q_N$. Then, we obtain $F^*\varphi \circ q_M = (e_1, \ldots, e_m)$.

Let Γ^k_{ij} resp. $\tilde{\Gamma}^k_{ij}$ be the Christoffel symbols of $\varphi \circ q_N$ resp. $F^*\varphi \circ q_M$. For any $i, j, k \in \{1, \ldots, m\}$ we have

$$\Gamma^k_{ij} = \tilde{\Gamma}^k_{ij} \qquad \Gamma^0_{ij}\, e_0 = -\Gamma^j_{i0}\, e_0 = \mathrm{II}(e_i, e_j) \qquad \Gamma^0_{i0} = 0.$$

Writing the Dirac operator in local coordinates we obtain for $i = 1, 2$

$$\begin{aligned} F^*\left(\nabla^{\Sigma N}_{e_i}[q_N, \sigma]\right) &= \left[q_M, \partial_{e_i}\sigma \circ F + \frac{1}{4}\sum_{j,k=0}^{m} \Gamma^k_{ij} E_j \cdot E_k \cdot \sigma \circ F\right] \\ &= \nabla^{\Sigma M}_{e_i} F^*[q_N, \sigma] + \frac{1}{2}\sum_{j=1}^{m} e_j \cdot (\mathrm{II}(e_i, e_j)) \cdot [q_N, \sigma]. \end{aligned}$$

Here $E_0, \ldots, E_m$ denotes the canonical basis of $\mathbb{R}^{m+1}$. □

PROPOSITION 5.3.2.

$$F^*\left(D^{\Sigma N}\Psi - \mathbf{n} \cdot \nabla^{\Sigma N}_{\mathbf{n}}\Psi\right) = D^{\Sigma M}F^*\Psi - \frac{m}{2}\, F^*\left(H\mathbf{n} \cdot \Psi\right)$$

Proof. This proposition follows from the previous one, by Clifford multiplication with e_i and summation over $i = 1, \ldots, m$. □

We obtain a theorem. Part (a) is a modification of a statement in [Bär98]. If $\alpha \neq 0$ and if the mean curvature is constant we will use a rotation to obtain a nicer form (Part (b)).

THEOREM 5.3.3. *We assume that the $m+1$-dimensional spin manifold (N, g) carries a non-trivial Killing spinor Ψ, $|\Psi| \equiv 1$ to the Killing constant $\alpha \in \mathbb{R}$. Let $M \to N$ be the immersion of an m-dimensional oriented manifold M into N and let D be the Dirac operator on M with respect to the spin structure pulled-back via F.*

(a) *If $\alpha = 0$ then the spinor*

$$\psi = \frac{1}{\sqrt{2}}\Big(F^*\Psi + \mathbf{n} \cdot F^*\Psi\Big)$$

is a solution of

$$D^{\Sigma M}\psi = \frac{m}{2}H\psi \qquad |\psi| \equiv 1.$$

(b) *If α is any real number and if H is constant, then there is $z = a + ib \in \mathbb{C}$, $|z| = 1$, such that*

$$\varphi = aF^*\Psi + b\,\mathbf{n} \cdot F^*\Psi$$

is a solution of

$$D^{\Sigma M}\varphi = \frac{m}{2}\sqrt{H^2 + 4\alpha^2}\,\varphi, \qquad |\varphi| \equiv 1. \tag{5.3.4}$$

Proof of Theorem 5.3.3. As Ψ is a Killing spinor we have

$$\nabla_V^{\Sigma N}\Psi = \alpha V \cdot \Psi \quad \text{for all vector fields } V.$$

As a consequence

$$D^{\Sigma N}\Psi - \mathbf{n} \cdot \nabla_{\mathbf{n}}^{\Sigma N}\Psi = -m\alpha\Psi.$$

Applying the previous proposition we obtain

$$\begin{aligned} D^{\Sigma M}F^*\Psi &= F^*\left(\frac{m}{2}H\mathbf{n} \cdot \Psi - m\alpha\Psi\right) \\ &= \frac{m}{2}H\mathbf{n} \cdot F^*\Psi - m\alpha F^*\Psi \end{aligned}$$

An easy calculation gives $D^{\Sigma M}\mathbf{n}\cdot = -\mathbf{n} \cdot D^{\Sigma M}$, and we obtain for

$$\psi := \frac{1}{\sqrt{2}}\Big(F^*\Psi + \mathbf{n} \cdot F^*\Psi\Big)$$

the equation

$$D^{\Sigma M}\psi = \frac{m}{2}\Big(H + 2\alpha\mathbf{n} \cdot \Big)\psi, \tag{5.3.5}$$

which provides (a), i.e. the claimed equation if $\alpha = 0$.

