The smallest Dirac eigenvalue in a spin-conformal class and cmc-immersions

Bernd Ammann *

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Abstract

Let us fix a conformal class \([g_0]\) and a spin structure \(\sigma\) on a compact manifold \(M\). For any \(g \in [g_0]\), let \(\lambda^+_1(g)\) be the smallest positive eigenvalue of the Dirac operator \(D\) on \((M, g, \sigma)\). In a previous paper we have shown that

\[
\lambda^+_{\min}(M, g_0, \sigma) := \inf_{g \in [g_0]} \lambda^+_1(g) \text{vol}(M, g)^{1/n} > 0.
\]

In the present article, we enlarge the conformal class by certain singular metrics. We will show that if \(\lambda^+_{\min}(M, g_0, \sigma) < \lambda^+_{\min}(S^n)\), then the infimum is attained on the enlarged conformal class. For proving this, we have to solve a system of semi-linear partial differential equations involving a nonlinearity with critical exponent:

\[
D \varphi = \lambda |\varphi|^{2/(n-1)} \varphi.
\]

The solution of this problem has many analogies to the solution of the Yamabe problem. However, our reasoning is more involved than in the Yamabe problem as the eigenvalues of the Dirac operator tend to \(+\infty\) and \(-\infty\).

Using the Weierstraß representation, the solution of this equation in dimension 2 provides a tool for constructing new periodic constant mean curvature surfaces.

Keywords: Dirac operator, eigenvalues, conformal geometry, critical Sobolev exponents

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1 Introduction

1.1 A supremum is attained

Let $M$ be a compact manifold, on which we fix a conformal class $[g_0]$ and a spin structure $\sigma$. Let $L^q(\text{im}_{C^\infty} D_{g_0})$ be the $L^q$-closure of the image of the Dirac operator. For $q \geq 2n/(n+1)$ we define

$$\mathcal{F}_q^{(M,g_0,\sigma)} : L^q(\text{im}_{C^\infty} D_{g_0}) \setminus \{0\} \rightarrow \mathbb{R}, \quad \mathcal{F}_q^{(M,g_0,\sigma)}(\psi) := \psi \mapsto \frac{\int \langle \psi, D^{-1}\psi \rangle}{\|\psi\|_{L^q}^2}. $$

where we assume that $D^{-1}(\psi)$ is chosen orthogonal to the kernel of $D$. In Lemma 5.1 we will see that this functional is bounded.

We are mainly interested in the case $q = 2n/(n+1)$. In this case the functional is conformally invariant and admits applications to conformal geometry.

**Theorem 1.1.** Let $q = 2n/(n+1)$. We assume that

$$\sup \mathcal{F}_q^{(M,g,\sigma)} > \sup \mathcal{F}_q^{(S^n, \text{can})},$$

where $(S^n, \text{can})$ is the sphere with the standard conformal structure. Then $\mathcal{F}_q^{(M,g,\sigma)}$ attains its supremum.

The value on the right hand side can be explicitly calculated to be

$$\sup \mathcal{F}_q^{(S^n, \text{can})} = \frac{2}{n} \omega_n^{1/n}. $$

Our interest in this theorem comes from two applications. The first one is an application to spectral theory, the second one is an application to the theory of constant mean curvature surfaces.

1.2 Application to spectral theory

For each metric $g \in [g_0]$ let $\lambda_1^+(g)$ be the smallest positive eigenvalue of the (classical) Dirac operator $D$ on $(M, g, \sigma)$. Because of Proposition 4.1 the dimension of the kernel of the Dirac operator is a conformal invariant.

The main subject of this application is the following infimum:

$$\lambda^+_{\min}(M, g_0, \sigma) := \inf_{g \in [g_0]} \frac{\lambda_1^+(g) \cdot \text{vol}(M, g)^{1/n}}{\text{vol}(S^n) \cdot \text{vol}(S^n, \text{can})}. $$
We will see that the infimum in this equation is attained for a metric that may have some singularities.

Let us summarize what has been known previously about the invariant $\lambda_{\min}^+$ from above.

If $D$ is invertible, then J. Lott has proven [Lot86] that $\lambda_{\min}^+$ is greater than zero. In [Amm03], we have shown that this even holds if $D$ is not invertible. Hijazi [Hij86] proved for $\dim M \geq 3$ that

$$
(\lambda_{\min}^+(M, g_0, \sigma))^2 \geq \frac{n}{4(n-1)} \lambda_Y(M, g_0).
$$

Christian Bär [Bär92] showed a 2-dimensional analogue, which is non-trivial iff $M$ has genus zero, i.e. $M \cong S^2$. Indeed he obtained

$$
(\lambda_{\min}^+(S^2, g_{\text{can}}))^2 := \inf_{g \in [g_{\text{can}}]} (\lambda_{\min}^+(S^2, g))^2 \text{area}(S^2, g) = 4\pi
$$

and the infimum is attained exactly for the metrics of constant Gauss curvature.

On the other hand, it is known [Amm03, AHM03a] that

$$
\lambda_{\min}^+(M, g_0, \sigma) \leq \lambda_{\min}^+(S^n),
$$

where $S^n$ carries the round metric.

The relation to Theorem 1.1 is based on the equation

$$
\lambda_{\min}^+(M, g_0, \sigma) = (\sup \mathcal{F}^{(M, g, \sigma)})^{-1}
$$

(see [Amm03] and or Proposition 5.5). As a consequence (1.2) is equivalent to

$$
\lambda_{\min}^+(M, g, \sigma) < \lambda_{\min}^+(S^n) = \frac{n}{2} \omega_n^{1/n}.
$$

We obtain

**Theorem 1.7.** Let $M$ be a compact manifold of dimension $\geq 2$ with a fixed conformal class $[g_0]$ and a spin structure $\sigma$. Assume (1.6) holds.

(A) There is a spinor field $\varphi \in C^{1,\alpha}(\Sigma M) \cap C^{\infty}(\Sigma(M \setminus \varphi^{-1}(0)))$ on $(M, g_0)$ such that

$$
D_{g_0} \varphi = \lambda_{\min}^+ |\varphi|^{2/(n-1)} \varphi, \quad \|\varphi\|_{2n/(n-1)} = 1.
$$

(B) There is a generalized conformal metric $g \in \mathcal{M}(g_0)$ such that

$$
\lambda_1^+(g) = \lambda_{\min}^+.
$$
If \( \dim M = 2 \), then the metric \( g \) is smooth and the set \( S_g \) of singularities of \( g \) is a discrete, finite set. Furthermore

\[
\#S_g < \text{genus}(M).
\]

In particular, if \( M \) is diffeomorphic to a 2-torus, then there are no singular points.

In order to explain the theorem, we have to define the space of generalized conformal metrics \( \mathcal{M}(g_0) \).

