ON THE GEOMETRY OF RIEMANNIAN MANIFOLDS WITH A LIE STRUCTURE AT INFINITY

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Abstract. A manifold with a "Lie structure at infinity" is a non-compact manifold M_0 whose geometry is described by a compactification to a manifold with corners M and a Lie algebra $\mathcal{V}$ of vector fields on M subject to constraints only on $M \smallsetminus M_0$. This definition recovers several classes of non-compact manifolds that were studied before: manifolds with cylindrical ends, manifolds that are Euclidean at infinity, conformally compact manifolds, and others. It hence provides a unified setting for the study of these classes of manifolds and of their geometric differential operators. The Lie structure at infinity on M_0 determines a complete metric on M_0 up to bi-Lipschitz equivalence. This leads to the natural problem of understanding the Riemannian geometry of these manifolds, which is the main question addressed in this paper. We prove, for example, that on a manifold with a Lie structure at infinity the curvature tensor and its covariant derivatives are bounded, by extending the Levi-Civita connection to an A^*-valued connection where the bundle A is uniquely determined by the Lie algebra $\mathcal{V}$. We study a generalization of the geodesic spray and give conditions for these manifolds to have positive injectivity radius. We also prove that the geometric operators are generated by the given Lie algebra of vector fields.

An important motivation for our study is to prepare the ground for the investigation of the analysis of geometric operators on manifolds with a Lie structure at infinity. The simplest examples of manifolds with a Lie structure at infinity are the manifolds with cylindrical ends. For these manifolds the corresponding analysis is that of totally characteristic operators on a compact manifold with boundary equipped with a "b-metric."

Contents

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Introduction

Geometric differential operators on complete, non-compact Riemannian manifold were extensively studied due to their applications to physics, geometry, number theory, and numerical analysis. Still, their properties are not as well understood as those of differential operators on compact manifolds, one of the main reason being that differential operators on non-compact manifolds do not enjoy some of the most useful properties enjoyed by their counterparts on compact manifolds.

For example, elliptic operators on non-compact manifolds are not Fredholm in general. (We use the term "elliptic" in the sense that the principal symbol is invertible outside the zero section.) Also, one does not have a completely satisfactory pseudodifferential calculus on an arbitrary complete, non-compact Riemannian manifold, which might allow us to decide whether a given geometric differential operator is bounded, Fredholm, or compact (see however [2] and the references within).

However, if one restricts oneself to certain classes of complete, non-compact Riemannian manifolds, one has a chance to obtain more precise results on the analysis of the geometric differential operators on those spaces. This paper is the first in a series of papers devoted to the study of such a class of Riemannian manifolds, the class of Riemannian manifolds with a "Lie structure at infinity" (see Definition 2.1). We stress here that few results on the geometry of these manifolds have a parallel in the literature, although there is a fair number of papers devoted to the analysis on *particular* classes of such manifolds [11, 12, 14, 15, 16, 39, 40, 57, 56, 66, 69, 72, 76, 78, 79, 81]. The philosophy of Cordes' comparison algebras [13], Kondratiev's approach to analysis on singular spaces [39], Parenti's work on manifolds that are Euclidean at infinity [66], and Melrose's approach to pseudodifferential analysis on singular spaces [57] have played an important role in the development of this subject.

A manifold M_0 with a Lie structure at infinity has, by definition, a natural compactification to a manifold with corners $M = M_0 \cup \partial M$ such that the tangent bundle $TM_0 \to M_0$ extends to a vector bundle $A \to M$ with some additional structure. We assume, for example, that the Lie bracket of vector fields on M_0 defines, by restriction, a Lie algebra structure on the space of sections of A such that the space $\mathcal{V} := \Gamma(A)$ of sections of A identifies with a Lie subalgebra of the Lie algebra of all vector field on M_0. The pair $(M, \mathcal{V})$ then defines a Lie structure at infinity on M_0. A simple, non-trivial class of manifolds with a Lie structure at infinity is that of manifolds with cylindrical ends. Let M_0 be a manifold with cylindrical ends. In this case, the compactification M is a manifold with boundary, $\mathcal{V}$ consists of all vector fields tangent to the boundary of M. This example plays a prominent role in the analysis of boundary value problems on manifolds with conical points [39, 41, 59, 55, 73, 74]. See the above references for earlier results.

Let $(M, \mathcal{V})$ be a Lie structure at infinity on M_0, $\mathcal{V} = \Gamma(A)$. The choice of a fiberwise scalar product on A gives rise to a fiberwise scalar product g on TM_0, i.e. a Riemannian metric on M_0. Since M is compact, any two such metrics g_1 and g_2 are equivalent, in the sense that there exists a positive constant $C > 0$ such that $C^{-1} g_1 \geq g_2 \geq C g_1$. One can thus expect that the properties of the Riemannian manifold (M_0, g) obtained by the above procedure depend only on the Lie structure at infinity on M_0 and not on the particular choice of a metric on A. However, as shown in the following example, a metric on M_0 does not determine a Lie structure at infinity on M_0.

Example 0.1. Let us compactify $\mathbb{R}$ by including $+\infty$ and $-\infty$:

$$\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}.$$

We define $\varphi : [-1, +1] \to \overline{\mathbb{R}}$, $\varphi(t) = \log(t+1) - \log(1-t)$, $\varphi(\pm 1) = \pm\infty$. The pullback of the differentiable structure on $[-1, 1]$ defines a differentiable structure on $\overline{\mathbb{R}}$. On $\overline{\mathbb{R}}$ we consider the Lie algebra of vector fields that vanish at $\pm\infty$. The product of these compactifications of $\mathbb{R}$ defines a Lie structure on $M_0 := \mathbb{R}^n$, in which the compactification M is diffeomorphic to the manifold with corners $[-1, 1]^n$ and the sections of A are all vector fields tangent to all hyperfaces. (The resulting Lie structure at infinity is that of the b-calculus (see Example 1.5)). Alternatively, one can consider the radial compactification of $\mathbb{R}^n$. The resulting Lie structure at infinity is described in Example 1.6, which is closely related to the so-called scattering calculus [56, 66].

We thus see that $\mathbb{R}^n$ fits into our framework and is in fact a manifold with a Lie structure at infinity for several distinct compactifications M.

Thus, although our motivation for studying manifolds with a Lie structure at infinity comes from analysis, this class of manifolds leads to some interesting questions about their geometry, and this paper (the first one in a series of papers on this subject) is devoted mainly to the issues and constructions that have a strong Riemannian geometric flavor. It is important to mention here that only very few results on the geometry of particular classes of Riemannian manifolds with a Lie structure at infinity were proved before, except some special examples (e.g. compact manifolds and manifolds with cylindrical ends). For example, we prove that M_0 is complete and has bounded curvature, in the sense that the Riemannian curvature R and all its covariant derivatives $\nabla^k R$, with respect to the Levi-Civita connection, are bounded. Also, under some mild assumptions on $(M, \mathcal{V})$, we prove that (M_0, g) has positive injectivity radius, and hence M_0 has bounded geometry. This is very convenient for the analysis on these manifolds. The main technique is based on generalizing the Levi-Civita connection to an "A^*-valued connection" on A. (An *A^*-valued connection on a bundle $E \to M$* is a differential operator $\nabla : E \to E \otimes A^*$ that satisfies all the usual properties of a connection, but with A replacing the tangent bundle, see Definition 1.20. This concept was first introduced in a slightly different form in [26] by Evens, Lu, and Weinstein. The right approach to the geometry of manifolds with a Lie structure at infinity requires us to replace the tangent bundle by A. This was noticed before in particular examples, see for instance [11, 46, 56, 57, 60].

The Lie structure at infinity on M_0 allows us to define a canonical algebra of differential operators on M_0, denoted $\mathrm{Diff}(\mathcal{V})$, as the algebra of differential operators generated by the vector fields in $\mathcal{V} = \Gamma(A)$ and multiplication by functions in $\mathcal{C}^\infty(M)$. If $E_0, E_1 \to M$ are vector bundles on M, then one can similarly define the spaces $\mathrm{Diff}(\mathcal{V}; E_0, E_1)$ (algebras if $E_0 = E_1$) of differential operators generated by $\mathcal{V}$ and acting on sections of E_0 with values sections of E_1. All geometric operators on M_0 (de Rham, Laplace, Dirac) will belong to one of the spaces $\mathrm{Diff}(\mathcal{V}; E_0, E_1)$, for suitable bundles E_0 and E_1. The proof of this result depends on our extension of the Levi-Civita connection to an A^*-valued connection.

Many questions in the analysis on non-compact manifolds or on the asymptotics of various families of operators can be expressed in terms of $\mathrm{Diff}(\mathcal{V})$. We refer to [15, 17, 44, 33, 43, 46, 51, 56, 57, 58] for just a few of the many possible examples in the literature. Indeed, let $\Delta = d^*d \in \mathrm{Diff}(\mathcal{V})$ be the scalar Laplace operator on M_0. Then Δ is essentially self-adjoint on $\mathcal{C}_c^\infty(M_0)$ by old results of Gaffney [29] and Roelcke [70] from 1951 and 1960. Assume that M_0 has positive injectivity radius, then $P(1+\Delta)^{-m/2}$ and $(1+\Delta)^{-m/2}P$ are bounded operators on $L^2(M_0)$, for any differential operator $P \in \mathrm{Diff}(\mathcal{V})$ of order at most m. Cordes [12, 13] defined the comparison algebra $\mathfrak{A}(M, \mathcal{V})$ as the norm closed algebra generated by the operators $P(1+\Delta)^{-m/2}$ and $(1+\Delta)^{-m/2}P$, with P a differential operator $P \in \mathrm{Diff}(\mathcal{V})$ of order at most m. The comparison algebra is useful because it leads to criteria for differential and pseudodifferential operators to be compact or Fredholm between suitable Sobolev spaces [43, 1, 2].

We expect manifolds with a Lie structure at infinity and especially the analytic tools (pseudodifferential and asymptotic analysis) that we have established in [1, 2] to play an important role for solving some problems in geometric analysis simultaneously for a large class of manifolds. Indeed, in special cases of manifolds with a Lie structure at infinity the solutions to quite a few interesting problems in geometric analysis rely heavily on those methods. For instance, consider asymptotically Euclidean manifolds, a special case of Example 1.6. In general relativity one is interested in finding solutions to the Einstein equations whose spatial part is asymptotically Euclidean. Integration of the first nontrivial coefficient in the asymptotic development of the metric at infinity yields the so-called "mass" of the solution [7]. The positive mass theorem states that any non-flat asymptotically Euclidean Riemannian manifold with non-negative scalar curvature has positive mass. An elegant proof of the positive mass theorem by Witten [67] uses Sobolev embeddings on such manifolds. The positive mass theorem provides the final step in the proof of the Yamabe conjecture on compact manifolds [71]: Any conformal class on a compact manifold M admits a metric with constant scalar curvature. In order to prove the conjecture in the locally conformally flat case, one replaces the metric g on M by a scalar-flat metric $u \cdot g$ on $M \setminus \{p\}$ where u is a function $u(x) \to \infty$ for $x \to p$, and a neighborhood of p provides the asymptotically euclidean end, and one applies the positive mass conjecture to this. On most non-compact manifolds, the Yamabe problem is still unsolved. However, special cases have been solved, e.g. on manifolds with cylindrical ends [3].

Both the geometry and the analysis of asymptotically hyperbolic manifolds have been the subject of articles in general relativity and the analysis of 3-manifolds, see [4, 6]. One can prove rigidity theorems [5] for asymptotically hyperbolic ends, or existence results for asymptotically hyperbolic Einstein metrics [48]. Similar rigidity problems for asymptotically complex hyperbolic ends are subject in [9, 10, 31].

Or take the construction of manifolds with with special holonomy $SU(m)$, $Sp(m)$ and G_2 where the analysis of weighted function spaces on manifolds which are quasi asymtotically locally euclidean [35, 36, 37] has been used.

In summary, our present program will lead to a unified approach to the analysis on various types of manifolds with a "good" asymptotic behavior at infinity.