If $\alpha \in \mathbb{R}$ is arbitrary and H is constant, then we set

$$\varphi = aF^*\Psi + b\,\mathbf{n} \cdot F^*\Psi$$

for $a, b \in \mathbb{R}$ that will be chosen later

$$\begin{aligned} D^{\Sigma M}\varphi &= aD^{\Sigma M}\psi + bD^{\Sigma M}\,\mathbf{n} \cdot \psi \\ &= \frac{m}{2}(-2\alpha a + bH)F^*\Psi + \frac{m}{2}(aH + 2\alpha b)\mathbf{n} \cdot F^*\Psi. \end{aligned}$$

Our goal is to choose a and b such that $D^{\Sigma M}\varphi = \sqrt{H^2 + 4\alpha^2}\,\varphi$. In order to achieve this, we use that the subalgebra $\mathbb{R} \oplus \mathbb{R}\mathbf{n}$ of the Clifford algebra is isomorphic to $\mathbb{C}$. We set $z = a + bi \in \mathbb{C}$. An easy calculation in $\mathbb{C}$ shows that $D^{\Sigma M}\varphi = \sqrt{H^2 + 4\alpha^2}\varphi$ iff

$$z = \frac{-2\alpha + Hi}{\sqrt{H^2 + 4\alpha^2}}\,\overline{z}.$$

This is satisfied if $2 \arg z = \arg(-2\alpha + Hi)$ and $|z| = 1$. We obtain (b). □

Definition. The spinor ψ resp. φ provided by the above theorem will be called the *spinor induced by the immersion F*.

5.3.3 Conformal immersions with possible branching points

As before, let N be an $m+1$-dimensional spin manifold which carries a Killing spinor with Killing constant $\alpha \in \mathbb{R}$. However, we release the condition that $F : (M, g_0) \to (N, g_N)$ is an isometric immersion. We only claim that it is conformal. If $m = 2$ we also admit a set of *branching points* of odd order.

Outside S we write $g = F^*(g_N) = f^2 g_0$, $f^2 = |dF|^2_{g_0} \in C^\infty(M)$, $f \in C^\infty(M \setminus S)$. Then we can apply Proposition 2.2.1 and transform the results of the previous section. We set $A_2 := A_1 \circ A$, where A denotes as before the isomorphism $\Sigma N \to \Sigma M|_N$. We directly obtain

THEOREM 5.3.6. *We assume that the spin manifold (N^{m+1}, g) carries a non-trivial Killing spinor Ψ, $|\Psi| \equiv 1$ to the Killing constant $\alpha \in \mathbb{R}$. Let $(M^m, g_0) \to (N, g)$ be a conformal immersion into N and let D be the Dirac operator on M with respect to g_0 and the spin structure pulled-back via F. Let m be even.*

(a) *If $\alpha = 0$ then the spinor*

$$\frac{1}{\sqrt{2}} A_2^{-1}\Big(F^*\Psi + \mathbf{n} \cdot F^*\Psi\Big)$$

is a solution of

$$D\psi = H|\psi|^{\frac{2}{m-1}}\,\psi \qquad |\psi| \equiv |dF|_{g_0}^{\frac{n-1}{2}}.$$

(b) *If α is any real number and if H is constant, then there is $z = a + ib \in \mathbb{C}$, $|z| = 1$, such that*

$$\varphi = A_2^{-1}\left(aF^*\Psi + b\,\mathbf{n}\cdot F^*\Psi\right)$$

is a solution to

$$D\varphi = \sqrt{H^2+4\alpha^2}\,|\varphi|^{\frac{2}{n-1}}\,\varphi, \qquad |\varphi| \equiv |dF|_{g_0}^{\frac{n-1}{2}}.$$

If $m = 2$, we also admitted branching points of odd order. A priori, φ and ψ are only defined outside S. However, if the order of the branching points is odd, then the induced spin structure (which a priori is only defined on $M \setminus S$) extends to M. Furthermore, we can extend φ and ψ by 0 in the branching points of F. Hence, the spinors φ and ψ vanish exactly in the branching points of F. As $m = \dim M = 2$, the regularity results in Section 4.4 imply that φ and ψ are smooth in the branching points.