**Definition.** Let \( \mathcal{M}(g_0) \) be the set of metrics in \( [g_0] \) with unit volume. We define the set of generalized conformal metrics \( \mathcal{M}(g_0) \) to be

\[
\mathcal{M}(g_0) := \{ g = f^{2/(n-1)} g_0 \mid f \geq 0, f \in C^1, \alpha(M) \cap C^\infty(M \setminus f^{-1}(0)) \forall \alpha \in (0, 1), \text{supp } f = M, \text{vol}(M, g) := \int f^{n/(n-1)} = 1 \}.
\]

We say that \( g \) is regular, if \( f^{-1}(0) = \emptyset \). Otherwise \( g \) is singular and \( S_g := f^{-1}(0) \) is called the singular set.

Note that singular metrics are not complete.

For any generalized conformal metric \( g \), we define the smallest positive Dirac eigenvalue on \((M, g, \sigma)\) as

\[
\lambda^+_1(g) := \inf \left\{ \frac{\|\psi\|^2_{L^2(M, g)}}{\int \langle \psi, D^{-1}_g \psi \rangle_g d\text{vol}_g} \mid \psi \in \text{im}_{C^\infty} D_g, \|\psi\|_{L^\infty} < \infty, \int \langle \psi, D^{-1}_g \psi \rangle_g d\text{vol}_g > 0 \right\}.
\]

Here \( \text{im}_{C^\infty} D_g \) is the image of the Dirac operator on \((M \setminus S_g, g)\) acting on smooth spinors. The scalar product \( \int \langle \cdot, \cdot \rangle_g d\text{vol}_g \) is the \( L^2 \)-scalar product on spinors on \( M \setminus S_g \). Obviously, this definition coincides with the smallest positive Dirac eigenvalue in the ordinary sense if \( g \) is regular. In Lemma 9.1 we show that

\[
\inf_{g \in \mathcal{M}(g_0)} \lambda^+_1(g) = \inf_{g \in \mathcal{M}(g_0)} \lambda^+_1(g).
\]

**Remark.** The largest negative eigenvalue. Denote the largest negative eigenvalue of the Dirac operator by \(-\lambda^-_1(g)\), i.e. \(-\lambda^-_1(g)\) is the negative eigenvalue closest to 0. All statements of the present article still hold if we replace the smallest positive eigenvalue \( \lambda^+_1(g) \) by \( \lambda^-_1(g) \). We omit the proofs as they are the same up to some sign changes.
1.3 Application to constant mean curvature surfaces

If the dimension of $M$ is 2, then (1.8) reads as $D\varphi = \lambda_{\min}^+ |\varphi|^2 \varphi$. The spinorial Weierstrass representation ([Wei66, KS96], see also [Abr89, Bär98, Fri98, Amm98]) tells us, that such a solution can be used to construct certain immersions with constant mean curvature.

Figure 1: A periodic branched conformal cmc surface, visualized by K. Grosse-Brauckmann with GRAPE

More exactly, let $(M, g)$ be a compact Riemann surface together with its universal covering $\pi : \tilde{M} \to M$. A periodic branched conformal cmc immersion based on $(M, g)$ is an smooth map $F : \tilde{M} \to \mathbb{R}^3$ together with finitely many points $p_1, \ldots, p_k$, the so-called branching points, and together with a homomorphism $h : \pi(M) \to \mathbb{R}^3$, the periodicity map such that

1. For any $\gamma \in \pi_1(M)$, and $x \in \tilde{M}$ one has
   
   $F(x \cdot \gamma) = F(x) + h(\gamma)$

   where $\cdot$ denotes the action of $\pi_1$ on $M$ via Deck transformation.

2. The restriction of $F$ to $\tilde{M} \setminus \pi^{-1}(\{p_1, \ldots, p_k\})$ is a conformal immersion.

3. $dF_q = 0$ for any $q \in \pi^{-1}(\{p_1, \ldots, p_k\})$. The order of the first non-vanishing term in the Taylor development of $dF$ in $q$ is called the branching index of $F$ at $q$. 

5
(4) \( F \left( \tilde{M} \setminus \pi^{-1}(\{p_1, \ldots, p_k\}) \right) \) is an immersed surface with constant mean curvature.

As an application of our theorem, we obtain a principle for constructing such surfaces.

**Principle for construction of cmc-surfaces.** Assume that the Riemann spin surface \((M, g, \sigma)\) carries a metric \(g\) such that the first positive eigenvalue of the Dirac operator is smaller than \(2\sqrt{\pi/\text{area}(M, g)}\). Then there is a periodic branched conformal cmc immersion \(F\) based on \((M, g)\). The regular homotopy class is determined by the spin-structure \(\sigma\) and all branching indices are even. In case \(M\) is


A similar statement also holds for immersions into \(S^3\) and into hyperbolic space \(H^3\), see [Amm].

1.4 Manifolds for which (1.6) holds

The inequality (1.6) has been proven for several classes of manifolds. It is known that it holds for not conformally flat manifolds of dimension \(\geq 7\) [AHM03a].

Inequality (1.6) has also been shown if \(M\) is conformally flat and if the mass endomorphism is not identically zero (after a possible change of orientation if \(\dim M \equiv 3 \mod 4\)) [AHM03b]. The mass endomorphism is a section of \(\text{End} \Sigma M\) defined as the zero order term of the development of the Green function for the Dirac operator at the diagonal with respect to a conformal coordinate map.

1.5 Relations to the Yamabe problem

Here, we want to note that several equations with similar critical nonlinearities have been the subject of intensive research. As an example, we want to mention the Yamabe problem. Yamabe, Trudinger and Aubin [Yam60, Tru68, Aub76] have shown that if the Yamabe invariant 
\[
\lambda_Y(M, [g_0])
\]

is smaller than the Yamabe invariant of the round sphere, then the Yamabe equation

\[
\left( 4 \frac{n-1}{n-2} \Delta + \text{scal} \right) u = \lambda_Y(M, [g_0]) u^{p-1}
\]

admits a positive smooth solution \(u\). By deep results, due to Aubin, Yau and Schoen [Aub76, SY79, Sch84, SY88], and summarized in [LP87], it has been shown that any manifold \(M\) which is not conformal to the round sphere actually satisfies

\[
\lambda_Y(M, [g_0]) < \lambda_Y(S^n).
\]

(1.10)
For spin manifolds a simpler proof is available in [Wit81, PT82, AH03]. A consequence of (1.10) is that any conformal class \([g_0]\) on \(M\) contains a metric with constant scalar curvature on \(M\).

The proofs of these statements have many analogies with the problem of this article. However, the relation is stronger than a mere analogy: From (1.4), (6.1) and (6.2) one can easily deduce that (1.6) implies (1.10).

Similar problems and statements also arise in the solution of the Yamabe problem on CR-manifolds, studied by Jerison and Lee [JL89].

1.6 Structure of the article

The structure of the article is as follows. At first, in Section 2, we will outline the proofs for the main statements without giving details. Then in Section 3 we collect several facts from the standard elliptic theory that will be needed later on. In Section 4 we recall the formula for the change of the Dirac operator under conformal change of the metric. Section 5 studies the functional \(F_q\). It is followed by a section in which we study the case of the sphere with its canonical conformal structure. We then need to find conditions under which the supremum \(F_q\) is attained. Section 7 proves the regularity theorem. We obtain \(C^{1,a}\)-solutions to the subcritical problem and we see that (1.6) implies the existence of a solution to the critical problem. Sections 8 and 9 deal with the singularities that may appear.