We now discuss the contents of each section. In Section 1 we introduce and study structural Lie algebras of vector fields and the equivalent concept of boundary tangential Lie algebroids. A structural Lie algebra of vector fields on a manifold with corners M gives rise to a canonical algebra $\mathrm{Diff}(\mathcal{V})$ of differential operators. We include numerous examples.

Then, in Section 2, we specialize to the case that the constraints are only on the boundary. This special case is called a "manifold with a Lie structure at infinity." The Lie structure at infinity defines a Riemannian metric on the interior of the manifold. This metric is unique up to bi-Lipschitz equivalence. Hence, the Lie structures at infinity is a tool for studying a large class of open Riemannian manifolds. We are interested in the analysis on such open manifolds.

Section 3 is devoted to the study of the geometry of Riemannian manifolds with a Lie structure at infinity. We will prove that these manifolds are complete and have bounded curvature (together with all its covariant derivatives). This depends on an extension of the Levi-Civita connection to an A^*-valued connection, the appropriate notion of connection in this setting. Then we investigate the question of whether a Riemannian manifold with a Lie structure at infinity has positive injectivity radius.

In Section 4 we introduce Dirac and generalized Dirac operators and prove that they belong to $\mathrm{Diff}(\mathcal{V}; W)$, where W is a Clifford module. The same property is shared by all geometric operators (Laplace, de Rham, signature) on the open manifold M_0.

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1. Structural Lie algebras and Lie algebroids

We introduce in this section the concept of structural Lie algebras of vector fields, which is then used to define manifolds with a Lie structure at infinity.

1.1. **Projective modules.** In this subsection, we recall some well-known facts about projective modules over $\mathcal{C}^\infty(M)$, where M is a compact manifold, possibly with corners.

Let V be a $\mathcal{C}^\infty(M)$ module with module structure $\mathcal{C}^\infty(M) \times V \ni (f, v) \mapsto fv \in V$. Let $x \in M$ and denote by $\mathfrak{p}_x$ the set of functions on M that vanish at $x \in M$. Then $\mathfrak{p}_x V$ is a complex vector subspace of V and $V/\mathfrak{p}_x V$ is called the *geometric fiber of V at x*. In general, the geometric fibers of V are complex vector spaces of varying dimensions.

A subset $S \subset V$ will be called a *basis* of V if every element $v \in V$ can be written uniquely as $v = \sum_{s \in S} f_s s$, with $f_s \in \mathcal{C}^\infty(M)$, $\#\{s \in S \,|\, f_s \neq 0\} < \infty$. (In our applications, S will always be a finite set, so we will not have to worry about this last condition.) A $\mathcal{C}^\infty(M)$-module is called *free (with basis S)* if it has a basis S. Unlike the general case, the geometric fibers of a free module have constant dimension, equal to the number of elements in the basis S. Note however, that if $f : V \to W$ is a morphism of free modules, the induced map between geometric fibers *may have non-constant rank*. For example, it is possible that f is injective, but the induced map on the geometric fibers is not injective on all fibers. An example is provided by $M = [0, 1]$, $V = W = \mathcal{C}^\infty([0, 1])$ and f being given by the multiplication with the coordinate function $x \in [0, 1]$. Then f is injective, but the induced map on the geometric fibers at 0 is 0.

A $\mathcal{C}^\infty(M)$–module V is called *finitely generated projective* if, by definition, there exists another module W such that $V \oplus W$ is free with a finite basis. We then have the following fundamental theorem of Serre and Swan [38]

Theorem 1.1 (Serre-Swan). *If V is a finitely generated, projective module over $\mathcal{C}^\infty(M)$, then the set $E := \cup_{x\in M}(V/\mathfrak{p}_x V) \times \{x\}$, the disjoint union of all geometric fibers of V, can be endowed with the structure of a finite-dimensional, smooth vector bundle $E \to M$ such that $V \simeq \Gamma(M;E)$. The converse is also true: $\Gamma(M;E)$ is a finitely generated, projective $\mathcal{C}^\infty(M)$-module for any finite-dimensional, smooth vector bundle $E \to M$.*

Suppose now that V is a $\mathcal{C}^\infty(M)$-module and that M is connected. Then V is a finitely generated, projective $\mathcal{C}^\infty(M)$-module if, and only if, there exists $k \in \mathbb{Z}_+$ satisfying the following condition:

For any $x \in M$, there exist $\varphi \in \mathcal{C}^\infty(M)$, $\varphi(x) = 1$, and k-elements $v_1, \ldots, v_k \in V$ with the property that for any $w \in V$ we can find $f_1, f_2, \ldots, f_k \in \mathcal{C}^\infty(M)$ such that

$$\varphi(f_1v_1 + f_2v_2 + \ldots + f_kv_k - w) = 0 \text{ in } V \tag{1}$$

and, moreover, the germs of $f_1, \ldots, f_k$ at x are uniquely determined.

A module V satisfying Condition (1) above is called *locally free of rank k*, and what we are saying here is that "locally free of rank k, for some k," is equivalent to "finitely-generated, projective." It is crucial here that the number of elements k is the same for any $x \in M$. In case M is not connected, the number k needs only be constant on the connected components of M.

Remark 1.2. The introduction of projective modules over $C^\infty(M)$ in Partial Differential Operators on non-compact manifolds was pioneered by Melrose [54] in the early 1980s.

1.2. **Manifolds with corners and structural Lie algebras.** We now fix our terminology and recall the definitions of the main concepts related to manifolds with corners.

In the following, by a *manifold* we shall always understand a C^∞-manifold *possibly with corners.* In contrast, a *smooth manifold* is a C^∞-manifold *without corners.* By definition, for every point p in a manifold with corners M, there is a coordinate neighborhood U_p of p and diffeomorphism φ_p to $[0,\infty)^k \times \mathbb{R}^{n-k}$, with $\varphi_p(p) = 0$, such that the transition functions are smooth (including on the boundary). The number k here clearly depends on p, and will be called the *boundary depth of p.* Hence points in the interior have boundary depth 0, points on the boundary of a manifold without corners have boundary depth 1, etc. Roughly speaking the boundary depth counts the number of boundary faces p is in.

Moreover, we assume that each hyperface H of M is an embedded submanifold and has a defining function, that is, there exists a smooth function $x_H \geq 0$ on M such that

$$H = \{x_H = 0\} \text{ and } dx_H \neq 0 \text{ on } H.$$

This assumption is just a simplifying assumption. We can deal with general manifolds with corners using the constructions from [60]. Note that a priori we do not fix a particular system of defining functions, but only use their existence occasionally.

If $F \subset M$ is an arbitrary face of M of codimension k, then F is an open component of the intersection of the hyperfaces containing it. Any set $x_1, \ldots, x_k$ of defining functions of the hypersurfaces containing F is called a *set of defining functions of F*; thus, F is a connected component of $\{x_1 = x_2 = \ldots = x_k = 0\}$.

This statement obviously does not depend on the choice of the defining functions x_j, $j = 1, \ldots, k$. We shall denote by ∂M the union of all non-trivial faces of M. Usually, we shall denote by M_0 the interior of M, that is, $M_0 := M \smallsetminus \partial M$.

A *submersion* $f : M \to N$, between two manifolds with corners M and N, is a differentiable map f such that df is surjective at all points and $df(v)$ is an inward pointing tangent vector of N if, and only if, v is an inward pointing vector M. It follows then that the fibers $f^{-1}(y)$ of f are smooth manifolds *without* corners, and that f preserves the boundary depth (i.e. the number of boundary faces a point p is in.) If x is a defining function of some hyperface of N, then $x_1 = x \circ f$ is such that $\{x_1 = 0\}$ is a union of hyperfaces of M and $dx_1 \neq 0$ on $\{x_1 = 0\}$.

Example 1.3. Let $A \to M$ be a smooth vector bundle. The *sphere bundle of* A, denoted $S(A)$, is defined, as usual, as the set of (positive) rays in the bundle A, that is, $S(A) = (A \setminus \{0\})/\mathbb{R}_+$. If we fix a smooth metric on A, then $S(A)$ identifies with the set of vectors of length one in A. Moreover, $S(A) \to M$ turns out to be a submersion of manifolds with corners.

A *submanifold with corners* N of a manifold with corners M is a submanifold $N \subset M$ such that N is a manifold with corners, and each hyperface H of N is a connected component of a set of the form $H' \cap N$, where H' is a hyperface of M intersecting N *transversally.*

The starting point of our analysis is a Lie algebra of vector fields on a manifold with corners. For reasons that will be clearer later, we prefer to keep this concept as general as possible, even if for the analysis on non-compact manifolds, only certain classes of Lie algebras of vector fields will be used.

Definition 1.4. A *structural Lie algebra of vector fields* on a manifold M (M possibly with corners) is a subspace $\mathcal{V} \subset \Gamma(TM)$ of the real vector space of vector fields on M with the following properties:

(i) $\mathcal{V}$ is closed under Lie brackets;
(ii) $\mathcal{V}$ is a finitely generated, projective $\mathcal{C}^\infty(M)$-module; and
(iii) the vector fields in $\mathcal{V}$ are tangent to all faces in M.

By (ii) we mean that $\mathcal{V}$ is closed for multiplication with functions in $\mathcal{C}^\infty(M)$ and the induced $\mathcal{C}^\infty(M)$-module structure makes it a finitely generated projective $\mathcal{C}^\infty(M)$-module.

Given a structural Lie algebra $\mathcal{V}$ of vector fields on a manifold with corners, we call the enveloping algebra $\mathrm{Diff}(\mathcal{V})$ of $\mathcal{V}$ the algebra of *$\mathcal{V}$-differential operators on M*. Note that any $\mathcal{V}$-differential operator $P \in \mathrm{Diff}(\mathcal{V})$ can be realized as a polynomial in vector fields in $\mathcal{V}$ with coefficients in $\mathcal{C}^\infty(M)$ acting on the space $\mathcal{C}^\infty(M)$

Let us give some examples for structural Lie algebras of vector fields. Some of these examples can also be found in [56]. We also give descriptions of the structural vector fields in local coordinates, because this will be helpful in the applications of the theory developed here. All of the following examples model the analysis on some non-compact manifold, except for the last one, which models the analysis of adiabatic families.

The following example is the simplest and most studied so far, however, it is quite important for us because it models the geometry of manifolds with cylindrical ends, and hence it is easier to grasp.

Example 1.5. Let M be a manifold with corners and

$$\mathcal{V}_b = \{X \in \Gamma(TM) : X \text{ is tangent to all faces of } M\}. \tag{2}$$

Then $\mathcal{V}_b$ is a structural Lie algebra of vector fields, and any structural Lie algebra of vector fields on M is contained in $\mathcal{V}_b$, by condition (iii) of the above definition. A vector field $X \in \mathcal{V}_b$ is called a *b-vector field* X. Fix $x_1, \dots, x_k$ and $y \in \mathbb{R}^{n-k}$ local coordinates near a point p on a boundary face of codimension k, with x_j defining functions of the hyperfaces through p. Then any b-vector field X is of the form

$$X = \sum_{j=1}^{k} a_j(x,y) x_j \partial_{x_j} + \sum_{j=1}^{n-k} b_j(x,y) \partial_{y_j}$$

on some neighborhood of p, with the coefficients a_j and b_j smooth everywhere (including the hyperfaces $x_j = 0$), for all j. This shows that the Lie algebra of b-vector fields is generated in a neighborhood U of p by $x\partial_x$ and ∂_y as a $\mathcal{C}^\infty(M)$-module. The differential operators in $\mathrm{Diff}(\mathcal{V}_b)$ are called *Fuchs type operators*, *totally characteristic*, or simply, and perhaps more systematically *b-differential operators*. The structural Lie algebra $\mathcal{V}_b$ and the analysis of the corresponding differential and pseudodifferential operators are treated in detail for instance in [20, 23, 34, 49, 55, 56, 75].