5.3.4 From a spinor to an immersion

It is a natural question to ask whether any solution of

$$D\varphi = \lambda|\varphi|^{p_D-2}\,\varphi \qquad p_D = 2m/(m-1) \tag{5.3.7}$$

on M^m is induced from a conformal immersion into an $m+1$-dimensional manifold carrying a Killing spinor. We will show now that if $m = 2$ and if M^m is simply connected, then any solution arises in this way. Thus, let M^m, $m = 2$, be a simply connected surface, and assume that the 3-manifold N carries a fixed Killing spinor to the Killing constant $\alpha \in \mathbb{R}$. Any immersion of M into N can be lifted to the universal covering of N. Hence, without loss of generality, we can assume that N is simply connected, i.e. it is the simply connected manifold $\mathcal{M}(\kappa)$ of constant sectional curvature $4\alpha^2$.

PROPOSITION 5.3.8. *Let (M, g_0) be a simply connected surface, and let φ be a solution to the non-linear spinor equation*

$$D\varphi = \lambda|\varphi|^2\varphi \qquad \lambda \in \mathbb{R}$$

Then if $\lambda^2 \geq \kappa \geq 0$, then there is a branched conformal immersion $F : M \to \mathcal{M}(\kappa)$ with constant mean curvature $\pm\sqrt{\lambda^2-\kappa}$, such that $|dF| = |\varphi|^2$ and φ is the spinor induced by F.

Proof. At first we assume that F is isometric and $|\varphi| \equiv 1$. In the case $\alpha = \kappa = 0$ the statement of the proposition has been proven in [KS96] and [Fri98]. For positive κ one has to do some modifications. As a first step we set

$$\vartheta = a\varphi + b\,\mathrm{vol}_M \cdot \varphi$$

for a and b chosen similarly as in the proof of Theorem 5.3.3, Part (b). This spinor satisfies

$$D\vartheta = \pm\sqrt{\lambda^2 - 4\alpha^2}\,\vartheta - 2\alpha \mathrm{vol}\cdot\vartheta.$$

It has been shown independently by Voss and Morel [Vos99, Satz 4.1] and [Mor02, Theorem 4.1] that there is an immersion F into $\mathcal{M}(\kappa)$ such that $\vartheta = F^*\Psi$ for a Killing spinor Ψ on $\mathcal{M}(\kappa)$.

In order to derive the non-isometric case from this, one removes the set of branching points S. After conformally rescaling by $g_1 := |\varphi|^{-4}g_0$, the isometric case provides an isometric immersion $\tilde{F}$ of the universal covering $\tilde{F} : (\widetilde{M\setminus S}, g_1)$ into $\mathcal{M}(\kappa)$ which pushes down to a map $F : M \to \mathcal{M}(\kappa)$ such that

$$\begin{array}{ccc} \widetilde{M\setminus S} & \stackrel{\tilde{F}}{\to} & \mathcal{M}(\kappa) \\ \downarrow & \underset{F}{\nearrow} & \\ M & & \end{array}$$

commutes.

5.4 Cmc-immersions into space-forms

Taking together the results of Chapter 4 and the results of the preceding sections we obtain immersions into $\mathcal{M}(\kappa)$ for $\kappa \geq 0$. After a possible change of orientation or of the spinor representation, we can restrict to the case $\alpha \leq 0$.

Let $(M, [g_0])$ be a compact Riemann surface, i.e. the orientation and the conformal class is fixed. We further assume that M carries a spin structure σ. (Choosing a spin structure σ on the surface M is in fact equivalent to choosing a square root of the complex line bundle $TM \to M$.)