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2 Proof of the main theorems

In this section we want to give proofs for the main theorems. For details we refer to statements in the following sections. As a consequence, this section also serves as an overview over the tools that we will need.

Proof of Theorem 1.7 (A). For any \(q \geq q_D := 2n/(n+1)\) we define

\[
\mu_q := \sup F_q
\]

where the infimum ranges over all smooth non-zero spinors on \((M, g_0, \sigma)\) in the image of \(D\). For \(q > q_D\), the problem is “subcritical” and we will see in Proposition 7.4...
that there is a weak solution $\psi_q$ of
\[
D^{-1}\psi_q - \mu_q|\psi_q|^{q-2}\psi_q \in \ker D, \quad \psi_q \in L^q(D), \quad \|\psi_q\|_{L^q} = 1. \tag{2.1}
\]

Our Regularity Theorem (Theorem 7.3) will imply that the solution is $C^{0,\alpha}$.

By a straightforward calculation we see that the following duality principle holds.

**Lemma 2.2 (Duality principle).**

Let $p, q > 1$, $\lambda, \mu \in \mathbb{R}^+$ with $p^{-1} + q^{-1} = 1$ and $\lambda \mu = 1$.

(i) If $\psi$ satisfies (2.1), then $\varphi := |\psi|^{q-2}\psi$ satisfies
\[
D\varphi = \lambda |\varphi|^{p-2}\varphi, \quad \varphi \in L^p(D), \quad \|\varphi\|_{L^p} = 1 \tag{2.3}
\]

(ii) If $\varphi$ satisfies (2.3), then $\psi := \mu D\varphi$ satisfies (2.1).

We will study the behavior of the solutions $\varphi_q := D^{-1}\psi_q$ for $q \to q_D$. Theorem 7.5 shows that the solutions $\varphi_q$ are uniformly bounded in $L^\infty$. Applying the Regularity Theorem (Theorem 7.3) once again, we see that $\psi_q$ is even uniformly bounded in $C^{0,\alpha}$, hence $\varphi_q$ is uniformly bounded in $C^{1,\alpha}$. After taking a subsequence $q_i \to q_D$, the spinor fields $\varphi_{q_i}$ converge to a $C^{1,\alpha}$-solution of (1.8).

**Proof of Theorem 1.7 (B).**

It is not difficult to see that statement (B) follows directly from statement (A). If we have a solution as in (A), then we set $g_1 := f^{2/(n-1)}g_0$ with $f = \langle \varphi, \varphi \rangle$. Note that $\text{vol}(M, g_1) = \int |\varphi|^{2n/(n-1)} = 1$, hence $g_1 \in \mathcal{M}(g_0)$.

The transformation formula for the Dirac operator under conformal changes (Proposition 4.1) implies that there is a spinor $\varphi_1$ on $(M, g_1, \sigma)$ such that
\[
D_{g_1}\varphi_1 = \lambda \varphi_1, \quad |\varphi_1|_{g_1} \equiv 1
\]
on $M \setminus S_{g_1}$. Then obviously, $\lambda_1^+(g_1) = \lambda_{\min}^+$ and (B) follows.

Statement (C) of Theorem 1.7 will follow from Proposition 8.4. With Proposition 5.5 we see that Theorem 1.1 is equivalent to Theorem 1.7 (B).

### 3 Elliptic regularity

In this section we want to collect some facts about elliptic regularity for Dirac operators. The proofs of these statements are analogous to proofs of the corresponding statements for the Laplace operators as done e.g. in [GT77] and [Ada75]. Details will be given in [Amm].

Let $(M, g)$ be a manifold with a spin structure $\sigma$.  

Definition (Sobolev spaces). Let us introduce the function spaces we need. For any smooth spinor \( \psi \), and \( q \in (1, \infty) \), \( k \in \mathbb{N} \), we define the \( H^q_k \)-norm of \( \psi \) as

\[
\| \psi \|_{H^q_k} = \left\| \bigotimes_{k \text{-times}} \nabla \psi \right\|_{L^q}.
\]

Obviously, the \( H^q_k \)-norms for different connections are equivalent.

If \( M \) is compact, an alternative way to introduce Sobolev norms on spinors is by setting

\[
\| \psi \|_{\tilde{H}^q_k} = \left\| D^k \psi \right\|_{L^q} + \| \pi \psi \|_1,
\]

where \( \pi \) is the \( L^2 \)-orthogonal projection to the kernel of \( D \), where \( \| \cdot \|_1 \) is an arbitrary norm on the kernel, and where \( |D|^k \) should be understood in the spectral sense, i.e. if \( \varphi \) is an eigenspinor of \( D \) to the eigenvalue \( \lambda \), then \( |D|^k \varphi = |\lambda|^k \). Such powers of differential operators are well-understood because \( M \) is compact (see [See67, Tay81]). One consequence of the properties of such operators is that the norms \( \tilde{H}^q_k \) are equivalent to the \( H^q_k \)-norms. The definition of the \( \tilde{H}^q_k \)-norms has the advantage that it extends to arbitrary \( k \in \mathbb{R} \).

We define the Sobolev space \( H^q_k(\Sigma M) = H^q_k \) as the completion of the smooth spinors with respect to this norm.

Definition (Hölder spaces). For \( \alpha \in (0,1] \), the Hölder-spaces \( C^{0,\alpha}(\Sigma M) \) and \( C^{1,\alpha}(\Sigma M) \) are defined to be the completions of \( C^\infty(\Sigma M) \) with respect to the Hölder norms

\[
\| \varphi \|_{C^{0,\alpha}} := \text{hö}l_\alpha(\varphi)
\]

\[
\| \varphi \|_{C^{1,\alpha}} := \| \varphi \|_{\text{sup}} + \text{hö}l_\alpha(\nabla \varphi)
\]

\[
\text{hö}l_\alpha(Q) := \sup\left\{ \frac{|Q(x) - P_\gamma Q(y)|}{d(x,y)^\alpha} \mid x, y \in M, x \neq y, P_\gamma \text{ is the parallel transport along a shortest geodesic } \gamma \text{ from } x \text{ to } y \right\}
\]

From elliptic theory, we know the following statements [Amm].

**Theorem 3.1 (Sobolev embedding theorem).** Let \( k, s \in \mathbb{R}, k \geq s \) and \( q, r \in (1, \infty) \).

(a) If

\[
\frac{1}{r} - \frac{s}{n} \geq \frac{1}{q} - \frac{k}{n} \tag{3.2}
\]

then \( H^q_k(\Sigma M) \) is continuously embedded into \( H^s_r(\Sigma M) \).
(b) (Rellich-Kondrakov theorem). If strict inequality holds in (3.2), then the inclusion \( H^q_k(\Sigma M) \hookrightarrow H^r_s(\Sigma M) \) is a compact operator.

(c) Suppose \( 0 < \alpha < 1, m \in \{0,1\} \) and
\[
\frac{1}{q} < \frac{k - m - \alpha}{n}.
\]
Then \( H^q_k(\Sigma M) \) is continuously embedded into \( C^{m,\alpha}(\Sigma M) \).