Example 1.6. Let M be a compact manifold with boundary and $x : M \to \overline{\mathbb{R}}_+$ a boundary defining function. Then the Lie algebra $\mathcal{V}_{sc} := x\mathcal{V}_b$ does not depend on the choice of x and the vector fields in $\mathcal{V}_{sc}$ are called *scattering vector fields*; with respect to local coordinates (x, y) near the boundary, scattering vector fields are generated by $x^2\partial_x$ and $x\partial_y$. An analysis of the scattering structure can be found in [56]. Since this structure models the analysis on asymptotically Euclidean spaces, let us be a little bit more precise and recall some basic definitions. A Riemannian metric g on the interior of M is called a *scattering metric* if, close to the boundary ∂M, it is of the form $g = \frac{dx^2}{x^4} + \frac{h}{x^2}$ where h is a smooth, symmetric 2-tensor on M which is non-degenerate when restricted to the boundary. Then scattering vector fields are exactly those smooth vector fields on M that are of bounded length with respect to g, and the corresponding Laplacian Δ_g is an elliptic polynomial in scattering vector fields. As a special case of this setting note that the radial compactification map

$$\mathrm{RC} : \mathbb{R}^N \longrightarrow S^N_+ := \{\omega = (\omega_0, \omega') \in S^N : \omega_0 \geq 0\} : z \longmapsto (1 + |z|^2)^{-1/2}(1, z)$$

identifies the Euclidean space $\mathbb{R}^N$ with the interior of the upper half-sphere S^N_+ such that the Euclidean metric lifts to a scattering metric on S^N_+.

The following example is one of the examples that we are interested to use in applications.

Example 1.7. Let M be a manifold with boundary ∂M, which is the total space of a fibration $\pi : \partial M \to B$ of smooth manifolds. We let

$$\mathcal{V}_e = \{X \in \Gamma(TM) : X \text{ is tangent to all fibers of } \pi \text{ at the boundary}\}$$

be the space of *edge vector fields*. In order to show that this is indeed a structural Lie algebra of vector fields, we have to show that it is closed under Lie brackets. Let $i : \partial M \to M$ be the inclusion. Assume that $X, Y \in \mathcal{V}_e$. Because

$$[X, Y]|_{\partial M} = [X|_{\partial M}, Y|_{\partial M}],$$

the commutator is again tangent to the fibers of π. If (x, y, z) are coordinates in a local product decomposition near the boundary, where x corresponds to the boundary defining function, y to a set of variables on the base B lifted through π, and z is a set of variables in the fibers of π, then edge vector fields are generated by $x\partial_x$, $x\partial_y$, and ∂_z. Using this local coordinate description is another way, to see immediately that the space of edge vector fields is in fact a Lie algebra. More importantly, it shows that it is a projective $\mathcal{C}^\infty(M)$-module. The analysis of the Lie algebra $\mathcal{V}_e$ is partly carried out in [51] and more recently in [46].

A special case of the edge structure is of particular importance for the analysis on hyperbolic space, so it deserves its own name:

Example 1.8. Let M be a compact manifold with boundary, and let $\mathcal{V}_0$ be the edge vector fields corresponding to the trivial fibration $\pi = \mathrm{id} : \partial M \to \partial M$, i. e., we have

$$\mathcal{V}_0 = \{X \in \Gamma(TM) : X|_{\partial M} = 0\}$$

which explains the name 0-*vector fields* for the elements in $\mathcal{V}_0$. With respect to local coordinates (x, y) near the boundary, 0-vector fields are generated by $x\partial_x$ and $x\partial_y$. Recall that a Riemannian manifold (M_0, g_0) is called *conformally compact* provided it is isometric to the interior of a compact manifold M with boundary equipped with a metric $g = \varrho^{-2}h$ in the interior, where h is a smooth metric on M and $\varrho : M \to \overline{\mathbb{R}}_+$ a boundary defining function. Note that 0-vector fields are the smooth vector fields on M that are of bounded length with respect to g; moreover, the Laplacian Δ_{g_0} is given as an elliptic polynomial in 0-vector fields. A particular example of conformally compact spaces is of course the hyperbolic space with compactification given by the ball model. Conformally compact spaces arise naturally in questions related to the Einstein equation [4, 48, 53], and the "AdS/CFT-correspondence." An analysis of 0-vector fields and the associated 0-differential and pseudodifferential operators was carried out for instance in [42, 51, 68]. Criteria for the Fredholmness of operators in $\mathrm{Diff}(\mathcal{V}_0)$, which is crucial in the approach to the study of Einstein's equations on conformally compact manifolds used in the above mentioned papers, were established for instance in [42, 43, 45, 46, 51, 52, 56, 68].

The structural Lie algebra of vector fields in the next example is a slight variation of the Lie algebra of edge vector fields, however, it is worth pointing out that this slight variation leads to a completely different analysis for the associated (pseudo)differential operators.

Example 1.9. Let M be as in Example 1.7 and $x : M \to \overline{\mathbb{R}}_+$ be a boundary defining function. Then $\mathcal{V}_{de} := x\mathcal{V}_e$ is a structural Lie algebra of vector fields; the corresponding structure is called the *double-edge structure*. With respect to local product coordinates as in Example 1.7, double-edge vector fields are generated by $x^2\partial_x$, $x^2\partial_y$, and $x\partial_z$. The analysis of the double-edge structure, which is in fact much simpler than the corresponding analysis of the edge structure, can be found for instance in [44].

The following example appears in the analysis of holomorphic functions of several variables.

Example 1.10. Let M be a smooth compact manifold with boundary ∂M and let $\Theta \in \mathcal{C}^\infty(M, \Lambda^1 T^*M)$ be a smooth 1-form such that $i^*\Theta \neq 0$ where $i : \partial M \hookrightarrow M$

is the inclusion of the boundary. Moreover, let x be a boundary defining function. Then

$$\mathcal{V}_\Theta := \{V \in \mathcal{V}_b : V = 0 \text{ at } \partial M \text{ and } \Theta(V) \in x^2\mathcal{C}^\infty(M)\}$$

is a structural Lie algebra of vector fields that is called the Θ-*structure*. For a local description as well as for an analysis of the Θ-structure we refer to [25].

All the above examples of structural Lie algebras of vector fields model the analysis on certain non-compact manifolds (giving rise to algebras of differential operators that replace the algebra of totally characteristic differential operators) on manifolds with cylindrical ends. The following example, however, models the analysis of a family of an adiabatic differential operators.

Example 1.11. Let N be a closed manifold that is the total space of a locally trivial fibration $\pi : N \to B$ of closed manifolds, let $TN/B \to N$ be the vertical tangent bundle, and let $M := N \times [0, \infty)_x$. Then

$$\mathcal{V}_a := \{V \in \Gamma(TM) : V(x) \in TN \text{ for all } x \in [0, \infty) \text{ and } V(0) \in \Gamma(TN/B)\}$$

is a structural Lie algebra of vector fields that is called the *adiabatic algebra*. If (y, z) are local coordinates on N, where again the set of variables y corresponds to variables on the base B lifted through π, and z are variables in the fibers, then adiabatic vector fields are generated by $x\partial_y$ and ∂_z. The adiabatic structure has been studied and used for instance in [62] and [63]

We shall sometimes refer to a structural Lie algebra of vector fields simply as *Lie algebra of vector fields*, when no confusion can arise. Because $\mathcal{V}$ is a finitely generated, projective $\mathcal{C}^\infty(M)$-module, using the Serre-Swan theorem [38] (recalled above, see Theorem 1.1) we obtain that there exists a vector bundle

$$A = A_{\mathcal{V}} \to M, \quad \text{such that } \mathcal{V} \cong \Gamma(A_{\mathcal{V}}), \tag{3}$$

naturally as $\mathcal{C}^\infty(M)$-modules. We shall identify from now on $\mathcal{V}$ with $\Gamma(A_{\mathcal{V}})$. The following proposition is due to Melrose.

Proposition 1.12. *If $\mathcal{V}$ is a structural Lie algebra of vector fields, then there exists a natural vector bundle map $\varrho : A_{\mathcal{V}} \to TM$ such that the induced map $\varrho_\Gamma : \Gamma(A_{\mathcal{V}}) \to \Gamma(TM)$ identifies with the inclusion map.*

Proof. Let $m \in M$. Then the fiber A_m at m of $A = A_{\mathcal{V}} \to M$ identifies with $\mathcal{V}/\mathfrak{p}_m\mathcal{V}$, where $\mathfrak{p}_m$ is the ideal of $\mathcal{C}^\infty(M)$ consisting of smooth functions on M that vanish at m. Recall now that $\mathcal{V}$ consists of vector fields on M. Then the map $A_m \to T_mM$ sends the class of $X \in \mathcal{V}$ to the vector $X(m) \in T_mM$. $\square$

Remark 1.13. The condition in Definition 1.4 that $\mathcal{V}$ has to be projective is essential. As an example consider $M = [0, 1]$ and let

$$\mathcal{V} := \Big\{ f(t)\partial_t \ : \ f : [0,1] \to \mathbb{R} \text{ smooth}\,, \ f(1) = 0\,, \ \text{ and } \\ t^k(d^m f/dt^m) \to 0 \text{ as } t \to 0 \text{ for all } k, m \in \mathbb{N} \cup \{0\}. \Big\}.$$

Then $\mathcal{V}$ is a $C^\infty(M)$-module. However, $\mathcal{V}$ is not a projective $C^\infty(M)$-module, as we can see by contradiction. Assume $\mathcal{V}$ were projective. Then there is a bundle

A over $[0,1]$ with $\mathcal{V} = \Gamma(A)$. Let s be a trivialization of A, i. e., $s(t) = f(t)\partial_t$ with f as above. Hence $\tilde{f}(t) = (1/t)f(t)$ also decays sufficiently fast; however

$$\Gamma(A) \not\ni \frac{1}{t}s(t) = \tilde{f}(t)\partial_t \in \mathcal{V}.$$

It is convenient for the following discussion to recall the definition of a Lie algebroid. General facts about Lie algebroids can be found in [17, 50] (a few basic facts are also summarized in [65]).

Definition 1.14. A *Lie algebroid* A over a manifold M is a vector bundle A over M, together with a Lie algebra structure on the space $\Gamma(A)$ of smooth sections of A and a bundle map $\varrho : A \to TM$, extended to a map $\varrho_\Gamma : \Gamma(A) \to \Gamma(TM)$ between sections of these bundles, such that

(i) $\varrho_\Gamma([X,Y]) = [\varrho_\Gamma(X), \varrho_\Gamma(Y)]$,
(ii) $[X, fY] = f[X,Y] + (\varrho_\Gamma(X)f)Y$, for any smooth sections X and Y of A and any smooth function f on M.

The map ϱ_Γ is called the *anchor of* A. If, in addition,

(iii) all vector fields $\varrho_\Gamma(\Gamma(A))$ are tangential to the faces,

then the Lie algebroid $A \to M$ is called a *boundary tangential* Lie algebroid.

We thus see that there exists an equivalence between the concept of a structural Lie algebra of vector fields $\mathcal{V} = \Gamma(A_\mathcal{V})$ and the concept of a boundary tangential Lie algebroid $\varrho : A \to TM$ such that $\varrho_\Gamma : \Gamma(A) \to \Gamma(TM)$ is injective and has range in $\mathcal{V}_b$. In order to shorten our notation, we will write Xf instead of $\varrho_\Gamma(X)f$ for the action of the sections of a Lie algebroid on functions if the meaning is clear from the context.

1.3. **Constructing new Lie algebroids from old ones.** Let $f : N \to M$ be a submersion of manifolds with corners in the above sense (which implies in particular that any fiber is a smooth manifold). Let $A = A_\mathcal{V}$ be a boundary tangential Lie algebroid over M.

Definition 1.15. The *thick pull-back* $f^\# A$ is the vector bundle over N which at the point $p \in N$ is defined to be

$$f^\# A_p := \big\{(v,w) \,|\, v \in A_{f(p)},\, w \in T_pN,\, f_*(w) = \varrho(v)\big\}$$

equipped with the vector bundle structure induced by $f^*(A) \oplus TN$.

Projection to the first component yields a surjective linear map $f^\# A_p \to A_{f(p)}$, denoted in the following by f_*, and projection onto the second component yields a linear map $f^\# A_p \to T_pN$, denoted by $f^\#\varrho$.

We obtain the commuting diagram

$$\begin{array}{ccc} f^\# A_p & \xrightarrow{f^\#\varrho} & T_pN \\ \downarrow f_* & & \downarrow f_* \\ A_{f(p)} & \xrightarrow{\varrho} & T_{f(p)}M. \end{array}$$

For example $f^\# TM = TN$.