Let $\lambda_1^+(M, g, \sigma)$ be the first positive eigenvalue of the Dirac operator on (M, g, σ). REcall that we defined in (2.4.1)

$$\lambda^+_{\min}(M, [g_0], \sigma) := \inf_{g\in[g_0]} \lambda_1^+(M, g, \sigma)\,\mathrm{vol}(M, g)^{1/n}.$$

Let $\widetilde{M}$ be the universal covering of M, and let $\Gamma = \pi_1(M)$ be the fundamental group, acting as Deck transformations on M.

Theorem 5.4.1. *If $\lambda^+_{\min}(M, [g_0], \sigma) < 2\sqrt{\pi}$, then there exists a conformal branched immersion $F : \widetilde{M} \to \mathbb{R}^3$ and a group homomorphism $h : \Gamma \to \mathbb{R}^3$ such that*

(1) $F(\gamma \cdot p) = F(p) + h(\gamma)$

(2) *If* M_1 *is a fundamental domain of* $\widetilde{M} \to M$*, then* $\mathrm{area}(F(M_1)) = 1$*, or more exactly* $\mathrm{area}(M_1, F^* g_{\mathrm{eucl}}) = 1$.

(3) *The mean curvature of* $F(M)$ *is constant and equals to* $\lambda^+_{\min}(M, [g_0], \sigma)$.

Remark. The Willmore integral of such an immersion F is

$$\int_{F(M)} H^2 = {\lambda^+_{\min}}^2 < 4\pi.$$

On the other hand, it is easy to prove that if M is a compact surface, then any immersion $F : M \to \mathbb{R}^3$ satisfies

$$\int_{F(M)} H^2 \geq 4\pi.$$

As a consequence we see that $h(\Gamma)$ contains nontrivial elements.

Analogously, we obtain immersions into $\mathcal{M}(\kappa)$, $\kappa > 0$. We view $\mathcal{M}(\kappa)$ as $\mathrm{SU}(2) = S^3$ equipped with the metric $\kappa^{-2} g_{\mathrm{can}}$.

THEOREM 5.4.2. *If* $\lambda^+_{\min}(M, [g_0], \sigma) < 2\sqrt{\pi}$*, then there exists a conformal branched immersion* $F : \widetilde{M} \to \mathcal{M}(\kappa)$ *and a group homomorphism* $h : \Gamma \to \mathrm{SU}(2)$ *such that*

(1) $F(\gamma \cdot p) = h(\gamma) \cdot F(p)$.

(2) *If* M_1 *is a fundamental domain of* $\widetilde{M} \to M$*, then* $\mathrm{area}(F(M_1)) = 1$*, or more exactly* $\mathrm{area}(M_1, F^*(\kappa^{-1} g_{\mathrm{can}})) = 1$.

(3) *The mean curvature of* $F(M)$ *is constant and equals to* H *with*

$$H^2 + \kappa = \lambda^+_{\min}(M, [g_0], \sigma)^2.$$

In the same manner as above, we conclude that $h(\Gamma)$ contains non-trivial elements.

Remark. Using Lawson's transform [Law70], we also obtain immersions into $\mathcal{M}(\kappa)$, $\kappa < 0$, for a periodicity map $h : \Gamma \to \mathrm{Isom}(\mathcal{M}(\kappa))$.

5.5 Open problems

In this section we want to formulate some open problems which are interesting for further investigations.

Problem No. 1

CONJECTURE 5.5.1 (Spinorial positive mass conjecture for surfaces)**.** *Let* (M, g) *be a compact Riemann surface of genus* ≥ 1 *with spin structure* σ*. Then*

$$\lambda^+_{\min}(M, [g], \sigma) < 2\sqrt{\pi}.$$

Problem No. 2
A theory for stationary points of $\mathcal{F}_q$ which are not suprema is required. Examples of such stationary points are obtained by closed cmc-immersions in $\mathbb{R}^3$, e.g. Wente tori.

Problem No. 3
If $\kappa \geq 0$, then we have seen that the period map h takes values in $\mathbb{R}^3$ or SU(2), which are both 3-dimensional subgroups of $\mathrm{Isom}(\mathcal{M}(\kappa))$. For $\kappa < 0$ we obtained periodic immersions into $\mathcal{M}(\kappa)$ via Lawson's transform. It is unclear whether there is a 3-dimensional subgroup of $\mathrm{Isom}(\mathcal{M}(\kappa))$ such that $h(\Gamma)$ is contained in this subgroup.