**THEOREM 3.3 (Interior \( L^p \) estimates).** Let \( \Omega \) be open in \( \mathbb{R}^n \), equipped with a Riemannian metric and the spin structure \( \sigma \) induced from \( \mathbb{R}^n \), and \( K \subset \Omega \) compact. We assume \( g(v,v) \geq \zeta g_{\text{eucl}}(v,v) \) for all \( v \in T\Omega \) and \( g \in C^{k+1} \) with \( \|g\|_{C^{k+1}} \leq Z \). Let \( D \) be the Dirac operator on \((\Omega,g,\sigma)\). Let \( \psi \) be in \( H^q_k(\Sigma \Omega) \) and let \( \varphi \) be a weak solution of
\[
D\varphi = \psi
\]
on \( \Omega \). Then \( \varphi \in H^q_{k+1}(\Sigma K) \) and
\[
\|\varphi\|_{H^q_{k+1}(\Sigma K)} \leq C \cdot \left( \|\psi\|_{H^q_k(\Sigma \Omega)} + \|\varphi\|_{L^q(\Sigma \Omega)} \right),
\]
where \( C = C(\zeta, Z, \Omega, K) \).

**THEOREM 3.4 (Interior Schauder estimates).** Let \( \Omega \) be open in \( \mathbb{R}^n \), equipped with a Riemannian metric and the spin structure induced from \( \mathbb{R}^n \), and \( K \subset \Omega \) compact. Let \( \psi \) be a \( C^{0,\alpha} \)-spinor on \( \Omega \). We assume \( g(v,v) \geq \zeta g_{\text{eucl}}(v,v) \) for all \( v \in T\Omega \) and \( \|g\|_{C^{k+1,\alpha}(\Omega)} \leq Z \). Then for any \( C^1 \)-solution \( \varphi \) of
\[
D\varphi = \psi
\]
we have \( \varphi \in C^{k+1,\alpha}(K) \) and
\[
\|\varphi\|_{C^{k+1,\alpha}(K)} \leq C \cdot (\|\psi\|_{C^{k,\alpha}(\Omega)} + \|\varphi\|_{C^{0}(\Omega)}),
\]
where \( C \) only depends on \( n, \alpha, \text{diam}(\Omega), \text{dist}(K,\partial\Omega), \zeta \) and \( Z \).

By gluing together the local versions via charts, we obtain global \( L^p \) and Schauder estimates.

**THEOREM 3.5 (Global \( L^p \) estimates).** Let \((M,g,\sigma)\) be a compact Riemannian spin manifold and \( \psi \in H^q_k(\Sigma M) \). Then any weak solution of
\[
D\varphi = \psi
\]

satisfies \( \varphi \in H^q_{k+1}(\Sigma M) \), and there is a constant \( C = C(M,g,\sigma) \) such that
\[
\|\varphi\|_{H^q_{k+1}} \leq C \left( \|\psi\|_{H^q_k} + \|\varphi\|_{L^q} \right).
\]
THEOREM 3.6 (Global Schauder estimates). Let \((M, g, \sigma)\) be a compact Riemannian spin manifold and \(\psi \in C^{k,\alpha}(\Sigma M)\). Then any solution of
\[
D\varphi = \psi,
\]
satisfies \(\varphi \in C^{k+1,\alpha}(\Sigma M)\), and there is a constant \(C = C(M, g, \sigma)\) such that
\[
\|\varphi\|_{C^{k+1,\alpha}} \leq C (\|\psi\|_{C^{k,\alpha}} + \|\varphi\|_{C^0}).
\]

4 Conformal changes and the Dirac operator

In this section we recall the formula for the Dirac operator under conformal changes of the metric.

PROPOSITION 4.1 ([Hit74, Hij86]). Let \(g = e^{2u}g_0\). There is an isomorphism of vector bundles \(F : \Sigma(M, g_0, \sigma) \rightarrow \Sigma(M, g, \sigma)\) which is a fiberwise isometry such that
\[
D_g(F(\varphi)) = F\left(e^{-\frac{n+1}{2}u}D_{g_0}e^{\frac{n+1}{2}u}\varphi\right).
\]

It is convenient to define for
\[
\widehat{F}(\varphi) = F(e^{-\frac{n+1}{2}u}\varphi).
\]

We will use this isomorphism \(\widehat{F}\) to identify spinors associated to conformal metrics. With this identification the transformation formula reads as
\[
D_g(e^u\psi) = D_{g_0}(\psi).
\]

A calculation now shows that \(F_q\) is conformally invariant, if and only of \(q = 2n/(n + 1)\).

5 The functional \(F_q\)

In this section we derive the main properties of \(F_q\).

Wet set
\[
\mu_q =: \mu_q(M, g_0, \sigma) := \sup_{\psi \in (\text{Im}C^\infty D_{g_0}) \setminus \{0\}} F_q(\psi),
\]
and as before \(q_D = \frac{2n}{n+1}\). Because \(D_{g_0}\) has positive eigenvalues, we see that \(\mu_q \in (0, \infty]\).
**Lemma 5.1.** Let \( q \in [q_D, \infty) \). Then

1. \( \mu_q < \infty \),
2. \( \mathcal{F}_q \) extends to a differentiable functional on \( L^q(\text{im}_{C^\infty} D_{g_0}) \setminus \{0\} \) and the derivation is given by

\[
\mathcal{F}_q' (\varphi) = \frac{2}{\|\psi\|^2_{L^q}} \int \langle D^{-1}\psi - \rho_{q,\psi}, |\psi|^{q-2}\psi, \varphi \rangle,
\]

where \( \rho_{q,\psi} = \mathcal{F}_q(\psi) \|\psi\|^{2-q}_{L^q} \).

**Proof.** The Embedding Theorem 3.1 implies that for any \( q \) there is a constant \( C_q \) such that

\[
\|\psi\|_{H^2_{-1/2}} \leq C_q \|\psi\|_{L^q} \quad \forall \psi \in \text{im}_{C^\infty} D_{g_0}.
\]

Hence

\[
\frac{\int \langle \psi, |D|^{-1}\psi \rangle}{\|\psi\|^2_{L^q}} \leq C'
\]

and hence (1). Similarly one proves that \( \mathcal{F}_q \) extends to \( L^q(\text{im}_{C^\infty} D_{g_0}) \setminus \{0\} \).