Lemma 1.16. *The thick pull-back $f^\# A$ is a boundary tangential Lie algebroid over N with anchor map given by $f^\#\varrho$.*

Proof. Let $\Gamma_{\mathrm{vert}} TN$ denote the bundle of vertical sections X, i. e., $f_* X = 0$. This bundle coincides by definition with the analogously defined bundle of vertical sections of A. The rows of the following commutative diagram are exact.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma_{\mathrm{vert}}(f^{\#}A) & \longrightarrow & \Gamma(f^{\#}A) & \longrightarrow & \Gamma(A) \otimes_{C^\infty(M)} C^\infty(N) & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma_{\mathrm{vert}}(TN) & \longrightarrow & \Gamma(TN) & \longrightarrow & \Gamma(TM) \otimes_{C^\infty(M)} C^\infty(N) & \longrightarrow & 0
\end{array}$$

The vertical arrows are inclusions. The horizontal arrows of the second row are Lie algebra homomorphisms. The space $\Gamma(A)$ is by definition a Lie subalgebra of $\Gamma(TM)$, thus $\Gamma(A) \otimes_{C^\infty(M)} C^\infty(N)$ is a Lie subalgebra of $\Gamma(TM) \otimes_{C^\infty(M)} C^\infty(N)$. A standard diagram chase then implies that $\Gamma(f^{\#}A)$ is also a Lie subalgebra of $\Gamma(TN)$.

The fact that A is projective [respectively, boundary tangential] immediately implies that $\Gamma(f^{\#}A)$ is also projective [respectively, boundary tangential]. □

Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras. Suppose that there is given an action by derivations of $\mathfrak{g}$ on $\mathfrak{h}$:

$$\varphi : \mathfrak{g} \to \mathrm{Der}(\mathfrak{h}). \tag{4}$$

Then we can define the semi-direct sum $\mathfrak{g} \ltimes_\varphi \mathfrak{h}$ as follows. As a vector space, $\mathfrak{g} \ltimes_\varphi \mathfrak{h} = \mathfrak{g} \oplus \mathfrak{h}$, and the Lie bracket is given by

$$[(X_1, Y_1), (X_2, Y_2)] := ([X_1, X_2], \varphi_{X_1}(Y_2) - \varphi_{X_2}(Y_1) + [Y_1, Y_2]), \tag{5}$$

for any $X_1, X_2 \in \mathfrak{g}$ and $Y_1, Y_2 \in \mathfrak{h}$. We shall usually omit the index φ denoting the action by derivations in the notation for the semi-direct sum.

We want to use this construction to obtain new Lie algebroids from old ones. Assume then that we are given two Lie algebroids $A, L \to M$ over the same manifold and that $\Gamma(A)$ acts by derivations on $\Gamma(L)$. Denote this action by φ, as above. We assume that the action of $\Gamma(A)$ on $\Gamma(L)$ is compatible with the $\mathcal{C}^\infty(M)$-module structure on $\Gamma(L)$, in the sense that

$$\varphi_X(fY) = X(f)Y + f\varphi_X(Y),$$

for any $X \in \Gamma(A)$, $Y \in \Gamma(L)$, and $f \in \mathcal{C}^\infty(M)$. Assume, for simplicity, that the structural map (anchor) $L \to TM$ is zero, then we can endow $\Gamma(A \oplus L) = \Gamma(A) \oplus \Gamma(L)$ with the semi-direct sum structure obtained from $\Gamma(A) \ltimes \Gamma(L)$ such that $A \oplus L$ becomes a Lie algebroid denoted $A \ltimes L$, and called the *semi-direct product of* L *by* A [32]. Thus

$$\Gamma(A \ltimes L) = \Gamma(A) \ltimes \Gamma(L). \tag{6}$$

In the language of Lie algebroids, the action of $\Gamma(A)$ on $\Gamma(L)$ considered above is called a representation of A on L. In a similar way, the action of $\Gamma(A)$ on $\Gamma(L)$ *by derivation*, considered above, deserves to and will be called a representation by derivations of A on L. If $A \to M$ is a tangential Lie algebroid, then $A \ltimes L \to M$ will also be one.

1.4. **Differential operators.** We will from now on assume that A denotes the vector bundle determined by the structural Lie algebra $\mathcal{V}$ and vice versa.

Definition 1.17. Let $\mathrm{Diff}(\mathcal{V})$ denote the algebra of differential operators generated by $\mathcal{V}$, where the vector fields are regarded as derivations on functions.

We also want to study differential operators with coefficients in vector bundles. Let $E_1 \to M$ and $E_2 \to M$ be two vector bundles. Embed $E_i \hookrightarrow M \times \mathbb{C}^{N_i}$, $i = 1, 2$. Denote by e_i a projection in $M_{N_i}(\mathcal{C}^\infty(M))$ whose (pointwise) range is E_i. Then we define

$$\mathrm{Diff}(\mathcal{V}; E_1, E_2) := e_2 M_{N_2 \times N_1}(\mathrm{Diff}(\mathcal{V}))e_1. \tag{7}$$

This definition of $\mathrm{Diff}(\mathcal{V}; E_1, E_2)$ is independent of the choices of the embeddings $E_i \hookrightarrow M \times \mathbb{C}^{N_i}$ and of the choice of e_i. Elements of $\mathrm{Diff}(\mathcal{V}; E_0, E_1)$ will be called *differential operators generated by* $\mathcal{V}$. In the special case $E_1 = E_2 = E$ we simply write $\mathrm{Diff}(\mathcal{V}; E)$, the *algebra of differential operators on E generated by* $\mathcal{V}$.

It is possible to describe the differential operators in $\mathrm{Diff}(\mathcal{V}; E)$ locally on M as follows.

Lemma 1.18. *A linear map $D : \Gamma(E) \to \Gamma(E)$ is in $\mathrm{Diff}(\mathcal{V}; E)$ if, and only if, for any trivialization $E|_U \cong U \times \mathbb{C}^N$, above some open subset $U \subset M$, the restriction $D|_U : \Gamma(E|_U) \cong \mathcal{C}^\infty(U) \otimes \mathbb{C}^m \to \Gamma(E|_U)$ can be written as a linear combination of compositions of operators of the form $X \otimes 1$ and f, with $X \in \Gamma(A)$ and f a smooth endomorphism of the vector bundle $E|_U$.*

Proof. In a trivialization of E above some open subset, we can assume that e is a constant matrix. ☐

Example 1.19. *De Rham differential generated by $\mathcal{V} = \Gamma(A)$.*
We define for a section ω of $\Lambda^q A^*$

$$(d\omega)(X_0, \dots, X_k) = \sum_{j=0}^{q} (-1)^j X_j(\omega(X_0, \dots, \hat{X}_j, \dots, X_k)) + \sum_{0 \le i < j \le q} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k).$$

This is well defined as $\Gamma(A)$ is closed under the Lie bracket. By choosing a local basis of A we see that this defines a differential operator $\Gamma(\Lambda^q A^*) \to \Gamma(\Lambda^{q+1} A^*)$ generated by $\mathcal{V} = \Gamma(A)$, the *de Rham differential*.

Assume now that $A|_{M_0} = TM_0$. The vector bundles $\Lambda^q T^* M_0$ extend to bundles $\Lambda^q A^*$ on M. The Cartan formula (e.g. [8]) says that on M_0 the de Rham differential is the de Rham differential of ordinary differential geometry.

Definition 1.20. Let $E \to M$ be a vector bundle. An *A^*-valued connection on E* is a differential operator

$$D : \Gamma(E) \to \Gamma(E \otimes A^*)$$

such that, for any $X \in \Gamma(A)$, the induced operator $D_X : \Gamma(E) \to \Gamma(E)$ satisfies the usual properties of a connection:

$$(i)\ D_X(f\xi) = fD_X(\xi) + X(f)\xi; \quad \text{and} \quad (ii)\ D_{fX}\xi = fD_X(\xi). \tag{8}$$

It is clear from (i) that the operator D_X is of first order.

Our definition of an A^*-valued connection is only slightly more restrictive than that of A-connection introduced in [26]. (In that paper, Evens, Lu, and Weinstein considered (ii) only up to homotopy.)

Clearly if D and D' are A^*-valued connections on E and, respectively, E', then $D'' := D \otimes 1 + 1 \otimes D'$ is an A^*-valued connection on $E \otimes E'$.

See also [21, 22]

2. Lie structures at infinity

In this section we introduce the class of manifolds with a Lie structure at infinity, and we discuss some of their properties. Our definition, Definition 2.1, formalizes some definitions from [56].

In some of the first papers on the analysis on open manifolds using Lie algebras of vector, for example [12, 13, 15, 79], the vector fields considered were required to vanish at infinity. In order to obtain more general results and in agreement with the more recent papers on the subject (for example [14, 56, 64, 80]), we do not make this assumption. As a consequence, the comparison algebras that result from our setting do not have in general the property that the commutators are compact.

2.1. **Definition.** In the following, ∂M denotes the union of all hyperfaces of a manifold with corners M.

Definition 2.1. A *Lie structure at infinity* on a smooth manifold M_0 is a pair $(M, \mathcal{V})$, where

(i) M is a compact manifold, possibly with corners and M_0 is the interior of M;
(ii) $\mathcal{V}$ is a structural Lie algebra of vector fields;
(iii) $\varrho_{\mathcal{V}} : A_{\mathcal{V}} \to TM$ induces an isomorphism on M_0, i. e., $\varrho_{\mathcal{V}}|_{M_0} : A_{\mathcal{V}}|_{M_0} \to TM_0$ is a fiberwise isomorphism.

If M_0 is compact without boundary, then it follows from the above definition that $M = M_0$ and $A_{\mathcal{V}} = TM$, so a Lie structure at infinity on M_0 gives no additional information on M_0. The interesting cases are thus the ones when M_0 is non-compact. Note that all the Examples 1.5 – 1.10 are in fact Lie structures at infinity on the interior of M.

Here is now an explicit test for a Lie algebra of vector fields $\mathcal{V}$ on a compact manifold with corners M to define a Lie structure at infinity on the interior M_0 of M. This characterization of Lie structures at infinity is in the spirit of our discussion of local basis (see Equation (1) and the discussion around it).

Proposition 2.2. *We have that the Lie algebra $\mathcal{V} \subset \Gamma(M;TM)$ defines a Lie structure at infinity on M_0 if, and only if, the following conditions are satisfied:*

(i) $\mathcal{V} \subset \mathcal{V}_b$, with $\mathcal{V}_b$ defined in Equation (2), and $\mathcal{C}^\infty(M)\mathcal{V} \subset \mathcal{V}$.
(ii) If $x \in M_0$, U is a compact neighborhood U of x in M_0, and Y is a vector field on M_0, then there exists $X \in \mathcal{V}$, such that $X|_U = Y|_U$.
(iii) If $x \in \partial M = M \smallsetminus M_0$, then we can find n linearly independent vector fields $X_1, X_2, \ldots, X_n \in \mathcal{V}$, $n = \dim M$, defined on a neighborhood U of x, such that for any $X \in \mathcal{V}$, there exist smooth functions $f_1, \ldots, f_n \in \mathcal{C}^\infty(U)$ uniquely determined by

$$X = \sum_{k=1}^{n} f_k X_k \quad on \ U .$$

(iv) There are functions $f_{ijk} \in \mathcal{C}^\infty(U)$ (in particular smooth on the boundary $\partial M \cap U$) such that the vector fields X_j from (iii) satisfy $[X_i, X_j] = \sum_{k=1}^n f_{ijk} X_k$ on U.

Proof. The proof is an immediate translation of the definition of a manifold with a Lie structure at infinity using the description of projective $CI(M)$ module given at the end of Subsection 1.1 (especially Equation (1)). □

2.2. **Riemannian manifolds with Lie structures at infinity.** We now consider Riemannian metrics on $A \to M$.