5.6 Examples and Visualizations

In this section we want to present some examples of surfaces obtained by our construction. The first two examples are already well studied and visualizations exist. The third one yields new periodic conformal immersions, but no visualization exist until now. As we consider it interesting to visualize them in the future, we added a subsection on software that could be used to visualize them in future.

5.6.1 Unduloids

visualized by N. Schmitt.

A (2-dimensional) torus with any Riemannian metric carries four different spin structures. For exactly one of them, the so-called *trivial spin structure*, the dimension of the kernel of the Dirac operator is 2, for all other spin structures, *the non-trivial spin structures*, the kernel is $\{0\}$. The terminology comes from the fact that if the metric is flat, then the trivial spin structure is the only one which is trivialized by parallel spinors.

Remark. We want to warn the reader that this terminology is a bit misleading if one considers spin-cobordism. The 2-dimensional spin cobordism group is $\mathbb{Z}_2$, and T^2 equipped with the **trivial** spin structure represents the **non-trivial** element

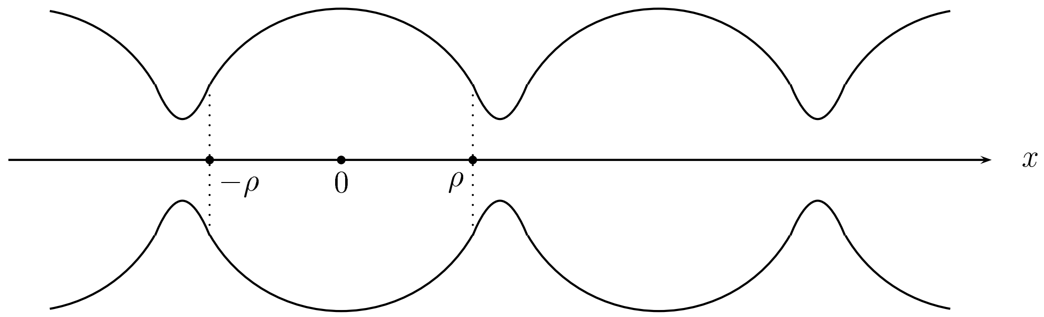


Figure 5.1: The curve generating $F : \mathbb{R}^2/\langle e_1\rangle \to \mathbb{R}^3$

in $\mathbb{Z}_2$, whereas T^2 with any non-trivial spin structure represents the trivial element in $\mathbb{Z}_2$.

Note that any torus is conformal to a flat one, and hence it can be written as $\mathbb{R}^2/\Gamma$, where Γ is a lattice acting on $\mathbb{R}^2$ via translations.

PROPOSITION 5.6.1. *Let $T^2 = \mathbb{R}^2/\Gamma$ be the torus with the canonical metric and with a spin structure. Assume that Γ is generated by two orthogonal vectors e_1 and e_2, and that the spin structure is such that there exists a parallel section of the spinor bundle along e_1, but not along e_2. Then the spinorial positive mass theorem (Conjecture 5.5.1) holds for this surface, i.e. $\lambda^+_{\min} < 2\sqrt{\pi}$.*

Proof. Let $F : \mathbb{R}^2/\langle e_1\rangle \to \mathbb{R}^3$, $F \in C^1$ be the periodic conformal immersion that parametrizes the surface of revolution obtained by rotating the curve drawn in Figure 5.1.

The convex parts are isometric to spheres of radius 1 with two caps removed, say $\{(x, y, z) \,|\, x^2 + y^2 + z^2 = 1, -\cos\rho \le x \le \cos\rho\}$ for $\rho \in (0, \pi/2)$ On the other hand, the concave part is a part of a catenoid, hence a minimal surface. One sees that such a catenoid can be glued in a C^1-manner for any $\rho \in (0, \pi/2)$. Furthermore it is a periodic conformal immersion of a flat torus as described above, and $|e_2|/|e_1|$ runs from 0 to ∞ when ρ runs from 0 to $\pi/2$.