Now take \( p \) with \( p^{-1} + q^{-1} = 1 \). If \( \psi \) is in \( L^q \), \( q \geq \frac{2n}{n+1} \), then \( D^{-1}\psi \) is in \( H^1_\varphi \) and hence in \( L^p \). Also \( |\psi|^{q-2}\psi \in L^p \). Hence the right hand side of (5.2) defines a continuous functional on \( L^q \) which we denote by \( \varphi \mapsto \text{RHS}_\varphi(\varphi) \). Similarly one sees that

\[
\mathcal{F}_q(\psi + \varphi) - \mathcal{F}_q(\psi) - \text{RHS}_\varphi(\varphi) \leq o(\|\varphi\|_{L^q}),
\]

hence \( \mathcal{F}_q \) is Fréchet differential with derivative \( \text{RHS}_\psi \). \( \square \)

**Proposition 5.3 (Properties of \( \mu_q \)).** The function \( [q_D, \infty) \to (0, \infty) \), \( q \mapsto \mu_q \) is

1. nonincreasing in \( q \),
2. is continuous from the right,
3. \( \mu_2 = (\lambda_1^+(g_0))^{-1} \).
4. \( \mu_{q_D} \) is conformally invariant, i.e. for \( g_1 \in [g_0] \)

\[
\mu_{q_D}(M, g_0, \sigma) = \mu_{q_D}(M, g_1, \sigma).
\]
5. \( \mu_{q_D}(M, g, \sigma) \geq \mu_{q_D}(S^n, g_{\text{can}}) \).

Here \( g_{\text{can}} \) is the metric on \( S^n \) of constant sectional curvature and volume 1.
Proof.

(1) This is evident as \( q \mapsto \|\psi\|_{L^q} \) is nondecreasing.

(2) For a given \( q \geq q_D \), we take a smooth spinor field \( \psi \) such that \( F_q(\psi) \geq \mu_q - \varepsilon \).

Observe that for \( q' \geq q \)

\[
F_{q'}(\psi) = \frac{\|\psi\|_{L^q}}{\|\psi\|_{L^{q'}}} F_q(\psi).
\]

The function \( q' \mapsto \|\psi\|_{L^{q'}} \) is continuous, hence if \( q' \) is sufficiently close to \( q \), then

\[
\mu_{q'} \geq F_{q'}(\psi) \geq F_q(\psi) - \varepsilon \geq \mu_q - 2\varepsilon.
\]

Because \( q \mapsto \mu_q \) is nonincreasing, the statement follows.

(3) follows directly, by decomposing \( L^2(\text{im}_C D_{g_0}) \) into eigenspaces for \( D \).

(4) This follows from section 4.

(5) This has been proven in [Amm03] unless \( \ker D \neq \{0\} \) and \( n = 2 \). The idea of the proof is as follows. In a small neighborhood \( B_\varepsilon(p) \) of a given point \( p \), we conformally blow up a sphere on which we construct a suitable test spinor. The corresponding Rayleigh quotient for \( D^2 \) is close to the smallest eigenvalue of \( D^2 \) on the sphere.

Hence, the statement follows for the case that \( D \) is invertible. If \( D \) is not invertible, we have to ensure that the test spinor is almost orthogonal to the kernel of \( D \). If \( n \geq 3 \), this can be done by choosing \( B_\varepsilon(p) \) sufficiently small.

The remaining case \( \ker D \neq \{0\} \) and \( n = 2 \) was shown in [AHM03a].

In Proposition 7.4 we will see that the supremum defining \( \mu_q \) is attained for \( q > q_D \) by a \( C^0,\alpha \)-function which in turn implies that the function

\[
[q_D, \infty) \rightarrow (0, \infty), \quad q \mapsto \mu_q
\]

is also continuous from the left.

**Corollary 5.4.** The smallest positive eigenvalue of the Dirac operator is bounded from below by \( \mu_{qD}^{-1} \):

\[
\lambda^+_1 \geq \mu_{qD}^{-1} \quad \lambda^+_\text{min} \geq \mu_{qD}^{-1}.
\]

**Proposition 5.5.**

\[
\lambda^+_\text{min} = \mu_{qD}^{-1}
\]

**Proof.** By definition of \( \lambda^+_\text{min} \) and \( \mu_{qD} \) and the previous corollary we have to show that

\[
\sup_{g \in [g_0]} \lambda^+_1(g)^{-1} \text{vol}(M, g)^{-1/n} \geq \sup F_{qD}.
\]

For any \( \varepsilon > 0 \) we take a \( \psi_\varepsilon \) with \( F_{qD}(\psi_\varepsilon) \geq \sup F_{qD} - \varepsilon \) and \( \|\psi_\varepsilon\|_{L^{qD}} = 1 \). After a small perturbation of \( \psi_\varepsilon \) we can assume that \( \psi_\varepsilon \) has no zeros. We set

\[
g_\varepsilon := |\psi_\varepsilon|^{4/(n+4)} g_0.
\]
Then \( \text{vol}(M, g_\varepsilon) = 1 \) and \(|\psi| \equiv 1\). Hence,

\[
\mathcal{F}_{Q_D}^{(M, g_\varepsilon, \sigma)}(\psi_\varepsilon) = \mathcal{F}_{Q_D}^{(M, g_\varepsilon, \sigma)}(\psi_\varepsilon) = \mathcal{F}_2^{(M, g_\varepsilon, \sigma)}(\psi_\varepsilon) \leq \lambda_1^+(g_\varepsilon)^{-1}.
\]

This implies the proposition.

## 6 The case of the sphere

For the sphere \( S^n \) with the canonical conformal structure can the invariant \( \lambda_{\text{min}}^+ \) is not hard to calculate. On the one hand, the Hijazi inequality tells us, that

\[
\lambda_{\text{min}}^+(S^n) \geq \frac{n}{4(n-1)} \lambda_Y(S^n) = \frac{n^2}{4} \omega_n^{2/n}.
\]  

(6.1)

On the other hand, the sphere of constant sectional curvature 1 carries a Killing spinor \( \psi \) to the constant \(-1/2\), i.e. it satisfies

\[
\nabla_X \psi = -(1/2)X \cdot \psi.
\]

This implies that the length of \( \psi \) is constant. Because of \( D\psi = (n/2)\psi \) we obtain

\[
\lambda_{\text{min}}^+(S^n) \leq \lambda_1^+(S^n, g_0) \text{vol}(S^n, g_0)^{1/n} \leq \frac{n}{2} \omega_n^{1/n}.
\]  

(6.2)
Together with Proposition 5.5 we obtain
\[ \lambda_{\min}^+(S^n) = \left( \sup F_{2n/(n+1)} \right)^{-1} = \frac{n}{2} \omega_n^{-1/n}. \]

In order to clarify the role of \( S^n \) we will prove the following proposition although it will not be needed in what follows.