Definition 2.3. A manifold M_0 with a Lie structure at infinity $(M, \mathcal{V})$ together with a Riemannian metric on $A = A_{\mathcal{V}}$, i. e., a smooth positive definite symmetric 2-tensor g on A, is called a *Riemannian manifold M_0 with a Lie structure at infinity.*

In particular g defines a Riemannian metric on M_0. The geometry of these metrics will be the topic of the next section. Note that the metrics on M_0 that we obtain are not restrictions of Riemannian metrics on M. In the following section, we will prove for example that (M_0, g) is a complete Riemannian metric. Any curve joining a point on the boundary ∂M to the interior M_0 is necessarily of infinite length.

Example 2.4. *Manifolds with cylindrical ends.*
A manifold M with cylindrical ends is obtained by attaching to a manifold M_1 with boundary ∂M_1 the cylinder $(-\infty, 0] \times \partial M_1$, using a tubular neighborhood of ∂M_1, where the metric is assumed to be a product metric. The metric on the cylinder is also assumed to be the product metric. Let t be the coordinate of $(-\infty, 0]$. By the change of variables $x = e^t$, we obtain that M is diffeomorphic to the interior of M_1 and $\mathcal{V} = \mathcal{V}_b$. Other changes of variables lead us to different Lie structures at infinity. Similarly, products of manifolds with cylindrical ends can be modeled by manifolds with corners and the structural Lie algebra of vector fields $\mathcal{V}_b$. This applies also to manifolds that are only locally at infinity products of manifolds with cylindrical ends.

2.3. **Bi-Lipschitz equivalence.** It turns out that the metric on a manifold with a Lie structure at infinity is essentially unique, namely any two such metrics are bi-Lipschitz equivalent (see the corollary below).

Lemma 2.5. *We assume that a manifold M_0 which is the interior of a compact manifold with corners M carries two Lie structures at infinity $(M, \mathcal{V}_1)$ and $(M, \mathcal{V}_2)$ satisfying $\mathcal{V}_1 \subset \mathcal{V}_2$. Furthermore, let g_j be Riemannian metrics on $A_{\mathcal{V}_j}$, $j = 1, 2$. Then there is a constant C such that*

$$g_2(X, X) \leq C g_1(X, X) \qquad \textit{for all} \quad X \in TM_0.$$

Proof. The pull-back of g_2 to $A_{\mathcal{V}_1}$ is a non-negative symmetric two-tensor on $A_{\mathcal{V}_1}$. The statement then follows from the compactness of M. □

As a consequence the volume element of g_2 is bounded by a multiple of the volume element of g_1. Furthermore, we have inclusions of L^p-functions: $L^p(M_0, g_1) \hookrightarrow L^p(M_0, g_2)$.

Corollary 2.6. *If two Riemannian metrics g_1, g_2 on M_0 are Riemannian metrics for the same Lie structure at infinity $(M, \mathcal{V})$, then they are bi-Lipschitz, i. e., there is a constant $C > 0$ with*

$$C^{-1} g_2(X, X) \leq g_1(X, X) \leq C g_2(X, X)$$

for all $X \in TM_0$. In particular, $C^{-1} d_2(x, y) \leq d_1(x, y) \leq C d_2(x, y)$, where d_i is the metric on M_0 associated to g_i.

Proof. The first part is clear. The proof of the last statement is obtained by comparing the metrics on a geodesic for one of the two metrics. □

3. Geometry of Riemannian manifolds with Lie structures at infinity

We now discuss some geometric properties of Riemannian manifolds with a Lie structure at infinity. We begin with a simple observation about volumes.

3.1. **Volume.** Let $d\mathrm{vol}$ be the volume element on M_0

Proposition 3.1. *Let M_0 be a Riemannian manifold with Lie structure $(M, \mathcal{V}, g)$ at infinity. Let $f \geq 0$ be a smooth function on M. If $\int_{M_0} f d\mathrm{vol} < \infty$, then f vanishes on each boundary hyperface of M. In particular the volume of any non-compact Riemannian manifold with a Lie structure at infinity is infinite.*

Proof. Because of Lemma 2.5, we can assume that $\Gamma(A)$ are the vector fields tangential to the boundary. For simplicity in notation, let us assume that M is a compact manifold with boundary. Let $d\mathrm{vol}'$ be the volume element on M associated to some metric on M that is smooth up to the boundary. Then $d\mathrm{vol} \geq C x^{-1} d\mathrm{vol}'$ for any boundary defining function x and a constant C.

So, if f is non-zero on ∂M with defining function x, then

$$\int_{M_0} f d\mathrm{vol} \geq \int_M f x^{-1} d\mathrm{vol}' = \infty.$$

□

3.2. **Connections and Curvature.** Most of the natural differential operators between bundles functorially associated to the tangent bundle extend to differential operators generated by $\mathcal{V}$, with the tangent bundle replaced by A. The main example is the Levi-Civita connection.

Lemma 3.2. *Let M_0 be a Riemannian manifold with a Lie structure $(M, \mathcal{V}, g)$ at infinity. Then the Levi-Civita connection $\nabla : \Gamma(TM_0) \to \Gamma(TM_0 \otimes T^*M_0)$ extends to a differential operator in $\mathrm{Diff}(\mathcal{V}; A, A \otimes A^*)$, also denoted by ∇. In particular, $\nabla_X \in \mathrm{Diff}(\mathcal{V}; A)$, for any $X \in \Gamma(A)$, and it satisfies*

$$\nabla_X(fY) = X(f)Y + f\nabla_X(Y) \quad \text{and} \quad X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

for all $X, Y, Z \in \Gamma(A)$ and $f \in \mathcal{C}^\infty(M)$. Moreover, the above equations uniquely determine ∇.

Proof. Suppose $X, Y \in \mathcal{V} = \Gamma(A) \subset \Gamma(TM)$. We shall define $\nabla_X Y$ on M_0 using the usual Levi-Civita connection ∇ on TM_0. We need to prove that there exists $X_1 \in \Gamma(A)$ whose restriction to M_0 is $\nabla_X Y$.

Recall (for example from [8]), that the formula for $\nabla_X Y$ is given by

$$\begin{aligned}(9)\quad 2\langle \nabla_X Y, Z\rangle = \langle [X,Y], Z\rangle - \langle [Y,Z], X\rangle + \langle [Z,X], Y\rangle \\ + X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle.\end{aligned}$$

Suppose $X, Y, Z \in \Gamma(A)$ in the above formula. We see then that the function $2\langle \nabla_X Y, Z\rangle$, which is defined a priori only on M_0, extends to a smooth function on M. Since the inner product $\langle\ ,\ \rangle$ is the same on A and on TM_0 (where they are both defined), we see that the above equation determines $\nabla_X Y$ as a smooth section of A. This completes the proof. □

The above lemma has interesting consequences about the geometry of Riemannian manifolds with Lie structures at infinity.

Using the terminology of A^*-valued connections (see Definition 1.20), Lemma 3.2 can be formulated as saying that the usual Levi-Civita connection on M_0 extends to an A^*-valued connection on A. Similarly, we get A^*-valued connections on A^* and on all vector bundles obtained functorially from A. We use this remark to obtain a canonical A^*-valued connection on the bundles $A^{*\otimes k} \otimes \Lambda^2 A^* \otimes \mathrm{End}(A)$. (Here $E^{\otimes k}$ denotes $E \otimes \ldots \otimes E$, k-times, as usual.)

We define the Riemannian curvature tensor as usual

$$R(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \in \Gamma(\mathrm{End}(TM_0)), \tag{10}$$

where $X, Y \in \Gamma(TM_0)$. We will regard R as a section of $\Lambda^2 T^* M_0 \otimes \mathrm{End}(TM_0)$. Then the covariant derivatives $\nabla^k R \in \Gamma(T^* M_0^{\otimes k} \otimes \Lambda^2 T^* M_0 \otimes \mathrm{End}(TM_0))$ are defined.

Corollary 3.3. *If M_0 and ∇ are as above (Lemma 3.2), then the Riemannian curvature tensor $R(X,Y)$ extends to an endomorphism of A, for all $X, Y \in \Gamma(A)$. Moreover, each covariant derivative $\nabla^k R$ extends to a section of $A^{*\otimes k} \otimes \Lambda^2 A^* \otimes \mathrm{End}(A)$ over M and hence they are all bounded on M_0.*

Proof. Fix $X, Y \in \Gamma(A)$. Then $[X,Y] \in \Gamma(A)$ and hence

$$\nabla_X, \nabla_Y, \nabla_{[X,Y]} \in \mathrm{Diff}(\mathcal{V}; A),$$

by Lemma 3.2. It follows that $R(X,Y) \in \mathrm{Diff}(\mathcal{V}; A)$. Since $R(X,Y)$ induces an endomorphism of TM_0 and M_0 is dense in M, it follows that $R(X,Y) \in \mathrm{End}(A)$.

Once we have obtained that $R \in \Gamma(\Lambda^2 A^* \otimes \mathrm{End}(A))$, we can apply the A^*-valued Levi-Civita connection to obtain

$$\nabla^k R \in \Gamma(A^{*\otimes k} \otimes \Lambda^2 A^* \otimes \mathrm{End}(A)).$$

The boundedness of $\nabla^k R$ follows from the fact that M is compact. □

The covariant derivative

$$\nabla_X : \Gamma(A^{\otimes k} \otimes A^{*\otimes j}) \to \Gamma(A^{\otimes k} \otimes A^{*\otimes (j+1)}), \quad X \in \Gamma(A), \tag{11}$$

will be called, by abuse of notation, the *A^*-valued Levi-Civita connection*, for any k and j. Sometimes, when no confusion can arise, we shall call this A^*-valued connection ∇ simply the *Levi-Civita connection.*

3.3. **Ehresmann connections.**

Definition 3.4. Let $\pi_N : N \to M$ be a submersion of manifolds with corners, and let $A \to M$ be a boundary tangential Lie algebroid. Smooth sections of the bundle $\bigwedge^p \pi_N{}^{\#}A \to N$ are called *A^*-valued p-forms on N.*

The fiber of A in $p \in M$ is denoted by A_p.

Definition 3.5. An *Ehresmann connection* on $\pi_N : N \to M$ with respect to A is a smooth field $x \mapsto \tau_x$, $x \in N$, of n-dimensional subspaces of $\pi_N{}^{\#}A$ such that $(\pi_N)_* : \pi_N{}^{\#}A \to A$ restricts to an isomorphism $\tau_x \to A_{\pi_N x}$, for all $x \in N$, the inverse $((\pi_N)_*)^{-1} : A_{\pi_N x} \to \tau_x$ is called a *horizontal lift.*

We chose the terminology "Ehresmann connection" to honor important work of Ehresmann's on the subject [24].

Example 3.6. (a) Let $\pi : V \to M$ be a vector bundle. The definition of an Ehresmann connection generalizes the notion of A^*-valued connection (in the sense of vector bundle connections). In fact, let ∇ be an A^*-valued connection. Then we obtain the Ehresmann connection as follows: For $X_0 \in V$, $p = \pi(X_0)$ we extend X_0 to a local section $X : U \to V$, where U is a neighborhood of p in M. We define the horizontal lift

$$\begin{array}{rcl} H_{X_0} : A_p & \to & (\pi^{\#}A)_{X_0} \\ Y & \mapsto & (X_*)_p(Y) - \nabla_Y X, \end{array}$$

where $\nabla_Y X \in V_p \subset (\pi^{\#}A)_{X_0}$. It is easy to check that this map does not depend on the extension X of X_0, that H_{X_0} is injective and that we have $(\pi_*)_{X_0} \circ H_{X_0} = \mathrm{id}$.

Then $\tau_{X_0} := \operatorname{im} H_{X_0}$ is an Ehresmann connection on V. The associated Ehresmann connection completely characterizes the A^*-valued connection. However, there are Ehresmann connections on V that do not come from A^*-valued connections (they are not "compatible" with the vector space structure).

(b) If the A^*-valued connection is metric with respect to a chosen metric on V, then the Ehresmann connection is tangential to the sphere bundle in V with respect to that metric.

3.4. **Geodesic flow.** For any boundary tangential Lie algebroid A equipped with a metric, let $S(A)$ be the unit tangent sphere in A,

$$S(A) := \{v \in A \,|\, \|v\| = 1\}\,.$$

The canonical projection map $\pi : S(A) \to M$ is a submersion of manifolds with corners. Let $\pi^{\#}A$ be the thick pull-back of A.