We calculate the *Willmore integral*

$$\mathcal{W}(F) = \int H^2 = 4\pi \sin\rho < 4\pi$$

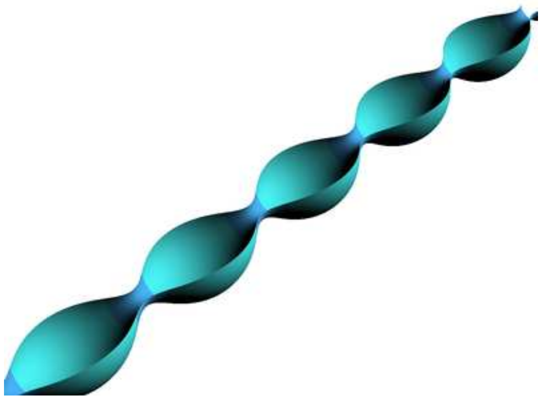


Figure 5.2: An unduloid in $\mathbb{R}^3$, visualized by Nick Schmitt.

where the integral is taken over a fundamental domain.

The standard Rayleigh quotient argument shows that for the metric induced from $\mathbb{R}^3$ the Dirac operator has an eigenvalue λ with λ^2area $< 4\pi$. The Dirac spectrum of surfaces is symmetric, hence $\lambda^+_{\min} < 2\sqrt{\pi}$. $\square$

As a consequence, we see that $\mathcal{F}_{4/3}$ attains its supremum, say in $\psi = D\varphi$. The Euler-Lagrange equations are $D\varphi = \lambda\,|\varphi|^2\varphi$. If ρ is sufficiently close to $\pi/2$, then one easily sees that the inf $\lambda_1^+\mathrm{area}(T^2)^{1/2}$ is **not** attained by a flat metric, and hence the length of φ is not constant. Hence, the spinorial Weierstrass representation induces a periodic conformal immersion into $\mathbb{R}^3$ which is not isometric. It is not known which cmc-immersion we obtain. However, we conjecture that one obtains the unduloid, visualized in Figure 5.2.

Similarly, we conjecture that in S^3 we obtain unduloids in S^3. Some unduloids are "closed", i.e. the immersion $\mathbb{R}^2/(r\mathbb{Z}\times\{0\})$, $r\in\mathbb{R}^+$ factors to a finite covering of the torus. Others are not closed, and the image is dense in a 3-dimensional set. An example of a closed one is visualized in Figure 5.3.

5.6.2 Triply periodic examples

studied and visualized by K. Grosse-Brauckmann.

We want to present some triply periodic examples in this subsection.

We start with a periodic minimal surface in $\mathbb{R}^3$ displayed in Figure 5.4. We view this surface as a minimal surface M in the 3-torus $T^3 = \frac{\mathbb{R}^3}{\mathbb{Z}^3}$.

Let O be the octahedral group, i.e. the symmetry group of the regular cube or of

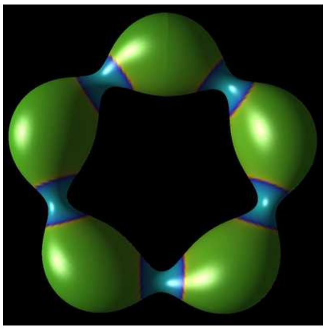


Figure 5.3: An unduloid in S^3, visualized by Nick Schmitt.