**PROPOSITION 6.3.** *On the standard sphere we have*
\[ \sup F_{2n/(n+1)} = \frac{2}{n} \omega_n^{-1/n}, \]
*and if the supremum is attained in \( \psi \), then \( \psi \) is the image of a Killing spinor to the Killing constant \(-1/2\) under an orientation preserving conformal diffeomorphism \( S^n \to S^n \).*

**Lemma 6.4.** *Let \( (M, g, \sigma) \) be an arbitrary Riemannian spin manifold (not necessarily complete or compact). Assume that there is a spinor \( \psi \) of constant length 1 and with \( D\psi = \lambda\psi \). Then*
\[ \text{scal} = 4 \frac{n-1}{n} \lambda^2 - 4|\tilde{\nabla}\psi|^2, \]
*where \( \tilde{\nabla}_X \psi := \nabla_X \psi - \frac{\lambda}{n} X \cdot \psi \) denotes the Friedrich connection on spinors.*

**Proof of the lemma.** The Friedrich connection is a metric connection, hence
\[ 0 = d^* d\langle \psi, \psi \rangle = 2 \text{Re}(\langle \tilde{\nabla}^* \tilde{\nabla} \psi, \psi \rangle - \langle \tilde{\nabla} \psi, \tilde{\nabla} \psi \rangle). \]
The Schrödinger-Lichnerowicz formula yields
\[ \left( D - \frac{\lambda}{n} \right)^2 = \tilde{\nabla}^* \tilde{\nabla} + \frac{\text{scal}}{4} - \frac{n-1}{n^2} \lambda^2. \]
This yields
\[ \frac{n-1}{n^2} \lambda^2 = \langle \tilde{\nabla} \psi, \tilde{\nabla} \psi \rangle + \frac{\text{scal}}{4}. \]

**Proof of the theorem.** It is evident from above that on the sphere the supremum is attained by Killing spinors to the Killing constant \(-1/2\). Because \( F_{qD} \) with \( qD = \frac{2n}{n+1} \) is conformally invariant, the supremum is also attained by the image of such a Killing spinor under an orientation preserving conformal map \( S^n \to S^n \).
We have to show that the supremum is not attained by any other spinor. For proving this, assume that $F_{2n/(n+1)} = \sup F_{2n/(n+1)}$, $\|\psi\|_{L^q} = 1$. As it is a supremum it satisfies the Euler-Lagrange equation (2.1).

On the open subset $U := M \setminus \psi^{-1}(0)$ we define the metric

$$g_1 := |\psi|^4 \frac{4}{n+1} g_{\text{can}}.$$  

Hence $\text{vol}(U', g_1) = 1$. In this metric (2.1) transforms into a solution of

$$D g_1 \psi = \lambda_{\text{min}}^+ \psi \quad |\psi|_{g_1} \equiv 1.$$  

Applying the lemma we obtain

$$\text{scal}_{g_1} \leq 4 \frac{n-1}{n} (\lambda_{\text{min}}^+)^2$$

According to standard results in the Yamabe problem, any metric in the form $g = f^2 g_0$, $f$ continuous on $M$ and smooth on $U' := M \setminus f^{-1}(0)$ satisfies

$$Y(M, [g_0]) \leq \frac{\int_{U'} \text{scal}_{g} \text{dvol}_g}{\text{vol}(U', g)^{\frac{n+2}{n}}}.$$  

Equality in this inequality implies that $f^{-1}(0) = \emptyset$, and that $g$ is a Yamabe metric. Applied to $M = S^n$, $f := |\psi|^\frac{2}{n}$ we obtain

$$n(n-1) \omega_n^{2/n} = Y(S^n, g_{\text{can}}) \leq 4 \frac{n-1}{n} (\lambda_{\text{min}}^+)^2 \int_U \text{dvol}_{g_1} = (n-1)n \omega_n^{2/n}.$$  

Hence, we have equality, which implies that $g_1$ is a Yamabe metric of volume 1 on $S^n$. This implies that there is an orientation preserving conformal map $h : S^n \to S^n$ such that $g_1 := \omega_n^{-2/n} h^* g_{\text{can}}$ and that $\psi$ is the pullback under $h$ of a Killing spinor on $(S^n, g_{\text{can}})$.  

\section{Solution of the equation}

If the supremum of $\mathcal{F}_q$ is actually attained by a function $\psi_q$ with $\|\psi_q\|_{L^q} = 1$, then because of (5.2) $\psi_q$ is a solution of

$$D^{-1} \psi - (\|\psi\|_{L^q}^{2-q} \mu_q \ |\psi|^{q-2} \psi) \in \ker D.$$  

(7.1)

Obviously, $\mathcal{F}_q(r \psi) = \mathcal{F}_q(\psi)$ for all $r \in \mathbb{R}^+$. As a consequence, any solution can be rescaled to one with $\|\psi\|_{L^q} = 1$. Hence, we study

$$D^{-1} \psi - \mu_q |\psi|^{q-2} \psi \in \ker D, \quad \|\psi\|_{L^q} = 1, \quad \psi \in L^q(D).$$  

(7.2)
THEOREM 7.3 (Regularity theorem). Suppose that $\psi \in L^q$, $q \geq q_D$ is a solution of equations (2.1). Suppose that there is an $r > q_D$ such that $\|\psi\|_{L^r} < \infty$. We choose $k, K > 0$ such that $\|\psi\|_{L^r} < k$ and $\mu_q \geq K$. Then for any $\alpha \in (0, 1)$ there is a constant $C$ depending only on $(M, g, \sigma)$, $q$, $r$, $K$, $k$ and $\alpha$ with
\[
\|\psi\|_{C^{0,\alpha}} \leq C \quad \text{and} \quad \|D^{-1}\psi\|_{C^{1,\alpha}} \leq C
\]

Proof. Without loss of generality we can assume $r < n$. We apply the Global $L^p$-estimates 3.5 to $2L^r$, and then the Sobolev embedding 3.1 and obtain
\[
D^{-1}(\psi) \in H^r_1 \hookrightarrow L^s
\]
with $1 - (n/r) = -n/s$, or equivalently $s = rn/(n - r)$. Hence, by equation 2.1, we know $\psi \in L^{s(q - 1)} \hookrightarrow L^{s(q_D - 1)}$. We set
\[
r' := s(q_D - 1) = \frac{n - 1}{n + 1} \frac{rn}{n - r}.
\]
The inequality $r > q_D$ implies $r' > r$. By iterating this bootstrap argument we obtain higher and higher regularity for $\psi$. The function
\[
r \mapsto \frac{n - 1}{n + 1} \frac{rn}{n - r}
\]
tends to $\infty$ if $r$ converges to $n$ from below. This shows that $\psi \in L^r$ for all $r \in (0, \infty)$. Thus $\psi \in H^r_1$ for all $r \in (0, \infty)$ and by the Sobolev embedding theorem 3.1 (c) one obtains $\psi \in C^{0,\alpha}$ for any $\alpha > 0$. And finally by the Schauder estimates 3.6 $D^{-1}\psi \in C^{1,\alpha}$. The uniform upper bound $C$ of the norms is now clear from the construction. \hfill \Box

Remark. Outside the zero locus $\psi^{-1}(0)$ we can continue the bootstrap argument and apply inductively the interior Schauder estimates Theorem 3.4. We conclude that $\psi$ is smooth on $M \setminus \psi^{-1}(0)$. Similarly, if $p := (q - 1)^{-1} + 1$ is an even integer, then $\varphi \mapsto |\varphi|^{p-2}\varphi$ is also smooth in $0$, and hence $\psi$ is smooth on $M$.

PROPOSITION 7.4. For any $q > q_D$ the supremum $\mu_q$ is attained by a spinor field $\psi_q \in C^{0,\alpha} \text{ which is a solution of (2.1)}$.