The manifold (with corners) $S(A)$ carries an Ehresmann connection and a horizontal lift H given by the Levi-Civita-connection on A.

Definition 3.7. The *geodesic spray* is defined to be the map

$$S(A) \ni X \longmapsto H_X(X) \in \pi^{\#}A, \tag{12}$$

which defines a section S of $\pi^{\#}A \to S(A)$. The flow of this vector field is called the *geodesic flow.*

Restricted to the interior of the manifold, these concepts recover the analogous concepts of ordinary Riemannian geometry.

By definition, the image of S through the anchor map is a vector field along $S(A)$ that is tangential to all the boundary faces of $S(A)$. These boundary faces are preimages of the boundary faces of M under π.

Lemma 3.8. *Let A be a boundary tangential Lie algebroid and let $X \in \Gamma(A)$. Then X is complete in the sense that the flow lines φ_t of X are defined on $\mathbb{R}$. The flow φ_t preserves the boundary depth. In particular, flow lines emanating from $N_0 := N \smallsetminus \partial N$ stay in N_0.*

Proof. For any boundary defining function x_H one has

$$\left.\frac{d}{dt}\right|_{t=0} x_H(\varphi_t) = dx_H(X) = 0,$$

hence the flow preserves the boundary depth. In particular, the flow preserves the boundary. Let $I = (a, b)$ be a maximal open interval on which one particular flow-line is defined. Let $t_i \to b$. Assume that $b < \infty$. Since N is compact, after passing to a suitable subsequence, we can assume that φ_{t_i} converges to $p \in N$. In a neighborhood of p, the flow exists, which contradicts the maximality of b. Hence $b = \infty$. The proof for $a = -\infty$ is similar. □

By applying this lemma to the geodesic flow on $N = S(A)$, we obtain two corollaries.

Corollary 3.9. *Let M_0 be a Riemannian manifold with a Lie structure $(M, \mathcal{V}, g)$ at infinity. Then M_0 is complete in the induced metric g.*

Corollary 3.10. *Let M_0 be a Riemannian manifold with a Lie structure $(M, \mathcal{V}, g)$ at infinity. Let $X \in A_p$, $p \in M$. Then the boundary depth of $\exp_p A_p$ equals the boundary depth of p.*

3.5. **Positive injectivity radius.** We already know that Riemannian manifolds with a Lie structure at infinity are complete and have bounded sectional curvature. For many analytic statements it is very helpful if we also know that the injectivity radius $\mathrm{inj}(M_0, g) = \inf_{p \in M_0} \mathrm{inj}_p$ is positive. For example see [40], where a "uniform bounded calculus of pseudodifferential operators" was defined on a manifold with bounded geometry. Hebey [30, Corollary 3.19] proves Sobolev embeddings for manifolds with bounded geometry, i. e., complete Riemannian manifolds with positive injectivity radius and bounded covariant derivatives $\nabla^k R$ of the Riemannian curvature tensor R. We will say more about this in a future paper.

Conjecture 3.11. *All Riemannian manifolds with Lie structures at infinity have positive injectivity radius.*

We now introduce two conditions on a Riemannian manifold M_0 with a Lie structure at infinity $(M, \mathcal{V})$, see Definitions 3.12 and 3.16, and prove that if any of these conditions holds, then the injectivity radius of M_0 is positive.

Definition 3.12. A manifold M_0 with a Lie structure at infinity $(M, \mathcal{V})$ is said to satisfy the *local closed extension property for* 1*-forms* if any $p \in \partial M$ has a small neighborhood $U \subset M$ such that any $\alpha_p \in A_p^*$ extends to a closed one-form on U.

Example 3.13. For the b-calculus, the local closed extension property holds, because in the notation of Example 1.5 the locally defined closed 1-forms dx^j/x^j and dy^k span $A_p^* = (T_p^b M)^*$ for any $p \in \partial M$.

Theorem 3.14. *Let M_0 be a manifold with a Lie structure at infinity $(M, \mathcal{V})$ which satisfies the local closed extension property. Then for any Riemannian metric g on A the injectivity radius of (M_0, g) is positive.*

Proof. We prove the theorem by contradiction. If the injectivity radius is zero, then, as the curvature is bounded, there is a sequence of geodesics loops $c_i : [0, a_i] \to M_0$, parametrized by arc-length, with $a_i \to 0$. Because of the compactness of $S(A)$ we can choose a subsequence such that $\dot{c}_i(0)$ converges to a vector $v \in S(A)$. Obviously, the base point of v has to be in ∂M.

By the local closed extension property, there is a closed 1-form α on a small neighborhood of the base-point of v such that $\alpha(v) \neq 0$. On the other hand, because of the closedness of α we get for sufficiently large i

$$0 = \int_0^1 \alpha(\dot{c}_i(a_i t))\, dt = \int_0^1 \alpha(\varphi_{a_i t}(\dot{c}_i(0))\, dt,$$

where $\varphi_t : S(A) \to S(A)$ denotes the geodesic flow. As $i \to \infty$, the integrand converges uniformly to $\alpha(v)$, thus we obtain the contradiction $\alpha(v) = 0$. □

In the remainder of this subsection we will prove another sufficient criterion.

Definition 3.15. Let $\varphi : [0, \infty)^{n-k} \times \mathbb{R}^k \to U \subset M$ be a local parametrization of M, i. e., φ^{-1} is a coordinate chart. Then for $v \in \mathbb{R}^n$, the local vector field $\varphi_*(v)$, i. e., the image of a constant vector field v on $\mathbb{R}^n$ is called a *coordinate vector field* with respect to φ.

Definition 3.16. A manifold M_0 with a Lie structure at infinity $(M, \mathcal{V})$ is said to satisfy the *coordinate vector field extension property* if $A_{\mathcal{V}}$ carries a Riemannian metric g such that for any $p \in \partial M$ there is a parametrization $\varphi : [0, \infty)^{n-k} \times \mathbb{R}^k \to U$ of a neighborhood U of p such that

(i) for any $v \in \mathbb{R}^n \setminus \{0\}$ the normalized coordinate vector field

$$X_v = \frac{\varphi_*(v)}{\sqrt{g(\varphi_*(v), \varphi_*(v))}}$$

which a priori is only defined on $U \cap M_0$ extends to a section of $A|_U$,

(ii) for linearly independent vectors v and w, $X_v(p)$ and $X_w(p)$ are linearly independent.

Note that Property (i) is equivalent to claiming that

$$f_v := \frac{1}{\sqrt{g(\varphi_*(v), \varphi_*(v))}}$$

extends to a smooth function on M.

Theorem 3.17. *Let M_0 be a manifold with a Lie structure at infinity $(M, \mathcal{V})$ that satisfies the coordinate vector field extension property. Then for any Riemannian metric g on A, the injectivity radius of (M_0, g) is positive.*

The theorem will follow right away from Proposition 3.19, Lemma 3.23, and Lemma 3.24, which we proceed to state and prove after the following definition.

In the following, balls in euclidean $\mathbb{R}^n$ will be called *flat balls* or sometimes even just *ϵ-ball*.

Definition 3.18. For $C \geq 1$ and $\varepsilon > 0$, we say that (M_0, g) is *locally C-bi-Lipschitz to an ε-ball* if each point $p \in M_0$ has a neighborhood that is bi-Lipschitz diffeomorphic to a flat ball of radius ε with bi-Lipschitz constant C.

Proposition 3.19. *Let (M_0, g) be a complete Riemannian manifold with bounded sectional curvature. Then the following conditions are equivalent:*

(1) (M_0, g) has positive injectivity radius.
(2) There are numbers $\delta_1 > 0$ and $C > 0$ such that any loop of length $\delta \leq \delta_1$ is the boundary of a disk of diameter $\leq C \cdot \delta$.
(3) There is $C > 0$ and $\varepsilon > 0$ such that (M_0, g) is locally C-bi-Lipschitz to a ball of radius ε.

Proof. (1)$\Longrightarrow$(3): Let $\varrho > 0$ be the injectivity radius of (M_0, g). Then $B_{\varrho/2}(p)$ is C-bi-Lipschitz to a flat ϵ-ball with C independent from p.

(3)$\Longrightarrow$(2): Under the condition of (3) any loop of length $\leq 2\varepsilon/C$ based in p lies completely inside $B_{\varepsilon/C}(p)$. On the other hand $B_{\varepsilon/C}(p)$ is contained in a neighborhood U of p which is C-bi-Lipschitz to a flat ball of radius ε. Inside a flat ball any short loop is the boundary of a disk of small diameter. Hence (2) follows from (3) with $\delta_1 := 2\varepsilon/C$ and with the same C in (2) as in (3).

(2)$\Longrightarrow$(1): Because sectional curvature is bounded from above, there is a $\varrho > 0$ such that there are no conjugate points along curves of length smaller than ϱ. For each $p \in M$, the exponential map is a local diffeomorphism from $B_\varrho(p)$ into M. We want to show that the exponential map is injective on any ball of radius $\varepsilon := \min\{\delta_1/2, \varrho/(4C)\}$. For this we assume that $\exp_p(q_1) = \exp_p(q_2)$, $q_1, q_2 \in B_\varepsilon(0) \subset T_pM$. Then

$$t \mapsto \begin{cases} \exp_p(tq_1) & \text{for } 0 \leq t \leq 1 \\ \exp_p((2-t)\,q_2) & \text{for } 1 \leq t \leq 2 \end{cases}$$

is a closed loop of length $\leq 2\varepsilon$. Because of the conditions in the proposition, this loop is the boundary of a disk of diameter $\leq \varrho/2$. Because the exponential map is a local diffeomorphism, it is not difficult to see that such disks lift to T_pM. Hence $q_1 = q_2$ which yields injectivity.

□

As a consequence of this proposition, the property of having positive injectivity radius is a bi-Lipschitz invariant inside the class of complete Riemannian manifolds with bounded curvature.

Corollary 3.20. *Let g_1 and g_2 be two complete metrics on M_0, such that there is $C > 0$ with*

$$C^{-1} g_1 \leq g_2 \leq C g_1,$$

$$|R_{g_1}| < C, \quad |R_{g_2}| < C.$$

Then (M_0, g_1) has positive injectivity radius if, and only if, (M_0, g_2) has positive injectivity radius.

Proof. Proposition 3.19 gives necessary and sufficient criteria for positive injectivity radius, that are bi-Lipschitz equivalent. □

Together with Corollary 2.6 we obtain.

Corollary 3.21. *Suppose that M_0 is a manifold with a Lie structure $(M,\mathcal{V})$ at infinity and let g and h be two metrics on A. Then $(M_0, g|_{M_0})$ has positive injectivity radius if, and only if, $(M_0, h|_{M_0})$ has positive injectivity radius.*

Definition 3.22. Let M_0 be a Riemannian manifold with a Lie structure $(M,\mathcal{V},g)$ at infinity. We say that the Lie structure at infinity is *controlled* if for all $p \in \partial M$ there is a parametrization $\varphi : [0,\infty)^{n-k} \times \mathbb{R}^k \to U$ around p, a $\delta > 0$ and a constant $C > 0$ such that for all $x \in M_0 \cap U$ and all $v \in \mathbb{R}^n$ the inequality

$$\max_{y \in B_\delta(x)} g_y\Big(\varphi_*(v)_y, \varphi_*(v)_y\Big) < C \min_{y \in B_\delta(x)} g_y\Big(\varphi_*(v)_y, \varphi_*(v)_y\Big)$$

holds. Here $B_\delta(x)$ denotes the ball of radius δ around x with respect to the metric g.