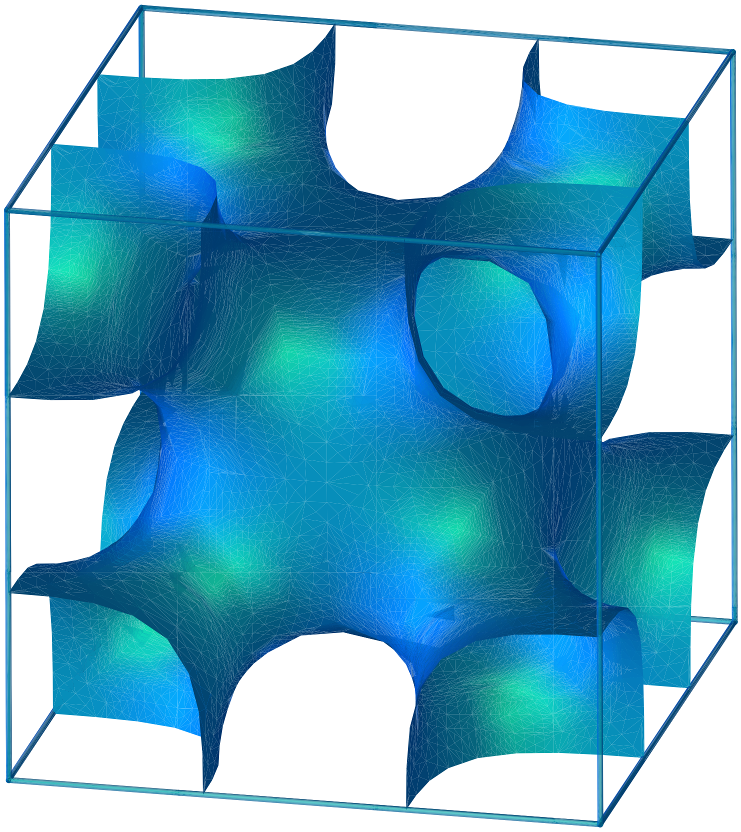


Figure 5.4: A periodic minimal surface, visualized by K. Grosse-Brauckmann with GRAPE

the regular octahedron. The group O also acts as isometries on M. The orientation preserving elements in O are denoted as O^+. Let O^* be the binary octahedral group, i.e. the preimage of O^+ under $\mathrm{Spin}(3) = \mathrm{SU}(2) \to \mathrm{SO}(3)$. Let M carry the induced spin structure. Then, O^* acts on the spin structure, and hence the spinor bundle is an $O^* \to O^+$-equivariant bundle.

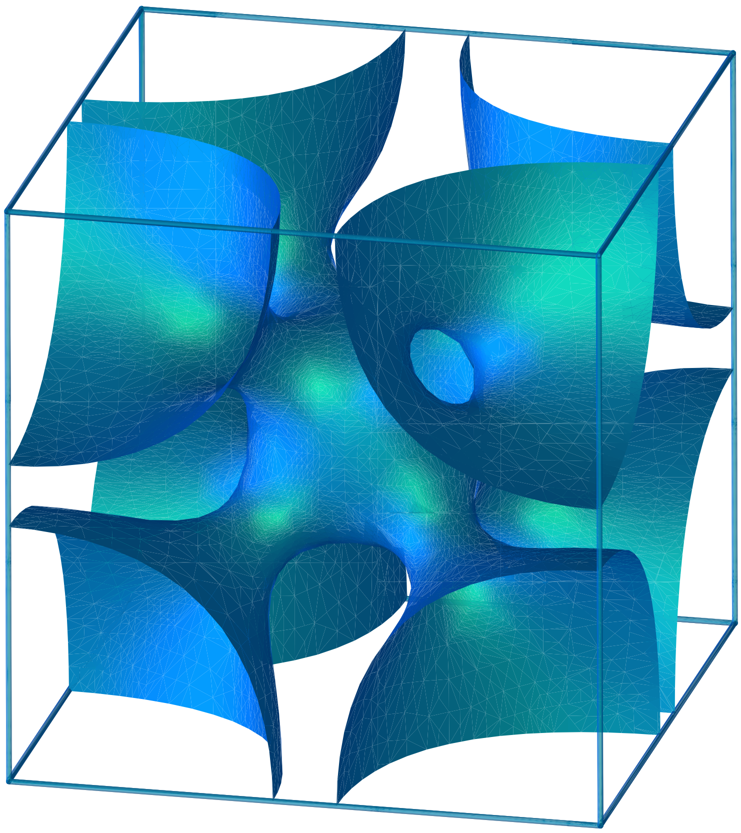


Figure 5.5: A periodic surface of constant mean curvature, visualized by K. Grosse-Brauckmann with GRAPE