Proof. Let $\psi_i$ be a maximizing sequence for $\mathcal{F}_q$, i.e. $\mathcal{F}_q(\psi_i) \to \mu_q$. We may assume $\|\psi_i\|_{L^q} = 1$. After taking a subsequence there is a $\psi_\infty \in L^q$ such that $\psi_i$ converges weakly to $\psi_\infty$ in $L^q$. Because the embedding $L^q \hookrightarrow H_{1/2}^2$ is compact, we can again choose a subsequence, and we obtain, in addition to the weak convergence in $L^q$, strong convergence to $\psi_\infty$ in $H_{1/2}^2$. Hence,
\[
\mu_q \leq \limsup \mathcal{F}_q(\psi_i) \leq \limsup \frac{\|\psi_i\|^2_{H_{1/2}^2}}{\|\psi_i\|_{L^q}^2} \leq \mathcal{F}_q(\psi_\infty) \leq \mu_q.
\]
As a consequence, we have equality in all inequalities, in particular $\|\psi_\infty\|_{L^q} = 1$. By the variational formula $\psi_\infty$ is a solution of

$$D^{-1}\psi_\infty - \mu_q|\psi_\infty|^{q-2}\psi_\infty \in \ker D.$$ 

Now the regularity theorem tells us that $\psi_\infty$ is actually $C^{0,\alpha}$.

**Theorem 7.5.** Let $\psi$ be a solution of (2.1) with $q \in (q_D, 2]$ and $\mu_q \geq \mu_{q_D}^Sn + \varepsilon$, $\varepsilon > 0$. Then there is a constant $C = C(M, g, \sigma, \varepsilon)$ such that

$$\|\psi\|_{C^0} < C.$$ 

**Proof.** Assume that such a constant does not exist. Then we find a sequence of solutions $\psi_k$ to (2.1) with $q = q_k$, $\mu_q = \mu_k \geq \mu_{q_D}^Sn + \varepsilon$ and

$$\|\psi_k\|_{C^0} \to \infty. \quad (7.6)$$ 

Let us assume for a moment that $q_\infty := \limsup q_k > q_D$. In this case, we can choose a subsequence with $q_k \to q_\infty$. Then the regularity theorem (Theorem 7.3) applied to a real number $r \in (q_D, q_\infty)$, says that $\|\psi_k\|_{C^0}$ is bounded, in contradiction to (7.6).

Hence, $\lim q_k = q_D$.

We now study

$$\varphi_k := \frac{1}{\mu_k} D^{-1}\psi_k,$$

which are solutions of the dual equation (2.3).

There is a sequence of points $s_k \in M$ with

$$m_k := |\varphi_k(s_k)| = \max\{\varphi_k(x) \mid x \in M\} \to \infty.$$ 

Since $M$ is compact, we can assume, after passing to a further subsequence, that $s_k$ converges to $p \in M$.

Now, we define rescaled geodesic normal coordinates $(\sigma_k)^{-1}$ via the formula

$$\sigma_k(x) = \exp_p \left( \delta_k x + \exp_p^{-1}(z_k) \right),$$

where $\delta_k = m_k^{2-p_k} \to 0$.

Then a straightforward calculation shows that

$$\tilde{\varphi}_k(x) := m_k^{-1} \varphi_k \circ \sigma_k(x)$$

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is a solution of
\[ D_k \tilde{\varphi}_k = \frac{1}{\mu_k} |\tilde{\varphi}_k|^{p_k - 2} \tilde{\varphi}_k, \]
where \( D_k \) is the Dirac operator associated to the metric \( g_k := \delta_k^{-2} \sigma_k^*(g) \). Writing
the metric \( g \) on \( M \) in geodesic normal coordinates centered in \( p \) one immediately
sees that on any closed ball the sequence \( g_k \) converges to the euclidean metric in the
\( C^\infty \)-topology.

Note that with respect to \( g_k \)
\[ \|\tilde{\varphi}_k\|_{C^0} \leq |\tilde{\varphi}_k(0)| = 1. \]

Hence, we may apply the interior \( L^p - \) and Schauder-estimates 3.3 and 3.4 to conclude
that
\[ \|\tilde{\varphi}_k\|_{C^{1,\alpha}} \leq C(R), \quad \forall k > k(R), \]
with constants \( C(R) \) and \( k(R) \).

Compare \( D \) with the Dirac operator \( D_{\text{flat}} \) on flat \( \mathbb{R}^n \) (see e.g. [Pfa02] or [AHM03a]
for explicit computations). We have
\[ \| (D_{\text{flat}} - D_k) \tilde{\varphi}_k \|_{C^{0,\alpha}(B_R(0))} \leq \tau_k \|\tilde{\varphi}_k\|_{C^{1,\alpha}(B_R(0))}, \]
with \( \tau_k \to 0. \)

We choose a sequence of radii \( R_m \to \infty \). After passing to a diagonal sequence, we
see that there is a spinor \( \tilde{\varphi}_\infty \) on \( \mathbb{R}^n \), such that \( \tilde{\varphi}_k|_{B_R(0)} \) converges to \( \tilde{\varphi}_\infty|_{B_R(0)} \in C^{1,\alpha}(B_R(0)) \).

For any \( \varepsilon > 0 \) and \( R > 0 \) there is a \( k_0(R, \varepsilon) \) such that
\[ \|\tilde{\varphi}_k\|_{L^p_k(B_R(0))} \leq 1 + \varepsilon \]
for all \( k \geq k_0 \). Because of the \( C^1 \)-convergence \( \tilde{\varphi}_k \to \tilde{\varphi}_\infty \), Fatou’s lemma yields
\[ \|\tilde{\varphi}_\infty\|_{L^q(B_R(0))} \leq 1 \]
for any \( R \), and finally for \( R = \infty \). Then \( \tilde{\varphi}_\infty \) is a solution of
\[ D_{\text{flat}} \tilde{\varphi}_\infty = \mu_{q_D} |\tilde{\varphi}_\infty|^{p_D - 2} \tilde{\varphi}_\infty, \]
and then \( \tilde{\psi}_\infty := \mu_{q_D} D_{\text{flat}} \tilde{\varphi}_\infty \) is a solution of
\[ (D_{\text{flat}})^{-1} \tilde{\psi}_\infty - \mu_{q_D} |\tilde{\psi}_\infty|^{q_D - 2} \tilde{\psi}_\infty \in \ker D. \]
We identify $\tilde{\psi}_\infty$ via stereographic projection and via $F$ with an $L^{q_D}$-spinor $\tilde{\psi}_\infty$ on $S^n$ with the identification provided in section 4. Note that

$$k b_1 k L q_{D} (S^n) = k e_1 k L q_{D} (\mathbb{R}^n),$$

$$Z S^n h D_1 b_1 i = Z n h D_1 e_1;$$

$$e_1 i = q_{D} k e_1 k q_{D} L q_{D} (n).$$

which is apparently a contradiction to our assumption $\mu_{q_D} \geq \mu_{q_D}^S + \varepsilon$. □

**PROPOSITION 7.7.** If there is a $q_0 > q_D$ and an $r > q_D$ such that for all $t \in (q_D, q_0)$ there is a solution $\psi_t$ of equation (2.1) with $q = t$ such that $\|\psi_t\|_{L^r}$ is bounded by a constant $C$ independent from $t$, then there is a sequence $t_i \to q_D$ such that $\psi_{t_i}$ converges in the $C^1$-topology to a solution of equation (2.1) with $q = q_D$.