Lemma 3.23. *Let M_0 be a manifold with a Lie structure at infinity $(M,\mathcal{V})$ satisfying the coordinate vector field extension property with the metric g. Then $(M,\mathcal{V},g)$ is controlled.*

Proof. Let φ be a parametrization of a neighborhood U of $p \in \partial M$ and let $\varphi_*(v)$ and $\varphi_*(w)$ be arbitrary coordinate vector fields. Because of Property (i) in Definition 3.16, the (local) functions $f_v := 1/\sqrt{g(\varphi_*(v),\varphi_*(v))}$ and $f_w := 1/\sqrt{g(\varphi_*(w),\varphi_*(w))}$ extend to the boundary, and $X_v := f_v\varphi_*(v)$ and $X_w := f_w\varphi_*(w)$ are sections of A. For linearly independent v and w we calculate

$$\begin{aligned}
[X_v, X_w] &= [f_v\varphi_*(v), f_w\varphi_*(w)] = X_v(f_w)\varphi_*(w) - X_w(f_v)\varphi_*(v) \\
&= X_v(\log f_w)X_w - X_w(\log f_v)X_v.
\end{aligned}$$

This is again a section of A, and because of Property (ii) in Definition 3.16, X_v and X_w are even linearly independent on the boundary ∂M. As a consequence, $X_v(\log f_w)$ is bounded. On the other hand, if $v = w$, then because f_w extends to the boundary $X_v(\log f_w) = f_v\varphi_*(v)(f_v)/f_v = \varphi_*(v)(f_v)$ is also bounded. Hence, in both cases:

$$|X_v(\log g(\varphi_*(w),\varphi_*(w)))| \leq C. \tag{13}$$

By summing up, one immediately sees that C can be chosen independently from the choice of v. Hence for fixed w, (13) holds uniformly for any unit vector X.

We take two arbitrary points $y_1, y_2 \in B_\delta(x)$, $x \in U \cap M_0$. We can assume that $B_\delta(x) \subset U$. We join y_1 and y_2 by a path $c : [0,\tilde{\delta}] \to M_0$, parametrized by arc-length, $\tilde{\delta} \leq 2\delta$. We estimate

$$\begin{aligned}
&\left|\log\left(\frac{g_{c(0)}(\varphi_*(w)_{c(0)}, \varphi_*(w)_{c(0)})}{g_{c(\tilde{\delta})}(\varphi_*(w)_{c(\tilde{\delta})}, \varphi_*(w)_{c(\tilde{\delta})})}\right)\right| \\
=\ &\left|\int_0^{\tilde{\delta}} \partial_{\dot{c}(t)}\ \left(\log g_{c(t)}(\varphi_*(w)_{c(t)}, \varphi_*(w)_{c(t)})\right)\, dt\right| \\
\leq\ &\tilde{\delta}\, C.
\end{aligned}$$

Hence, the quotient is bounded by an expression that depends only on p, U, δ and global data. Thus, $(M,\mathcal{V},g)$ is controlled. ☐

Lemma 3.24. *If the boundary tangential Lie algebroid is controlled, then there is $C > 0$ and $\varepsilon > 0$ such that (M_0, g) is locally C-bi-Lipschitz to an ε-ball.*

Proof. On the ball $B_\delta(x)$ we regard two metrics: the original metric g and the metric $\tilde{g}$ with $g_x = \tilde{g}_x$ and $\tilde{g}_x$ is constant in the local coordinate chart.

These metrics are bi-Lipschitz on B with a bi-Lipschitz constant C_1. Thus, the ball $\tilde{B}_x$ of radius δ/C_1 around x with respect to the metric $\tilde{g}$ is a flat ball. Hence, $(\tilde{B}_x)_{x\in M_0}$ are neighborhoods that are uniformly bi-Lipschitz to δ/C_1-balls. □

This completes the proof of Theorem 3.17. We continue with some examples and applications.

Example 3.25. The examples 1.5 to 1.10 satisfy the coordinate vector field extension property and hence have positive injectivity radius.

Example 3.26. Let $M = [0,\infty) \times \mathbb{R} \times \mathbb{R} \ni (t,x,y)$. The vector fields $X = t^2\partial_t$, $Y = e^{-C/t}\partial_x$ and $Z = e^{-C/t}\partial_y$ span a free $C^\infty(M)$-module that is closed under Lie-brackets. Hence, it is a structural Lie algebra of vector fields $\mathcal{V}$. We define a metric g by claiming that X, Y, Z are orthonormal, then $(M, g, \mathcal{V})$ satisfies the coordinate vector field extension property.

Example 3.27. Let $M = [0,\infty) \times \mathbb{R} \times \mathbb{R} \ni (t,x,y)$. The vector fields $X = t^2\partial_t$, $Y = e^{-C/t}\big(\sin(1/t)\partial_x + \cos(1/t)\partial_y\big)$ and $Z = e^{-C/t}\big(\cos(1/t)\partial_x - \sin(1/t)\partial_y\big)$ span a free $C^\infty(M)$-module that is closed under Lie-brackets. Hence it is a structural Lie algebra of vector fields $\mathcal{V}$. We define a metric by claiming that X, Y, Z are orthonormal. In this example the normalized coordinate vector fields $e^{-C/t}\partial_x$ and $e^{-C/t}\partial_y$ are *not* contained in $\mathcal{V}$. Hence $(M, \mathcal{V})$ does not satisfy the coordinate vector field extension property. However, $(M_0 := M\backslash\partial M, g)$ is isometric to the interior of the previous example. Corollary 3.20 together with Theorem 3.17 will show that (M_0, g) has positive injectivity radius, although the conditions in Theorem 3.17 are not satisfied for $(M, \mathcal{V})$ directly.

3.6. **Adjoints of differential operators.** We shall fix in what follows a metric on A, thus we obtain a Riemannian manifold (M_0, g) with a Lie structure at infinity, which will remain fixed throughout this section.

Let us now discuss adjoints of operators in $\mathrm{Diff}(\mathcal{V})$. The metric on M_0 defines a natural volume element μ on M_0, and hence it defines also a Hilbert space $L^2(M_0, d\mu)$ with inner product $(g_1, g_2) := \int_{M_0} g_1\overline{g_2}d\mu$. The formal adjoint $D^\sharp$ of a differential operator D is then defined by the formula

$$(Dg_1, g_2) = (g_1, D^\sharp g_2), \quad \forall g_1, g_2 \in \mathcal{C}^\infty_{\mathrm{c}}(M_0). \tag{14}$$

We would like to prove that $D^\sharp \in \mathrm{Diff}(\mathcal{V})$ provided that $D \in \mathrm{Diff}(\mathcal{V})$. To check this, we first need a Lemma. Fix a local orthonormal basis $X_1, \ldots, X_n$ of A (on some open subset of M). Then $\nabla_{X_i} X = \sum c_{ij}(X)X_j$, for some smooth functions $c_{ij}(X)$. Then $\mathrm{div}(X) := -\sum c_{jj}(X)$ is well defined and gives rise to a smooth function on M. See [27], Chapter IV. A.

Lemma 3.28. *Let $X \in \Gamma(A)$ and $f \in \mathcal{C}^\infty_c(M_0)$. Then*

$$\int_{M_0} X(f)\mu = \int_{M_0} f\,\mathrm{div}(X)\mu.$$

In particular, the formal adjoint of X is $X^\sharp = -X + \mathrm{div}(X) \in \mathrm{Diff}(\mathcal{V})$.

Proof. We know ([27], Example 4.6) that

$$\mathrm{div}(fX) = f\,\mathrm{div}(X) - X(f).$$

The divergence theorem (e.g. [27], Chapter IV.A.) states for $X \in \Gamma(A)$ and compactly supported functions f

$$0 = \int_{M_0} \operatorname{div}(fX)\,\mu = \int_{M_0} f \operatorname{div}(X)\,\mu - \int_{M_0} X(f)\,\mu,$$

so

$$\int_{M_0} X(f)\mu = \int_{M_0} f \operatorname{div}(X)\mu.$$

Now, if we set $f = g_1\overline{g_2}$, we see directly that

$$\int_{M_0} g_1\overline{X(g_2)}\,\mu + \int_{M_0} X(g_1)\overline{g_2}\,\mu = \int_{M_0} X(g_1\overline{g_2})\,\mu = \int_{M_0} \operatorname{div}(X) g_1\overline{g_2}\,\mu.$$

This implies the formula for the adjoint of X. □

Corollary 3.29. *Let (M_0, g) be a Riemannian manifold with a Lie structure at infinity. The algebra $\mathrm{Diff}(\mathcal{V})$ is closed under taking formal adjoints. Similarly, if E is a hermitian vector bundle on M, then $\mathrm{Diff}(\mathcal{V}; E)$ is also closed under taking formal adjoints.*

Proof. The formal adjoint of a vector field $X \in \mathcal{V}$, when regarded as a differential operator on M_0, is given by $X^\sharp = -X + \operatorname{div}(X)$. The adjoint of $f \in \mathcal{C}^\infty(M)$ is given by $f^\sharp = \overline{f}$. Since $\mathrm{Diff}(\mathcal{V})$ is generated as an algebra by operators of the form X and f, with X and f as above, and $(D_1D_2)^\sharp = D_2^\sharp D_1^\sharp$, this proves that $\mathrm{Diff}(\mathcal{V})$ is closed under taking adjoints.

If E is a hermitian vector bundle, then we can choose the embedding $E \to M \times \mathbb{C}^N$ to preserve the metric. Then the projection e onto the range of E is a self-adjoint projection in $M_N(\mathrm{Diff}(\mathcal{V}))$. The equation $e^* = e$ satisfied by e guarantees that $\mathrm{Diff}(\mathcal{V}; E) := eM_N(\mathrm{Diff}(\mathcal{V}))e$ is also closed under taking formal adjoints. □

Similarly, we obtain the following easy consequence.

Corollary 3.30. *If $E_0, E_1 \to M$ are hermitian vector bundles, then the adjoint of an operator $P \in \mathrm{Diff}(\mathcal{V}; E_0, E_1)$ is in $\mathrm{Diff}(\mathcal{V}; E_1, E_0)$.*

Proof. Write $E := E_0 \oplus E_1$ and use the resulting natural matrix notation for operators in $\mathrm{Diff}(\mathcal{V}; E)$. □

4. Geometric operators

In this section we will see that the Hodge Laplacian $(d + d^*)^2$ on forms and the classical Dirac operator on a Riemannian (spin) manifold M_0 with a Lie structure at infinity $(M, \mathcal{V})$ are differential operators generated by $\mathcal{V} = \Gamma(A)$. (See also [46] for some similar results).

Both the classical Dirac operator and $d + d^*$ are generalized Dirac operators. We will show that any generalized Dirac operator is a differential operator generated by $\mathcal{V}$. Our approach follows closely that in [28].

4.1. **Hodge-Laplacians.**

Recall from Example 1.19 that the de Rham differential defines an element $d \in \mathrm{Diff}(\mathcal{V}; \Lambda^p A^*, \Lambda^{p+1} A^*)$.

Proposition 4.1. *On a Riemannian manifold M_0 with a Lie structure at infinity $(M, \mathcal{V})$, the Hodge-Laplace operator*

$$\Delta_p = d^*d + dd^* = (d + d^*)^2 \in \mathrm{Diff}(\mathcal{V}; \Lambda^p A^*) \tag{15}$$

that is, it is a differential operator generated by $\mathcal{V}$.

Proof. This follows directly from Corollary 3.30 and the construction in Example 1.19. □

4.2. **Principal bundles and connection-1-forms.** Let $E \to M$ be a vector bundle of rank k carrying a metric and an orientation. In this subsection, we will show that giving a metric A^*-valued connection on E is equivalent to giving a A^*-valued connection-1-form on the frame bundle of E. Our approach generalizes the case of Riemannian manifolds (see e.g. [47, II, §4]), hence we will omit some details.

For simplicity, we will assume from now on that the vector bundle $A \to M$ is orientable.

Let P be a principal G-bundle. We denote the Lie algebra of G with $\mathfrak{g}$. The most important example will be the bundle of oriented orthonormal frames of the bundle E, denoted by $\pi_P : \mathrm{P_{SO}}(E) \to M$, which is a principal $\mathrm{SO}(k)$-bundle. Differentiating the action of G gives rise to the canonical map

$$\mathfrak{g} \to \Gamma(TP), \quad V \mapsto \widetilde{V}.$$

Definition 4.2. An A^*-valued connection-1-form ω is an $\mathfrak{g} \otimes A^*$-valued 1-form on $\mathrm{P_{SO}}(A)$ satisfying the compatibility conditions

$$\omega(\widetilde{V}) = V \quad \text{and} \quad g^*\omega = \mathrm{Ad}_{g^{-1}}\, \omega \quad \text{for all } \ V \in so(\mathfrak{g}).$$

If $\mathfrak{g} \subset so(k)$, we write $\omega = (\omega_{ij})$ with respect to the standard basis of $so(k)$. In particular, the ω_{ij} are A^*-valued 1-forms on $\mathrm{P_{SO}}(A)$ satisfying $\omega_{ij} = -\omega_{ji}$.