The parallel spinors on T^3 induce a 2-dimensional O^*-invariant space of harmonic spinors on M. We now deform the conformal structure inside the class of O^+-equivariant conformal structures. Let us assume that we perturb the original metric g on M to a metric g'. As M has genus ≥ 3, one knows [Mai97] that there are such deformations with $\dim\ker D_{g'} < \dim\ker D_g$. Because eigenvalues of the Dirac operator are continuous in the C^1-topology, we know that if we choose g' sufficiently C^1-close to g, then we have

$$\lambda^+_{\min}(M, [g'], \sigma) < 2\sqrt{\pi}.$$

One can strengthen this result by looking only at the O^*-invariant spinors and the same arguments as above imply that the O^*-invariant version of $\lambda^+_{\min}$ (i.e. the infimum of $\lambda^+_1 \mathrm{area}^{1/2}$ running over the O^*-invariant metrics) is smaller than $2\sqrt{\pi}$. All constructions in Chapter 4 are Γ^* equivariant. Hence, one can adapt the methods in Chapter 4 to the equivariant setting and we obtain a Γ^*-invariant solution to equation (4.2.7).

Hence, the results of Chapter 4 imply that there is a solution to equation (4.2.7). As all our constructions are Γ^*-equivariant, we can even assume that the obtained solution is Γ^*-equivariant.

This solution yields a cmc-surface, which is again Γ^+-equivariant, see e.g. Figure 5.5.

Similarly, but with slightly more technical efforts, one deals with non-orientation preserving symmetries.

5.6.3 New examples

Now, we will give some examples of new conformal cmc-surfaces.

We start with the periodic minimal surface from the last subsection, given as a conformal embedding $(M, [g]) \to \mathbb{R}^3/\mathbb{Z}^3$. As before let $[g']$ be close to $[g]$, but we release the symmetry condition. Once again if $[g']$ is sufficiently close to $[g]$ and if $\dim\ker D_{g'} < \dim\ker D_g$, then

$$\lambda^+_{\min}(M, [g'], \sigma) < 2\sqrt{\pi}.$$

Chapter 4 implies that (4.2.7) has a solution, and we obtain a conformally immersed surface of constant mean curvature. The rank of the period group of the immersed surface might even be larger than 3, in which case the image would be everywhere dense.

A completely other construction yields that any surface of arbitrary genus γ carries a spin-conformal class with

$$\lambda^+_{\min}(M, [g], \sigma) < 2\sqrt{\pi}. \tag{5.6.2}$$

The construction generalizes the construction in Proposition 5.6.1. The idea behind this construction is to take not only a line of unit spheres, but a periodic configuration of unit spheres, where the period group is a group of rank γ, say generated by $e_1, \ldots, e_\gamma$. We choose the lengths of the generators e_i appropriately (in particular, not too far away from 2), we cut off small neighborhoods of $\pm e_i/|e_i|$ and their translates via the period map, and glue in minimal surfaces joining each sphere S_k to $e_i + S_k$. The induced spinor Φ is a spinor such that $\mathcal{F}_q(\Phi) > (2\sqrt{\pi})^{-1}$. Hence, we have a test spinor that shows (5.6.2)

5.6.4 Software for visualizing cmc-surfaces

At end, we want to note that there is a growing number of interesting and free software which is extremely helpful to visualize minimal and cmc-surfaces in the space-forms H^3, $\mathbb{R}^3$ and S^3. See at

`http://www.berndammann.de/softwarelinks`

for links to these programs.

A classical program is Ken Brakke's surface evolver. The program works with triangulated surfaces, and numerical routines try to minimize the area of a surface under various constraints. This admits drawing minimal and cmc-surfaces. These programs can handle periodic surfaces very efficiently.

Much more adapted to the spinorial Weierstrass representation are various programs of the software packages of GANG, the Center for Geometry, Analysis, Numerics and Graphics (GANG) at the University of Massachusetts, Amherst, Massachusetts, USA. Nick Schmitt and his collaborators have programmed many programs (minlab, cmclab and others) for visualizing minimal and cmc-surfaces with the help of the spinorial Weierstrass representation.

Two other important packages for visualizations are MESH and GRAPE.

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Keyword Index

Symbol Index

List of Publications

Articles

- Bernd Ammann, *Minimal geodesics and nilpotent fundamental groups*, Geom. Dedic. **67**, *129–148* (1997).
- Bernd Ammann und Christian Bär, *The Dirac operator on nilmanifolds and collapsing circle bundles*, Ann. Global Anal. Geom. **16**, *221–253* (1998).
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