**Proof.** For $q$ sufficiently close to $q_D$, we know because of Proposition 5.3 that $\mu_q$ is bounded from below by a positive constant. Thus, we can apply the regularity theorem (Theorem 7.3) which tells us that $(\psi_t)$ is uniformly bounded in $C^{0, \alpha}$. By elliptic regularity, $D^{-1}(\psi_t)$ is uniformly bounded in $C^{1, \alpha}$. Hence, for a sequence $(t_i)$ with $t_i > q_D$, converging to $q_D$, the spinor fields $D^{-1}(\psi_{t_i})$ converge in the $C^1$-topology to a $C^1$-spinor field $\varphi_{q_D}$. Then $D \varphi_{q_D}$ is a solution of equation (2.1) with $q = q_D$. □

8 The size of the singular set

In this section we recall the weak Unique Continuation Property. We obtain directly that the set on which a solution of (7.1) vanishes does not contain any non-empty open set.

**THEOREM 8.1 (Weak Unique Continuation Property [BBMW, Theorem 2.1]).** Let $P$ be a locally bounded function on a connected Riemannian manifold $M$ and let $\varphi$ be a solution of

$$D \varphi = P \cdot \varphi$$

that vanishes on a nonempty open set. Then $\varphi$ is identically 0.

**COROLLARY 8.2.** The singular set of $C^1$-solutions to equation (2.3) does not contain any nonempty open set.

In case that the function $P$ is smooth, there is a stronger result by Christian Bär.
THEOREM 8.3 (Nodal sets for Dirac Operators [Bär97]). Let $M^n$ be compact and connected and let $\varphi$ be a solution of

$$D\varphi = P\varphi$$

where $P$ is a smooth endomorphism. Then the zero set of $\varphi$ has at most Hausdorff dimension $n - 2$. If $n = 2$, then the zero set is discrete.

In the case $n = 2$ this theorem implies that the zero set of a solution to (1.8) is discrete.

PROPOSITION 8.4. On a compact spin surface $(M, g, \sigma)$ of genus $\gamma$ let $\varphi$ be a solution of equation

$$D\varphi = \lambda |\varphi|^2 \varphi \quad ||\varphi||_{L^2} = 1.$$ 

Then the number of zeros of $\varphi$ is at most $\gamma - 1 + \frac{1}{4\pi}$.

In particular, this implies part (C) of Theorem 1.7.

Proof. We set $g_1 := |\varphi|^4 g$. Then outside the zero set we know by Lemma 6.4 that the curvature of $g_1$ is at most $\lambda^2$. Furthermore $\text{vol}(M, g_1) = 1$. Let $\varphi(p) = 0$. The integral of the geodesic curvature with respect to $g_1$ over small simply closed loop around $p$ is close to $-2(2j_p + 1)\pi$, where $j_p$ is the order of the first non-vanishing term in the Taylor expansion of $\varphi$ in $p$. We remove small open disks around the zeros of $\varphi$ from $M$, and we obtain a surface with boundary $M'$. With the Gauss-Bonnet theorem we obtain

$$2\pi \chi(M') = \int_{M'} K_{g_1} + \int_{\partial M'} k_{g_1} \leq \lambda^2 - \sum (2j_p + 1)2\pi.$$ 

And hence $2\pi (2 - 2\gamma) = 2\pi \chi(M) \leq \lambda^2 - 4\pi \sum j_p$, which implies the proposition. □

9 The smallest positive Dirac eigenvalue on singular spaces

As before we fix a spin structure $\sigma$ and a conformal structure $[g_0]$ on a compact manifold $M$. Let $g = f^{4/(n-1)} \cdot g_0$ be a generalized conformal metric. We define the smallest positive Dirac eigenvalue on $(M, g, \sigma)$ by (1.9). It is evident, that if $g$ is regular (i.e. $f > 0$), then $\lambda_1^+(g)$ is the smallest positive eigenvalue of the Dirac operator.
LEMMA 9.1. The following identities hold

$$\inf_{g \in \mathcal{M}(g_0)} \lambda_1^+(g) = \inf_{g \in \mathcal{M}(g_0)} \lambda_1^+(g) = \mu_{qD}^{-1}.$$ 

Proof. Obviously,

$$\inf_{g \in \mathcal{M}(g_0)} \lambda_1^+(g) \geq \inf_{g \in \mathcal{M}(g_0)} \lambda_1^+(g).$$

From the previous section we already know that \( \lambda_1^+(g) \geq \mu_{qD}^{-1} \) for all \( \mathcal{M}(g_0) \). Let us show that the argument even holds on \( \overline{\mathcal{M}(g_0)} \). For this we write \( g = e^{2u} g_0 \) on \( M \setminus S_g \), \( u \in C^\infty(M \setminus S_g) \). Using the conformal change formula we see that

$$\lambda_1^+(g) = \inf \left\{ \frac{\int \langle h^{-1} \psi, \psi \rangle \, d\text{vol}_{g_0}}{\int \langle \psi, D_{g_0}^{-1} \psi \rangle \, d\text{vol}_{g_0}} \mid \psi \in \text{im} C^\infty D_{g_0}, \quad \| h^{-(n+1)/2} \psi \|_{L^\infty} < \infty, \right.$$ 

$$\int \langle \psi, D_{g_0}^{-1} \psi \rangle \, d\text{vol}_{g_0} > 0 \right\}.$$

In particular all test spinors have finite \( \| \psi \|_{L^\infty} \). Applying Hölder’s inequality and using \( \int h^n = 1 \), we see that \( (h^{-1} \psi, \psi)_{g_0} \geq \| \psi \|_{L^2}^2 \). Hence,

$$\lambda_1^+(g) \geq \mu_{qD}^{-1} \quad \forall g \in \overline{\mathcal{M}(g_0)}.$$

On the other hand, let \( \psi \) be a spinor field such that \( \mathcal{F}_{qD}(\psi) \geq \mu_{qD} - \varepsilon \) with arbitrary small \( \varepsilon > 0 \). We can assume that \( \psi \) is a smooth spinor field without zeros. Then for \( g := |\psi|^{4/(n+1)} g_0 \) we obtain \( \lambda_1^+(g) \leq \mathcal{F}_{qD}(\psi)^{-1} \). Hence

$$\inf_{g \in \mathcal{M}(g_0)} \lambda_1^+(g) \leq \mu_{qD}^{-1}.$$

\( \Box \)

Remark. Much recent research deals with Laplacians acting on functions on singular spaces. The function Laplacian can be defined on a much larger category of spaces, i.e. on metric spaces carrying a measure with certain compatibility conditions. (See e.g. [Gro99], [KMS01], [KS01] and the references therein.)

References


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Author’s addresses:

Bernd Ammann  
Fachbereich Mathematik  
Universität Hamburg  
Bundesstraße 55  
20146 Hamburg  
Germany

E-Mail: ammann@math.uni-hamburg.de
WWW: http://www.math.uni-hamburg.de/home/ammann