Here "A^*-valued" is in the sense of Definition 3.4. Any A^*-valued connection-1-form on P gives rise to a G-invariant Ehresmann connection on the bundle P via $\tau = \{V \in {\pi_P}^{\#}A \,|\, \omega(V) = 0\}$. It is easy to check that this yields a one-to-one correspondence between G-invariant Ehresmann connections and connection-1-forms.

Proposition 4.3. *Let $E \to M$ be a vector bundle. For any A^*-valued connection-1-form on $\mathrm{P_{SO}}(E)$ there is a unique metric connection on E satisfying the formula*

$$\nabla e_i = \sum_{j=1}^{n} \mathcal{E}^*\omega_{ji} \otimes e_j$$

where $\mathcal{E} = (e_1, \ldots, e_n)$ is a local section of $\mathrm{P_{SO}}(E)$. Conversely, any metric connection on E arises from such an A^*-valued connection-1-form.

Note that $\mathcal{E}^*\omega_{ij}$ is a well-defined A^*-valued 1-form on M.

The proof is straightforward and runs completely analogous to [47, II, Proposition 4.4] with ordinary 1-forms replaced by A^*-valued 1-forms. As a result, we conclude that the Levi-Civita connection on A determines an $\mathrm{SO}(n)$-invariant Ehresmann connection and an A^*-valued connection-1-form on $\mathrm{P_{SO}}(A)$.

4.3. **Spin structures and spinors.** The results of the previous subsection now allow us to define the classical Dirac operator in a coordinate-free definition manner.

Definition 4.4. A *spin structure* on $(M, \mathcal{V})$ is given by a *Spin*(n)-principal bundle P_{Spin} over M together with bundle map $\theta : P_{\mathrm{Spin}} \to \mathrm{P_{SO}}(A)$ that is $\mathrm{Spin}(n) \to \mathrm{SO}(n)$-equivariant.

The (thick) pull-back of any $\mathrm{SO}(n)$-invariant Ehresmann connection on the principal $\mathrm{SO}(n)$ bundle $\mathrm{P}_{\mathrm{SO}}(A) \to M$ with respect to A defines a $\mathrm{Spin}(n)$-invariant Ehresmann connection on $P_{\mathrm{Spin}} \to M$ with respect to A. Similarly, by using the standard identification of the Lie algebra of $\mathrm{SO}(n)$ with the Lie algebra of $\mathrm{Spin}(n)$, any A^*-valued connection-1-form on $\mathrm{P}_{\mathrm{SO}}(A)$ pulls back to an A^*-valued connection-1-form on P_{Spin}.

Definition 4.5. Let P be a principal bundle with respect to the Lie group G. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H. Let $\varrho : G \to H$ be an inclusion of Lie groups. Then any A^*-valued connection-1-form on P defines an *induced* A^*-valued connection-1-form on $P \times_\varrho H$ via the formula

$$[X; Y] \mapsto \varrho_*(\omega(X)) + Y \qquad \text{for all } X \in TP \text{ and } Y \in TH$$

This definition does not depend on the choice of representative $[X; Y]$ as the map is invariant under the action of G on $P \times H$.

Now, let $\sigma_n : \mathrm{Spin}(n) \to \mathrm{SU}(\Sigma_n)$ be the complex spinor representation, e.g. the restriction of an odd irreducible complex representation of the Clifford algebra on n-dimensional space [47]. The complex dimension of Σ_n is $d_n := 2^{[n/2]}$.

Definition 4.6. Let M_0 be an n-dimensional Riemannian manifold with a Lie structure at infinity $(M, \mathcal{V}, g)$ carrying a spin structure $P_{\mathrm{Spin}}(A) \to \mathrm{P}_{\mathrm{SO}}(A)$. The *spinor bundle* is the associated vector bundle $\Sigma M := P_{\mathrm{Spin}}(A) \times_{\sigma_n} \Sigma_n$ on M.

Any metric A^*-valued connection on A gives rise to an A^*-valued connection on ΣM as follows: Proposition 4.3 defines an A^*-valued connection-1-form on $\mathrm{P}_{\mathrm{SO}}(A)$ which can be pull-backed to $P_{\mathrm{Spin}}(A)$. With Definition 4.5 applied to $\varrho = \sigma_n : \mathrm{Spin}(n) \to \mathrm{SU}(d_n) \subset \mathrm{SO}(2d_n)$ we obtain an A^*-valued connection-1-form on $P_{\mathrm{Spin}}(A) \times_{\sigma_n} \mathrm{SO}(2d_n)$ compatible with complex multiplication. Another application of Proposition 4.3 yields a complex A^*-valued connection on ΣM.

In particular, the Levi-Civita-connection on A defines then a metric connection on ΣM, the so-called *Levi-Civita-connection.*

Recall that the spinor representation Σ_n admits a $\mathrm{Spin}(n)$-equivariant linear map

$$\mathbb{R}^n \otimes \Sigma_n \to \Sigma_n : \quad X \otimes \varphi \mapsto X \cdot \varphi.$$

satisfying

$$\big(X \cdot Y + Y \cdot X + 2g(X, Y)\big) \cdot \varphi = 0 \tag{16}$$

for all $X, Y \in \mathbb{R}^n$ and all $\varphi \in \Sigma_n$, the so-called *Clifford multiplication relations.* By forming the associated bundles this gives rise to a bundle map $A \otimes \Sigma M \to \Sigma M$, called *Clifford multiplication.* Equation (16) is satisfied for all $X, Y \in A, \varphi \in \Sigma M$ in the same base point.

4.4. **Generalized Dirac operators.**

We now discuss Clifford modules in our setting.

Definition 4.7. A *Clifford module* over M is a complex vector bundle $W \to M$ equipped with a positive definite product $\langle \cdot, \cdot \rangle$, anti-linear in the second argument, an A^*-valued connection ∇^W, and a linear bundle map $A \otimes W \to W$, $X \otimes \varphi \mapsto X \cdot \varphi$ called *Clifford multiplication* such that

(1)

$$\big(X \cdot Y + Y \cdot X + 2g(X, Y)\big) \cdot \varphi = 0$$

(2) ∇^W is metric, i. e.,

$$\partial_X \langle \psi, \varphi \rangle = \langle \nabla^W_X \psi, \varphi \rangle + \langle \psi, \nabla^W_X \varphi \rangle$$

for $X \in \Gamma(A)$, $\varphi, \psi \in \Gamma(W)$,

(3) Clifford multiplication with vectors is skew-symmetric, i. e.,

$$\langle X \cdot \psi, \varphi \rangle = \langle \psi, X \cdot \varphi \rangle$$

for $\varphi, \psi \in \Gamma(W)$, $X \in \Gamma(A)$,

(4) Clifford multiplication is parallel, i. e.,

$$\nabla^W_X (Y \cdot \varphi) = (\nabla^W_X Y) \cdot \varphi + Y \cdot (\nabla^W_X \varphi)$$

for $X \in \Gamma(A)$, $Y \in \Gamma(A)$, $\varphi \in \Gamma(W)$.

The *generalized Dirac operator* associated to a Clifford module W is the first order operator $\not{D}^W$ obtained by the following composition:

$$\Gamma(W) \xrightarrow{\nabla^W} \Gamma(W \otimes A^*) \xrightarrow{\mathrm{id}\otimes\#} \Gamma(W \otimes A) \xrightarrow{\cdot} \Gamma(W)$$

$$\not{D}^W := \cdot \circ (\mathrm{id} \otimes \#) \circ \nabla^W.$$

The last map is Clifford multiplication and $\# : A^* \to A$ is the isomorphism given by g.

The principal symbol of any generalized Dirac operator is elliptic, as for any non zero vector X, Clifford multiplication $X\cdot$ is an invertible element of $\mathrm{End}(\Sigma M)$.

Example 4.8. For any $p \in M$, we define the Clifford algebra $\mathrm{Cl}(A_p)$ as the universal commutative algebra generated by A_p subject to the relation

$$X \cdot Y + Y \cdot X + 2g(X,Y)1 = 0.$$

Let $\mathrm{Cl}(A)$ be the Clifford-bundle of (A, g), i. e., the bundle whose fibers at the point $p \in M$ is the Clifford algebra $\mathrm{Cl}(A_p)$. The A^*-valued connection on A extends to an A^*-valued connection on $\mathrm{Cl}(A)$. Let $W = \mathrm{Cl}(A)$, equipped with the module structure given by left multiplication. After identifying with the canonical isomorphism $\mathrm{Cl}(A) \cong \Lambda^*(A)$, $e_{i_1} \cdot \ldots \cdot e_{i_k} \mapsto e^b_{i_1} \wedge \ldots \wedge e^b_{i_1}$ for an orthonormal basis (e_i) with dual (e^b_i), the generalized Dirac operator on this bundle is the de Rham operator $d + d^*$.

Example 4.9. If M is spin then the spinor bundle from Definition 4.6 is also a Clifford module. The corresponding Dirac operator is called the *(classical) Dirac operator*.

Example 4.10. If M is even dimensional and if A carries a Kähler structure, then the Dolbeault operator $\sqrt{2}\left(\overline{\partial} + \overline{\partial}^*\right)$ acting on $(0,*)$-forms is a generalized Dirac operator.

Example 4.11. Let W be any complex vector bundle with a positive definite scalar product and a Clifford multiplication such that (1) and (3) of Definition 4.7 are satisfied. Then there is always a connection ∇^W on W satisfying the compatibility conditions (2) and (4). This can be seen as follows: If M is spin then we can write

$$W = \Sigma M \otimes V$$

where V is isomorphic to the homomorphisms from ΣM to W that are $\mathrm{Cl}(A)$ equivariant. V carries a compatible metric. After choosing any metric A^*-valued connection ∇^V on V, the product connection on W satisfies (2) and (4).

If M is not spin, the connection can be constructed locally on a open covering in the same way, and the connection can then be glued together by using a partition of unity, hence we obtain the statement.

For any two sections σ_1 and σ_2 of W, we let

$$(\sigma_1, \sigma_2) := \int_M \langle \sigma_1, \sigma_2 . \rangle$$

This expression is not always defined. However, it is well-defined scalar product on generalized L^2-spinor fields, i.e., generalized spinor fields with $\int_M \langle \sigma_i, \sigma_i \rangle < \infty$. It is also well-defined, if one of the sections s_i or s_j has compact support and the other is locally L^2.

For the benefit of the reader, let us recall the following basic result (see for example [28]).

Proposition 4.12 ([28])**.** *Generalized Dirac operators $D\!\!\!/$ on complete Riemannian manifolds are formally self-adjoint and essentially self-adjoint. More concretely, for smooth sections σ_i we have*

$$(D\!\!\!/\sigma_1, \sigma_2) = (\sigma_1, D\!\!\!/\sigma_2)$$

if at least one of the sections σ_1 or σ_2 has compact support, and the maximal and minimal extension of $D\!\!\!/$ coincide, hence $D\!\!\!/$ extends uniquely to a self-adjoint operator densely defined on the L^2-sections of W.

For any choice of a connection as in the above theorem, the resulting Dirac operator is generated by $\mathcal{V}$.

Theorem 4.13. *Let $W \to M$ be a Clifford module. Then the Dirac operator on W is generated by $\mathcal{V}$:*

$$D\!\!\!/^W \in \mathrm{Diff}(\mathcal{V}; W).$$

Proof. The Dirac operator is the composition of Clifford multiplication and the A^*-valued connection ∇^W on W. Clifford multiplication is a zero order differential operator generated by $\mathcal{V}$. The A^*-valued connection ∇^W on W is a first order differential operator generated by $\mathcal{V}$. Hence the Dirac operator is also a first order differential operator generated by $\mathcal{V}$. □

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